CHAPTER 7

THE ADMITTANCE MODEL AND NETWORK CALCULATIONS

The typical power transmission network spans a large geographic area and involves a large number and variety of network components. The electrical characteristics of the individual components are developed in previous chapters and now we are concerned with the composite representation of those components when they are interconnected to form the network. For large-scale system analysis the network model takes on the form of a *network matrix* with elements determined by the choice of parameter.

There are two choices. The current flow through a network component can be related to the voltage drop across it by either an admittance or an impedance parameter. This chapter treats the admittance representation in the form of a *primitive model* which describes the electrical characteristics of the network components. The primitive model neither requires nor provides any information about how the components are interconnected to form the network. The steady-state behavior of all the components acting together as a system is given by the *nodal admittance matrix* based on nodal analysis of the network equations.

The nodal admittance matrix of the typical power system is large and sparse, and can be constructed in a systematic building-block manner. The building-block approach provides insight for developing algorithms to account for network changes. Because the network matrices are very large, *sparsity*

techniques are needed to enhance the computational efficiency of computer programs employed in solving many of the power system problems described in later chapters.

The particular importance of the present chapter, and also Chap. 8, which develops the ***nodal impedance matrix***, becomes evident in the course of power-flow and fault analysis of the system.

7.1 BRANCH AND NODE ADMITTANCES

In per-phase analysis the components of the power transmission system are modeled and represented by passive impedances or equivalent admittances accompanied, where appropriate, by active voltage or current sources. In the steady state, for example, a generator can be represented by the circuit of either Fig. 7.1(*a*) or Fig. 7.1(*b*). The circuit having the constant emf E_s, series impedance Z_a, and terminal voltage V has the voltage equation

$$E_s = IZ_a + V \tag{7.1}$$

Dividing across by Z_a gives the current equation for Fig. 7.1(*b*)

$$I_s = \frac{E_s}{Z_a} = I + VY_a \tag{7.2}$$

where $Y_a = 1/Z_a$. Thus, the emf E_s and its series impedance Z_a can be interchanged with the current source I_s and its shunt admittance Y_a, provided

$$I_s = \frac{E_s}{Z_a} \quad \text{and} \quad Y_a = \frac{1}{Z_a} \tag{7.3}$$

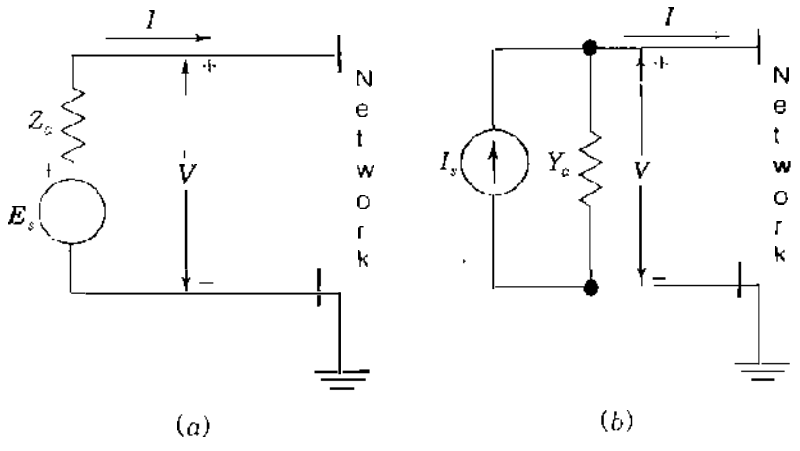


FIGURE 7.1
Circuits illustrating the equivalence of sources when $I_S = E_S/Z_a$ and $Y_a = 1/Z_a$.

Sources such as E_s and I_s may be considered externally applied at the nodes of the transmission network, which then consists of only passive branches. In this chapter subscripts a and b distinguish branch quantities from node quantities which have subscripts m, n, p, and q or else numbers. For network modeling we may then represent the typical branch by either the *branch impedance* Z_a or the *branch admittance* Y_a, whichever is more convenient. The branch impedance Z_a is often called the *primitive impedance*, and likewise, Y_a is called the *primitive admittance*. The equations characterizing the branch are

$$V_a = Z_a I_a \qquad \text{or} \qquad Y_a V_a = I_a \tag{7.4}$$

where Y_a is the reciprocal of Z_a and V_a is the voltage drop across the branch in the direction of the branch current I_a. Regardless of how it is connected into the network, the typical branch has the two associated variables V_a and I_a related by Eqs. (7.4). In this chapter we concentrate on the branch admittance form in order to establish the nodal admittance representation of the power network, and Chap. 8 treats the impedance form.

In Sec. 1.12 rules are given for forming the bus admittance matrix of the network. Review of those rules is recommended since we are about to consider an alternative method for $\mathbf{Y}_{\text{bus}}$ formation. The new method is more general because it is easily extended to networks with mutually coupled elements. Our approach first considers each branch separately and then in combination with other branches of the network.

Suppose that only branch admittance Y_a is connected between nodes (m) and (n) as part of a larger network of which only the reference node appears in Fig. 7.2. Current injected into the network at any node is considered *positive* and current leaving the network at any node is considered *negative*. In Fig. 7.2 current I_m is that portion of the total current injected into node (m) which passes through Y_a. Likewise, I_n is that portion of the current injected into node (n) which passes through Y_a. The voltages V_m and V_n are the voltages of nodes

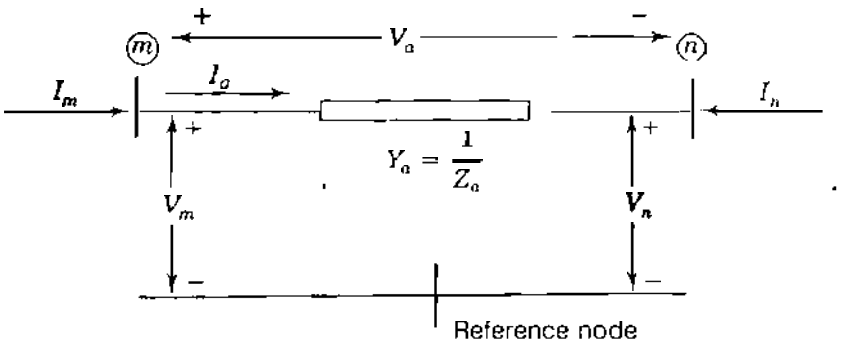


FIGURE 7.2
Primitive branch voltage drop V_a, branch current I_a, injected currents I_m and I_n, node voltages V_m and V_n with respect to network reference.

(m) and (n), respectively, measured with respect to network reference. By Kirchhoff's law $I_m = I_a$ at node (m) and $I_n = -I_a$ at node (n). These two current equations arranged in vector form are

$$\begin{bmatrix} I_m \\ I_n \end{bmatrix} = \begin{matrix} (m) \\ (n) \end{matrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} I_a \tag{7.5}$$

In Eq. (7.5) the labels or pointers (m) and (n) associate the direction of I_a *from* node (m) *to* node (n) with the entries 1 and -1, which are then said to be in row (m) and row (n), respectively. In a similar manner, the voltage drop in the direction of I_a has the equation $V_a = V_m - V_n$, or expressed in vector form

$$V_a = \begin{matrix} (m) & (n) \\ [\ 1 & -1\ ] \end{matrix} \begin{bmatrix} V_m \\ V_n \end{bmatrix} \tag{7.6}$$

Substituting this expression for V_a in the admittance equation $Y_a V_a = I_a$ gives

$$Y_a \begin{matrix} (m) & (n) \\ [\ 1 & -1\ ] \end{matrix} \begin{bmatrix} V_m \\ V_n \end{bmatrix} = I_a \tag{7.7}$$

and premultiplying both sides of Eq. (7.7) by the column vector of Eq. (7.5), we obtain

$$\begin{matrix} (m) \\ (n) \end{matrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} Y_a \begin{matrix} (m) & (n) \\ [\ 1 & -1\ ] \end{matrix} \begin{bmatrix} V_m \\ V_n \end{bmatrix} = \begin{bmatrix} I_m \\ I_n \end{bmatrix} \tag{7.8}$$

which, simplifies to

$$\begin{matrix} & \begin{matrix} (m) & (n) \end{matrix} \\ \begin{matrix} (m) \\ (n) \end{matrix} & \left[\begin{array}{cc} Y_a & -Y_a \\ -Y_a & Y_a \end{array}\right] \end{matrix} \begin{bmatrix} V_m \\ V_n \end{bmatrix} = \begin{bmatrix} I_m \\ I_n \end{bmatrix} \tag{7.9}$$

This is the *nodal admittance equation* for branch Y_a and the coefficient matrix is the *nodal admittance matrix*. We note that the off-diagonal elements equal the negative of the branch admittance. The matrix of Eq. (7.9) is singular because neither node (m) nor node (n) connects to the reference. In the particular case where one of the two nodes, say, (n), is the reference node, then node voltage

V_n is zero and Eq. (7.9) reduces to the 1×1 matrix equation

$$\begin{matrix} & \textcircled{m} \\ \textcircled{m} & \left[Y_a \right] \end{matrix} V_m = I_m \tag{7.10}$$

corresponding to removal of row $\textcircled{n}$ and column $\textcircled{n}$ from the coefficient matrix.

Despite its straightforward derivation, Eq. (7.9) and the procedure leading to it are important in more general situations. We note that the branch voltage V_a is transformed to the node voltages V_m and V_n, and the branch current I_a is likewise represented by the current injections I_m and I_n. The coefficient matrix relating the node voltages and currents of Eq. (7.9) follows from the fact that in Eq. (7.8)

$$\begin{matrix} \textcircled{m} \\ \textcircled{n} \end{matrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{matrix} \begin{matrix} \textcircled{m} & \textcircled{n} \end{matrix} \\ \begin{bmatrix} 1 & -1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} \textcircled{m} & \textcircled{n} \end{matrix} \\ \begin{matrix} \textcircled{m} \\ \textcircled{n} \end{matrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{matrix} \tag{7.11}$$

This 2×2 matrix, also seen in Eq. (1.64), is an important *building block* for representing more general networks, as we shall soon see. The row and column pointers identify each entry in the coefficient matrix by node numbers. For instance, in the first row and second column of Eq. (7.11) the entry -1 is identified with nodes $\textcircled{m}$ and $\textcircled{n}$ of Fig. 7.2 and the other entries are similarly identified.

Thus, the coefficient matrices of Eqs. (7.9) and (7.10) are simply *storage* matrices with row and column labels determined by the end nodes of the branch. Each branch of the network has a similar matrix labeled according to the nodes of the network to which that branch is connected. To obtain the overall nodal admittance matrix of the entire network, we simply combine the individual branch matrices by adding together elements with identical row and column labels. Such addition causes the sum of the branch currents flowing from each node of the network to equal the total current injected into that node, as required by Kirchhoff's current law. In the overall matrix, the off-diagonal element Y_{ij} is the negative sum of the admittances connected between nodes $\textcircled{i}$ and $\textcircled{j}$ and the diagonal entry Y_{ii} is the algebraic sum of the admittances connected to node $\textcircled{i}$. Provided at least one of the network branches is connected to the reference node, the net result is Y_{bus} of the system, as shown in the following example.

Example 7.1. The single-line diagram of a small power system is shown in Fig. 7.3. The corresponding reactance diagram, with reactances specified in per unit, is shown in Fig. 7.4. A generator with emf equal to $1.25\angle 0°$ per unit is connected through a transformer to high-voltage node ③, while a motor with internal voltage equal to $0.85\angle -45°$ is similarly connected to node ④. Develop the nodal

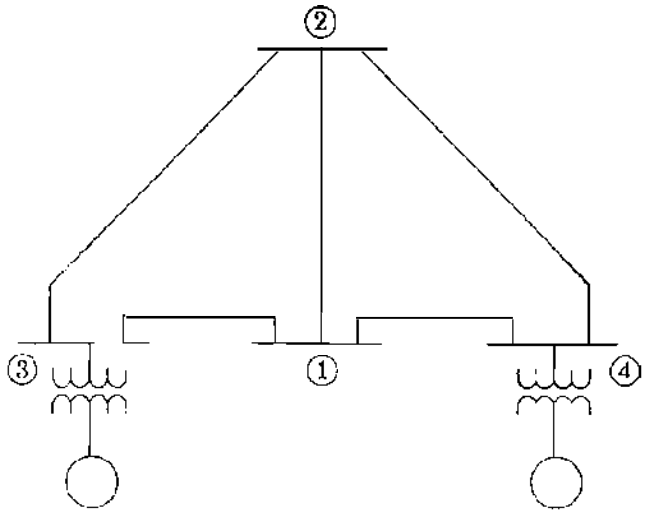


FIGURE 7.3
Single-line diagram of the four-bus system of Example 7.1. Reference node is not shown.

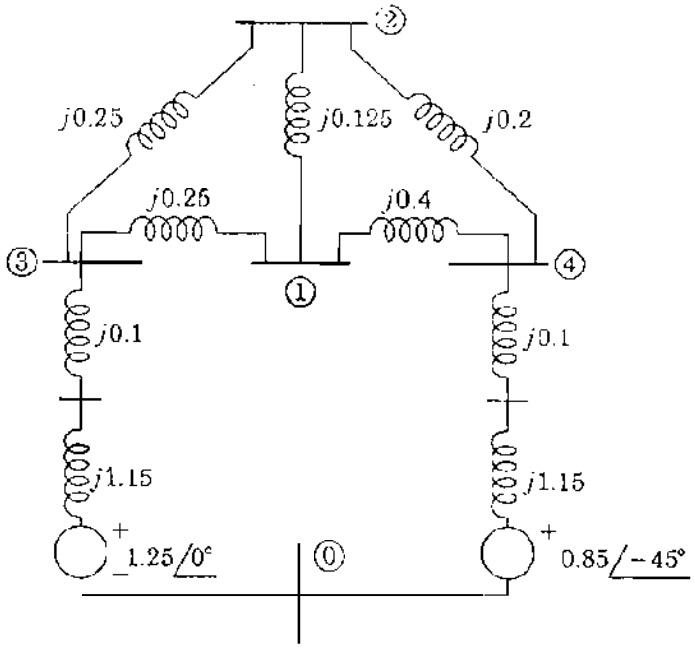


FIGURE 7.4
Reactance diagram for Fig. 7.3. Node ⓪ is reference, reactances and voltages are in per unit.

admittance matrix for each of the network branches and then write the nodal admittance equations of the system.

Solution. The reactances of the generator and the motor may be combined with their respective step-up transformer reactances. Then, by transformation of sources the combined reactances and the generated emfs are replaced by the equivalent current sources and shunt admittances shown in Fig. 7.5. We will treat the current sources as external injections at nodes ③ and ④ and name the seven passive branches according to the subscripts of their currents and voltages. For example, the branch between nodes ① and ③ will be called branch c. The admittance of each branch is simply the reciprocal of the branch impedance and Fig. 7.5 shows the resultant admittance diagram with all values in per unit. The two branches a and g connected to the reference node are characterized by Eq. (7.10), while Eq.

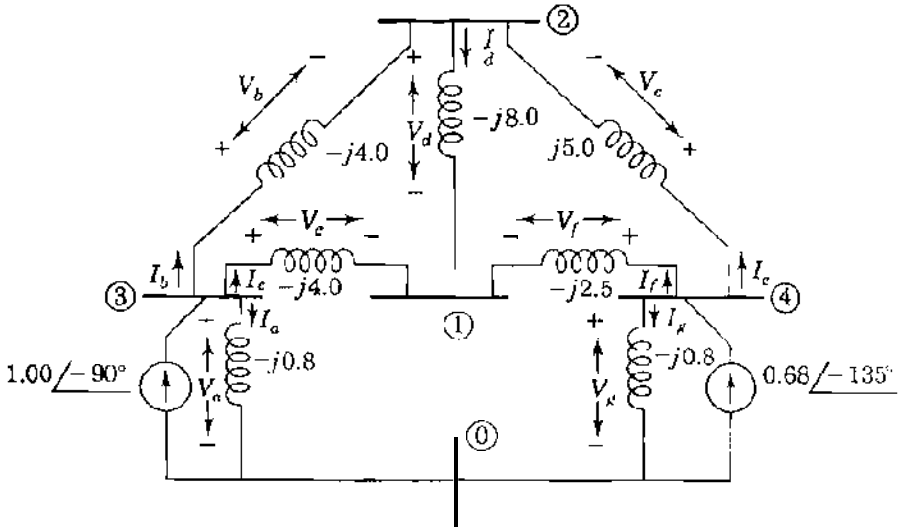


FIGURE 7.5
Per-unit admittance diagram for Fig. 7.4 with current sources replacing voltage sources. Branch names a to g correspond to the subscripts of branch voltages and currents.

(7.9) applies to each of the other five branches. By setting m and n in those equations equal to the node numbers at the ends of the individual branches of Fig. 7.5, we obtain

$$\begin{array}{c}③\\③\end{array}\begin{bmatrix}1\end{bmatrix}Y_a \qquad \begin{array}{c}\\③\\②\end{array}\begin{array}{c}③\quad ②\\\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}Y_b \qquad \begin{array}{c}\\③\\①\end{array}\begin{array}{c}③\quad ①\\\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}Y_c \qquad \begin{array}{c}④\\④\end{array}\begin{bmatrix}1\end{bmatrix}Y_g$$

$$\begin{array}{c}②\\①\end{array}\begin{array}{c}②\quad ①\\\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}Y_d \qquad \begin{array}{c}④\\②\end{array}\begin{array}{c}④\quad ②\\\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}Y_e \qquad \begin{array}{c}④\\①\end{array}\begin{array}{c}④\quad ①\\\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{array}Y_f$$

The order in which the labels are assigned is not important here, provided the columns and rows follow the *same* order. However, for consistency with later sections let us assign the node numbers in the directions of the branch currents of Fig. 7.5, which also shows the numerical values of the admittances. Combining together those elements of the above matrices having identical row and column labels gives

$$\begin{array}{c}\\①\\②\\③\\④\end{array}\begin{array}{c}①\qquad\qquad ②\qquad\qquad ③\qquad\qquad ④\\\begin{bmatrix}(Y_c+Y_d+Y_f) & -Y_d & -Y_c & -Y_f\\ -Y_d & (Y_b+Y_d+Y_e) & -Y_b & -Y_e\\ -Y_c & -Y_b & (Y_a+Y_b+Y_c) & 0\\ -Y_f & -Y_e & 0 & (Y_e+Y_f+Y_g)\end{bmatrix}\end{array}$$

which is the same as $\mathbf{Y}_{bus}$ of Eq. (1.62) since Figs. 1.23 and 7.5 are for the same network. Substituting the numerical values of the branch admittances into this matrix, we obtain for the overall network the nodal admittance equations

$$\begin{bmatrix} -j14.5 & j8.0 & j4.0 & j2.5 \\ j8.0 & -j17.0 & j4.0 & j5.0 \\ j4.0 & j4.0 & -j8.8 & 0.0 \\ j2.5 & j5.0 & 0.0 & -j8.3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

where V_1, V_2, V_3, and V_4 are the node voltages measured with respect to the reference node and $I_1 = 0$, $I_2 = 0$, $I_3 = 1.00\angle 90^\circ$ and $I_4 = 0.68\angle -135^\circ$ are the external currents injected at the system nodes.

The coefficient matrix obtained in the above example is exactly the same as the bus admittance matrix found in Sec. 1.12 using the usual rules for $\mathbf{Y}_{bus}$ formation. However, the approach based on the building-block matrix has advantages when extended to networks with mutually coupled branches, as we now demonstrate.

7.2 MUTUALLY COUPLED BRANCHES IN $\mathbf{Y}_{bus}$

The procedure based on the building-block matrix is now extended to two mutually coupled branches which are part of a larger network but which are not inductively coupled to any other branches. In Sec. 2.2 the primitive equations of such mutually coupled branches are developed in the form of Eq. (2.24) for impedances and Eq. (2.26) for admittances. The notation is different here because we are now using numbers to identify nodes rather than branches.

Assume that branch impedance Z_a connected between nodes (m) and (n) is coupled through mutual impedance Z_M to branch impedance Z_b connected between nodes (p) and (q) of Fig. 7.6. The voltage drops V_a and V_b due to the branch currents I_a and I_b are then given by the primitive impedance equation corresponding to Eq. (2.24) in the form

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} Z_a & Z_M \\ Z_M & Z_b \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \tag{7.12}$$

in which the coefficient matrix is symmetrical. As noted in Sec. 2.2, the mutual impedance Z_M is considered positive when currents I_a and I_b enter the terminals marked with dots in Fig. 7.6(*a*); the voltage drops V_a and V_b then have the polarities shown. Multiplying Eq. (7.12) by the inverse of the primitive impedance matrix

$$\begin{bmatrix} Z_a & Z_M \\ Z_M & Z_b \end{bmatrix}^{-1} = \frac{1}{Z_a Z_b - Z_M^2} \begin{bmatrix} Z_b & -Z_M \\ -Z_M & Z_a \end{bmatrix} = \begin{bmatrix} Y_a & Y_M \\ Y_M & Y_b \end{bmatrix} \tag{7.13}$$

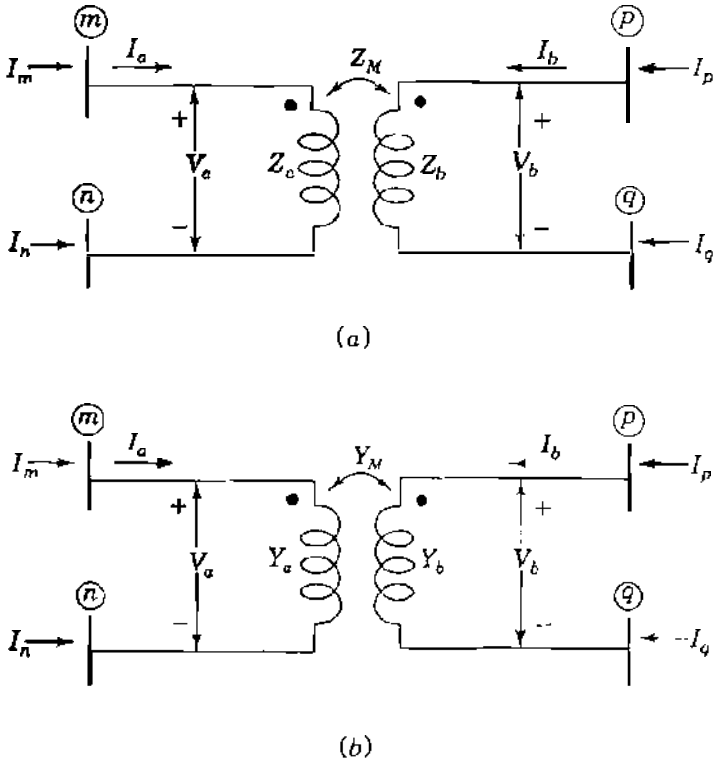


FIGURE 7.6
Two mutually coupled branches with (*a*) impedance parameters and (*b*) corresponding admittances.

we obtain the admittance form of Eq. (2.26) for the two branches

$$\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_b \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} \tag{7.14}$$

which is also symmetrical. The admittance matrix of Eq. (7.14), called the *primitive admittance matrix* of the two coupled branches, corresponds to Fig. 7.6(*b*). The primitive self-admittance Y_a equals $Z_b/(Z_aZ_b - Z_M^2)$ and similar expressions from Eq. (7.13) apply to Y_b and the primitive mutual admittance Y_M. We may write the voltage-drop equations $V_a = V_m - V_n$ and $V_b = V_p - V_q$ of Fig. 7.6 in the matrix form

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} V_m - V_n \\ V_p - V_q \end{bmatrix} = \begin{matrix} \\ \end{matrix} \begin{matrix} \textcircled{m} & \textcircled{n} & \textcircled{p} & \textcircled{q} \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_m \\ V_n \\ V_p \\ V_q \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_m \\ V_n \\ V_p \\ V_q \end{bmatrix} \tag{7.15}$$

in which the first row of the coefficient matrix **A** is associated with branch admittance Y_a and the second row relates to branch admittance Y_b. The node voltages V_m, V_n, V_p, and V_q are measured with respect to the network reference. In Fig. 7.6 the branch current I_a is related to the injected currents by the two

node equations $I_m = I_a$ and $I_n = -I_a$; similarly, branch current I_b is related to the currents I_p and I_q by the two node equations $I_p = I_b$ and $I_q = -I_b$. These four current equations arranged in matrix form become

$$\begin{bmatrix} I_m \\ I_n \\ I_p \\ I_q \end{bmatrix} = \begin{matrix} (m) \\ (n) \\ (p) \\ (q) \end{matrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \mathbf{A}^T \begin{bmatrix} I_a \\ I_b \end{bmatrix} \tag{7.16}$$

with the coefficient matrix equal to the transpose of that in Eq. (7.15). From Eq. (7.15) we substitute for the voltage drops of Eq. (7.14) to find

$$\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_b \end{bmatrix} \mathbf{A} \begin{bmatrix} V_m \\ V_n \\ V_p \\ V_q \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} \tag{7.17}$$

and premultiplying both sides of this equation by the matrix $\mathbf{A}^T$ of Eq. (7.16), we obtain

$$\underbrace{\mathbf{A}^T}_{4\times 2} \underbrace{\begin{bmatrix} Y_a & Y_M \\ Y_M & Y_b \end{bmatrix}}_{2\times 2} \underbrace{\mathbf{A}}_{2\times 4} \begin{bmatrix} V_m \\ V_n \\ V_p \\ V_q \end{bmatrix} = \begin{bmatrix} I_m \\ I_n \\ I_p \\ I_q \end{bmatrix} \tag{7.18}$$

When the multiplications indicated in Eq. (7.18) are performed, the result gives the nodal admittance equations of the two mutually coupled branches in the matrix form

$$\begin{matrix} \\ (m) \\ (n) \\ (p) \\ (q) \end{matrix} \begin{matrix} \begin{matrix} (m) & (n) & (p) & (q) \end{matrix} \\ \left[\begin{array}{cc|cc} Y_a & -Y_a & Y_M & -Y_M \\ -Y_a & Y_a & -Y_M & Y_M \\ \hline Y_M & -Y_M & Y_b & -Y_b \\ -Y_M & Y_M & -Y_b & Y_b \end{array}\right] \end{matrix} \begin{bmatrix} V_m \\ V_n \\ V_p \\ V_q \end{bmatrix} = \begin{bmatrix} I_m \\ I_n \\ I_p \\ I_q \end{bmatrix} \tag{7.19}$$

The two mutually coupled branches are actually part of a larger network, and so the 4×4 matrix of Eq. (7.19) forms part of the larger nodal admittance matrix of the overall system. The pointers (m), (n), (p), and (q) indicate the rows and columns of the *system* matrix to which the elements of Eq. (7.19) belong. Thus, for example, the quantity entered in row (n) and column (p) of the system

nodal admittance matrix is $-Y_M$ and similar entries are made from the other elements of Eq. (7.19).

The nodal admittance matrix of the two coupled branches may be formed directly by inspection. This becomes clear when we write the coefficient matrix of Eq. (7.19) in the alternative form

$$\left[\begin{array}{cc|cc} & \begin{matrix}\textcircled{m} & \textcircled{n}\end{matrix} & & \begin{matrix}\textcircled{p} & \textcircled{q}\end{matrix} \\ \begin{matrix}\textcircled{m}\\ \textcircled{n}\end{matrix} & \left[\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right] Y_a & \begin{matrix}\textcircled{m}\\ \textcircled{n}\end{matrix} & \left[\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right] Y_M \\ \hline & \begin{matrix}\textcircled{m} & \textcircled{n}\end{matrix} & & \begin{matrix}\textcircled{p} & \textcircled{q}\end{matrix} \\ \begin{matrix}\textcircled{p}\\ \textcircled{q}\end{matrix} & \left[\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right] Y_M & \begin{matrix}\textcircled{p}\\ \textcircled{q}\end{matrix} & \left[\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right] Y_b \end{array}\right] \tag{7.20}$$

To obtain Eq. (7.20), we multiply each element of the primitive admittance matrix by the 2×2 building-block matrix. The labels assigned to the rows and columns of the multipliers in Eq. (7.20) are easily determined. First, we note that the self-admittance Y_a is measured between nodes $\textcircled{m}$ and $\textcircled{n}$ with the dot at node $\textcircled{m}$. Hence, the 2×2 matrix multiplying Y_a in Eq. (7.20) has rows and columns labeled $\textcircled{m}$ and $\textcircled{n}$ *in that order*. Then, the self-admittance Y_b between nodes $\textcircled{p}$ and $\textcircled{q}$ is multiplied by the 2×2 matrix with labels $\textcircled{p}$ and $\textcircled{q}$ in the order shown since node $\textcircled{p}$ is marked with a dot. Finally, the labels of the matrices multiplying the mutual admittance Y_M are assigned row by row and then column by column so as to align and agree with those already given to the self-inductances. In the nodal admittance matrix of Eqs. (7.19) and (7.20) the sum of the columns (and of the rows) adds up to zero. This is because none of the nodes $\textcircled{m}$, $\textcircled{n}$, $\textcircled{p}$, and $\textcircled{q}$ has been considered as the reference node of the network. In the special case where one of the nodes, say, node $\textcircled{n}$, is in fact the reference, V_n is zero and column n in Eq. (7.19) does not need to appear; furthermore, I_n does not have to be explicitly represented since the reference node current is not an independent quantity. Consequently, when node $\textcircled{n}$ is the reference, we may eliminate the row and column of that node from Eqs. (7.19) and (7.20).

It is important to note that nodes $\textcircled{m}$, $\textcircled{n}$, $\textcircled{p}$, and $\textcircled{q}$ are often not distinct. For instance, suppose that nodes $\textcircled{n}$ and $\textcircled{q}$ are one and the same node. In that case columns $\textcircled{n}$ and $\textcircled{q}$ of Eq. (7.19) can be combined together since $V_n = V_q$, and the corresponding rows can be added because I_n and I_q are parts of the common injected current. The following example illustrates this situation.

Example 7.2. Two branches having impedances equal to $j0.25$ per unit are coupled through mutual impedance $Z_M = j0.15$ per unit, as shown in Fig. 7.7. Find

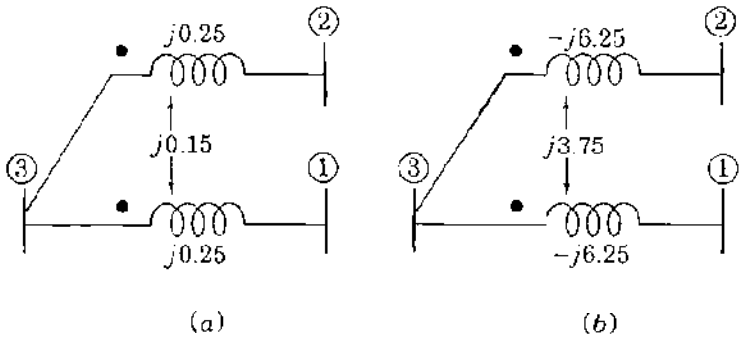


FIGURE 7.7
The two mutually coupled branches of Example 7.2, their (*a*) primitive impedances and (*b*) primitive admittances in per unit.

the nodal admittance matrix for the mutually coupled branches and write the corresponding nodal admittance equations.

Solution. The primitive impedance matrix for the mutually coupled branches of Fig. 7.7(*a*) is inverted as a single entity to yield the primitive admittances of Fig. 7.7(*b*), that is,

$$\begin{bmatrix} j0.25 & j0.15 \\ j0.15 & j0.25 \end{bmatrix}^{-1} = \begin{bmatrix} -j6.25 & j3.75 \\ j3.75 & -j6.25 \end{bmatrix}$$

First, the rows and columns of the building-block matrix which multiplies the primitive self-admittance between nodes ① and ③ are labeled ③ and ① in that order to correspond to the dot marking node ③. Next, the rows and columns of the 2×2 matrix multiplying the self-admittance between nodes ② and ③ are labeled ③ and ② in the order shown because node ③ is marked. Finally, the pointers of the matrices multiplying the mutual admittance are aligned with those of the self-admittances to form the 4×4 array similar to Eq. (7.20) as follows:

$$\left[\begin{array}{cc|cc} & \begin{matrix}③ & ①\end{matrix} & & \begin{matrix}③ & ②\end{matrix} \\ \begin{matrix}③\\①\end{matrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j6.25) & \begin{matrix}③\\①\end{matrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j3.75) \\ \hline & \begin{matrix}③ & ①\end{matrix} & & \begin{matrix}③ & ②\end{matrix} \\ \begin{matrix}③\\②\end{matrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (j3.75) & \begin{matrix}③\\②\end{matrix} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (-j6.25) \end{array}\right]$$

Since there are only three nodes in Fig. 7.7, the required 3×3 matrix is found by adding the columns and rows of common node ③ to obtain

$$\begin{array}{c} \\ ① \\ ② \\ ③ \end{array} \begin{array}{c} \begin{matrix} ① & \quad ② & \quad ③ \end{matrix} \\ \begin{bmatrix} -j6.25 & j3.75 & j6.25 - j3.75 \\ j3.75 & -j6.25 & -j3.75 + j6.25 \\ j6.25 - j3.75 & -j3.75 + j6.25 & -j6.25 - j6.25 + 2(j3.75) \end{bmatrix} \end{array}$$

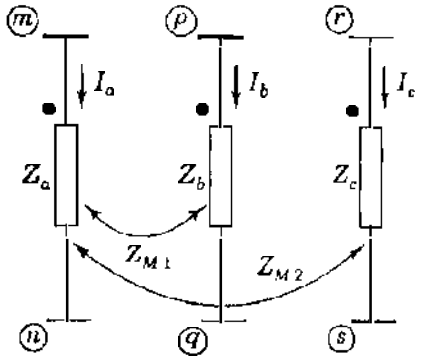


FIGURE 7.8
Three branches with mutual coupling Z_{M1} between branches a and b and Z_{M2} between branches a and c.

The new diagonal element representing node ③, for instance, is the sum of the four elements $(-j6.25 - j6.25 + j3.75 + j3.75)$ in rows ③ and columns ③ of the previous matrix. The three nodal admittance equations in vector-matrix form are then written

$$\begin{bmatrix} -j6.25 & j3.75 & j2.50 \\ j3.75 & -j6.25 & j2.50 \\ j2.50 & j2.50 & -j5.00 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

where V_1, V_2, and V_3 are the voltages at nodes ①, ②, and ③ measured with respect to reference, while I_1, I_2, and I_3 are the external currents injected at the respective nodes.

As in Sec. 7.1, the coefficient matrix of the last equation can be combined with the nodal admittance matrices of the other branches of the network in order to obtain the nodal admittance matrix of the entire system.

For three or more coupled branches we follow the same procedure as above. For example, the three coupled branches of Fig. 7.8 have primitive impedance and admittance matrices given by

$$\begin{bmatrix} Z_a & Z_{M1} & Z_{M2} \\ Z_{M1} & Z_b & 0 \\ Z_{M2} & 0 & Z_c \end{bmatrix}^{-1} = \begin{bmatrix} Y_a & Y_{M1} & Y_{M2} \\ Y_{M1} & Y_b & Y_{M3} \\ Y_{M2} & Y_{M3} & Y_c \end{bmatrix} \tag{7.21}$$

The zeros in the Z-matrix arise because branches b and c are not directly coupled. For all nonzero values of the current I_a of Fig. 7.8, branches b and c are *indirectly* coupled through branch a, as shown by nonzero Y_{M3} of the primitive admittance matrix.

Therefore, to form $\mathbf{Y}_{bus}$ for a network which has mutually coupled branches, we do the following in sequence:

1. Invert the primitive impedance matrices of the network branches to obtain the corresponding primitive admittance matrices. A single branch has a 1×1 matrix, two mutually coupled branches have a 2×2 matrix, three mutually coupled branches have a 3×3 matrix, and so on.
2. Multiply the elements of each primitive admittance matrix by the 2×2 building-block matrix.
3. Label the two rows and the two columns of each *diagonal* building-block matrix with the end-node numbers of the corresponding self-admittance. For mutually coupled branches it is important to label in the order of the *marked (dotted)—then—unmarked (undotted)* node numbers.
4. Label the two rows of each *off-diagonal* building-block matrix with node numbers aligned and consistent with the row labels assigned in (3); then label the columns consistent with the column labels of (3).
5. Combine, by adding together, those elements with identical row and column labels to obtain the nodal admittance matrix of the overall network. If one of the nodes encountered is the reference node, omit its row and column to obtain the system $\mathbf{Y}_{bus}$.

7.3 AN EQUIVALENT ADMITTANCE NETWORK

We have demonstrated how to write the nodal admittance equations for one branch or a number of mutually coupled branches which are part of a larger network. We now show that such equations can be interpreted as representing an equivalent admittance network with no mutually coupled elements. This may be useful when forming $\mathbf{Y}_{bus}$ for an original network having mutually coupled elements.

The currents injected into the nodes of Fig. 7.6 are described in terms of node voltages and admittances by Eq. (7.19). For example, the equation for the current I_m at node ③ is given by the first row of Eq. (7.19) as follows:

$$I_m = Y_a V_m - Y_a V_n + Y_M V_p - Y_M V_q \tag{7.22}$$

Adding and subtracting the term $Y_M V_m$ on the right-hand side of Eq. (7.22) and combining terms with common coefficients, we obtain the Kirchhoff's current equation at node ③

$$I_m = \underbrace{Y_a(V_m - V_n)}_{I_{ma}} + \underbrace{(-Y_M)(V_m - V_p)}_{I_{mp}} + \underbrace{Y_M(V_m - V_q)}_{I_{mq}} \tag{7.23}$$

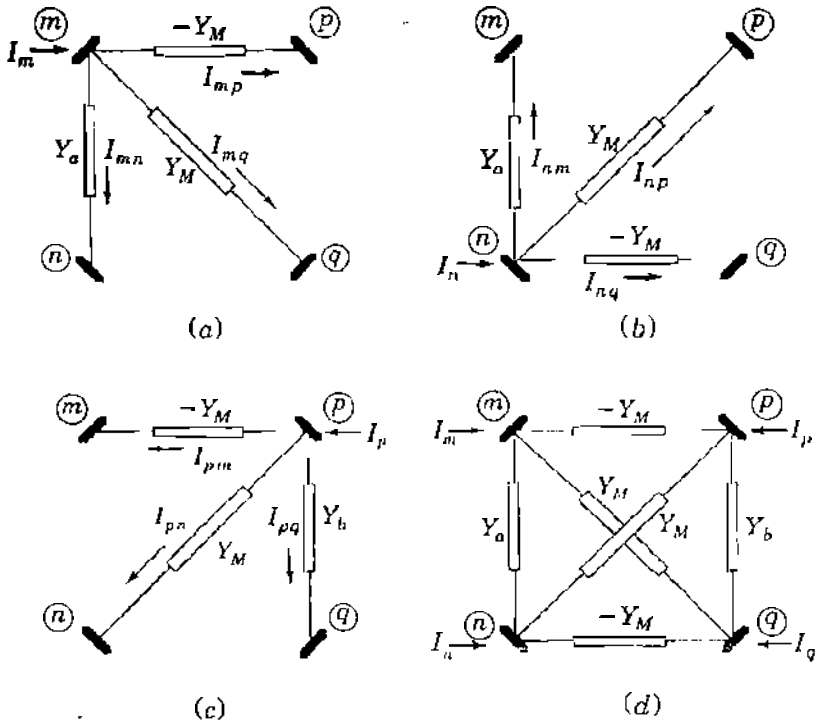


FIGURE 7.9
Developing the nodal admittance network of two mutually coupled branches.

The double subscripts indicate the directions of the currents I_{mn}, I_{mp}, and I_{mq} from node (3) to each of the other nodes (n), (p), and (q), respectively, of Fig. 7.9(*a*). A similar analysis of the second and third rows of Eq. (7.19) leads to equations for the currents I_n and I_p in the form

$$I_n = \underbrace{Y_a(V_n - V_m)}_{I_{nm}} + \underbrace{(-Y_M)(V_n - V_q)}_{I_{nq}} + \underbrace{Y_M(V_n - V_p)}_{I_{np}} \tag{7.24}$$

$$I_p = \underbrace{Y_a(V_p - V_q)}_{I_{pq}} + \underbrace{(-Y_M)(V_p - V_m)}_{I_{pm}} + \underbrace{Y_M(V_p - V_n)}_{I_{pn}} \tag{7.25}$$

and these two equations represent the partial networks of Figs. 7.9(*b*) and (*c*). The fourth row of Eq. (7.19) does not yield a separate partial network because it is not independent of the other rows. Combining the three partial networks without duplicating branches, we obtain an equivalent circuit in the form of the lattice network connected among nodes (m), (n), (p), and (q) in Fig. 7.9(*d*). This lattice network has no mutually coupled branches, but it is equivalent in every respect to the two original coupled branches of Fig. 7.6 since it satisfies

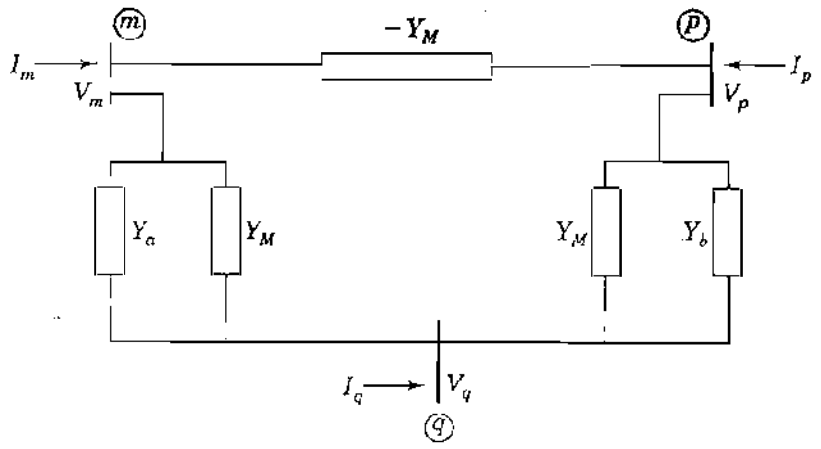


FIGURE 7.10
The nodal admittance network of two mutually coupled branches connected to nodes (m), (p), and (q).

Eq. (7.19). Accordingly, the standard rules of circuit analysis may be applied to the equivalent. For example, if the two coupled branches are physically connected among three independent nodes as in Example 7.2, we may regard nodes (n) and (q) of Fig. 7.9(*d*) as one and the same node, which we simply join together, as shown in Fig. 7.10. The three-bus equivalent circuit of Fig. 7.10 then yields the nodal equations for the original branches.

Thus, each physical branch or mutually coupled pair gives rise to an equivalent admittance network to which the usual rules of circuit analysis apply. The following example illustrates the role of the equivalent circuit in forming $\mathbf{Y}_{\text{bus}}$.

Example 7.3. Replace branches *b* and *c* between node-pairs (1)–(3) and (2)–(3) of Fig. 7.5 by the mutually coupled branches of Fig. 7.7. Then, find $\mathbf{Y}_{\text{bus}}$ and the nodal equations of the new network.

Solution. The admittance diagram of the new network including the mutual coupling is shown in Fig. 7.11. From Example 7.2 we know that the mutually coupled branches have the nodal admittance matrix

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array} \begin{array}{c} \begin{array}{ccc} (1) & (2) & (3) \end{array} \\ \begin{bmatrix} -j6.25 & j3.75 & j2.50 \\ j3.75 & -j6.25 & j2.50 \\ j2.50 & j2.50 & -j5.00 \end{bmatrix} \end{array}$$

which corresponds to the equivalent circuit shown encircled in Fig. 7.12. The remaining portion of Fig. 7.12 is drawn from Fig. 7.5. Since mutual coupling is not evident in Fig. 7.12, we may apply the standard rules of $\mathbf{Y}_{\text{bus}}$ formation to the

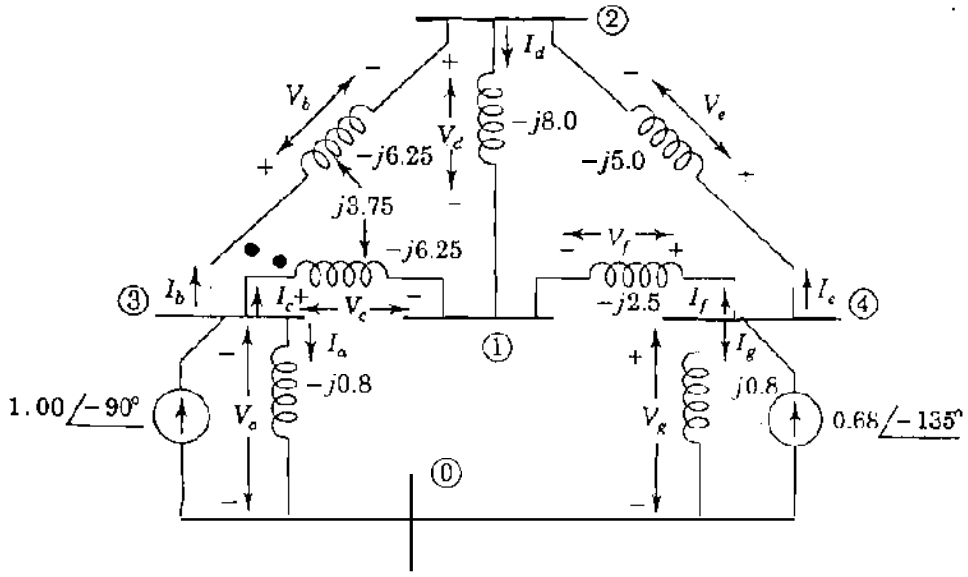


FIGURE 7.11
Per-unit admittance diagram for Example 7.3.

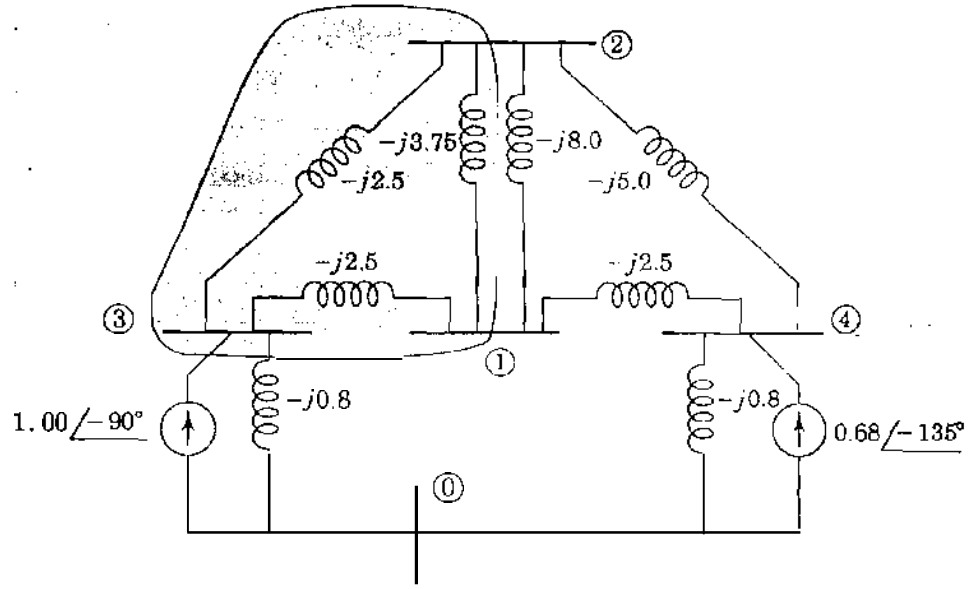


FIGURE 7.12
Nodal admittance network for Example 7.3. The shaded portion represents two mutually coupled branches connected between buses ①, ②, and ③.

overall network, which leads to the nodal admittance equations

$$\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array}\begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -j16.75 & j11.75 & j2.50 & j2.50 \\ j11.75 & -j19.25 & j2.50 & j5.00 \\ j2.50 & j2.50 & -j5.80 & 0 \\ j2.50 & j5.00 & 0 & -j8.30 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

Note that the two admittances between nodes ① and ② combine in parallel to yield

$$Y_{12} = -(-j3.75 - j8.00) = j11.75$$

7.4 MODIFICATION OF $\mathbf{Y}_{bus}$

The building-block approach and the equivalent circuits of Sec. 7.3 provide important insights into the manner in which each branch self- and mutual admittance contributes to the entries of $\mathbf{Y}_{bus}$ and the corresponding equivalent network of the overall system. As a result, it is clear that $\mathbf{Y}_{bus}$ is merely a systematic means of combining the nodal admittance matrices of the various network branches. We simply form a large array with rows and columns ordered according to the sequence in which the nonreference nodes of the network are numbered, and within it we combine entries with matching labels drawn from the nodal admittance matrices of the individual branches. Consequently, we can easily see how to *modify* the system $\mathbf{Y}_{bus}$ to account for branch additions or other changes to the system network. For instance, to modify $\mathbf{Y}_{bus}$ of the existing network to reflect the addition of the branch admittance between the nodes ⓜ and ⓝ, we simply add Y_a to the elements $\mathbf{Y}_{mm}$ and $\mathbf{Y}_{nn}$ of $\mathbf{Y}_{bus}$ and subtract Y_a from the symmetrical elements Y_{mn} and Y_{nm}. In other words, to incorporate the new branch admittance Y_a into the network, we add to the existing $\mathbf{Y}_{bus}$ the change matrix $\Delta\mathbf{Y}_{bus}$ given by

$$\Delta\mathbf{Y}_{bus} = \begin{array}{c} \\ ⓜ \\ ⓝ \end{array}\begin{array}{c} \begin{array}{cc} ⓜ & ⓝ \end{array} \\ \begin{bmatrix} Y_a & -Y_a \\ -Y_a & Y_a \end{bmatrix} \end{array} \tag{7.26}$$

Again, we recognize $\Delta\mathbf{Y}_{bus}$ as a storage matrix with rows and columns marked ⓜ and ⓝ. Using Eq. (7.26), we may *change* the admittance value of a single branch of the network by adding a new branch between the same end nodes ⓜ and ⓝ such that the parallel combination of the old and new branches yields the desired value. Moreover, to *remove* a branch admittance Y_a already connected between nodes ⓜ and ⓝ of the network, we simply add the branch admittance $-Y_a$ between the same nodes, which amounts to subtracting the

elements of $\Delta\mathbf{Y}_{\text{bus}}$ from the existing $\mathbf{Y}_{\text{bus}}$. Equation (7.20) shows that a *pair* of mutually coupled branches can be removed from the network by subtracting the entries in the change matrix

$$\Delta\mathbf{Y}_{\text{bus}} = \begin{array}{c} \\ \textcircled{m} \\ \textcircled{n} \\ \textcircled{p} \\ \textcircled{q} \end{array} \begin{array}{c} \begin{array}{cccc} \textcircled{m} & \textcircled{n} & \textcircled{p} & \textcircled{q} \end{array} \\ \begin{bmatrix} Y_a & -Y_a & Y_M & -Y_M \\ -Y_a & Y_a & -Y_M & Y_M \\ Y_M & -Y_M & Y_b & -Y_b \\ -Y_M & Y_M & -Y_b & Y_b \end{bmatrix} \end{array} \tag{7.27}$$

from the rows and columns of $\mathbf{Y}_{\text{bus}}$ corresponding to the end nodes $\textcircled{m}$, $\textcircled{n}$, $\textcircled{p}$, and $\textcircled{q}$. Of course, if only one of the two mutually coupled branches is to be removed from the network, we could first remove all the entries for the mutually coupled pair from $\mathbf{Y}_{\text{bus}}$ using Eq. (7.27) and then add the entries for the branch to be retained using Eq. (7.26). Other strategies for modifying $\mathbf{Y}_{\text{bus}}$ to reflect network changes become clear from the insights developed in Secs. 7.1 through 7.3.

Example 7.4. Determine the bus admittance matrix of the network of Fig. 7.5 by removing the effects of mutual coupling from $\mathbf{Y}_{\text{bus}}$ of Fig. 7.11.

Solution. The $\mathbf{Y}_{\text{bus}}$ for the entire system of Fig. 7.11 including mutual coupling is found in Example 7.3 to be

$$\mathbf{Y}_{\text{bus}} = \begin{array}{c} \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{array} \begin{array}{c} \begin{array}{cccc} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{array} \\ \begin{bmatrix} -j16.75 & j11.75 & j2.50 & j2.50 \\ j11.75 & -j19.25 & j2.50 & j5.00 \\ j2.50 & j2.50 & -j5.80 & 0 \\ j2.50 & j5.00 & 0 & -j8.30 \end{bmatrix} \end{array}$$

To remove completely the effect of mutual coupling from the network $\mathbf{Y}_{\text{bus}}$, we proceed in two steps by (*a*) first removing the two mutually coupled branches altogether and (*b*) then restoring each of the two branches without mutual coupling between them. (*a*) To remove the two mutually coupled branches from the network, we subtract from the system $\mathbf{Y}_{\text{bus}}$ the entries in

$$\Delta\mathbf{Y}_{\text{bus, I}} = \begin{array}{c} \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{array} \begin{array}{c} \begin{array}{cccc} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{array} \\ \begin{bmatrix} -j6.25 & j3.75 & j2.50 & \cdot \\ j3.75 & -j6.25 & j2.50 & \cdot \\ j2.50 & j2.50 & -j5.00 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

corresponding to the encircled portion of Fig. 7.12.

(*b*) Now we must reconnect to the network the uncoupled branches, each of which has an admittance $(j0.25)^{-1} = -j4.0$ per unit. Accordingly, to reconnect the branch between nodes ① and ③, we add to $\mathbf{Y}_{bus}$ the change matrix

$$\Delta\mathbf{Y}_{bus,2} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array} (-j4.0)$$

and similarly for the branch between nodes ② and ③ we add

$$\Delta\mathbf{Y}_{bus,3} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array} (-j4.0)$$

Appropriately subtracting and adding the three change matrices and the original $\mathbf{Y}_{bus}$ give the new bus admittance matrix for the uncoupled branches

$$\mathbf{Y}_{bus(new)} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -j14.5 & j8.0 & j4.0 & j2.5 \\ j8.0 & -j17.0 & j4.0 & j5.0 \\ j4.0 & j4.0 & j8.8 & 0 \\ j2.5 & j5.0 & 0 & -j8.3 \end{bmatrix} \end{array}$$

which agrees with Example 7.1.

7.5 THE NETWORK INCIDENCE MATRIX AND $\mathbf{Y}_{bus}$

In Secs. 7.1 and 7.2 nodal admittance equations for each branch and mutually coupled pair of branches are derived independently from those of other branches in the network. The nodal admittance matrices of the individual branches are then combined together in order to build $\mathbf{Y}_{bus}$ of the overall system. Since we now understand the process, we may proceed to the more formal approach which treats all the equations of the system simultaneously rather than separately. We will use the example system of Fig. 7.11 to establish the general procedure.

Two of the seven branches in Fig. 7.11 are mutually coupled as shown. The mutually coupled pair is characterized by Eq. (7.14) and the other five

branches by Eq. (7.4). Arranging the seven branch equations into an array format, we obtain

$$\begin{bmatrix} -j0.80 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -j6.25 & j3.75 & \cdot & \cdot & \cdot & \cdot \\ \cdot & j3.75 & -j6.25 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -j8.00 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -j5.00 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -j2.50 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j0.80 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \\ V_g \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \\ I_e \\ I_f \\ I_g \end{bmatrix} \tag{7.28}$$

The coefficient matrix is the primitive admittance matrix formed by inspection of Fig. 7.11. Each branch of the network contributes a diagonal entry equal to the simple reciprocal of its branch impedance except for branches b and c, which are mutually coupled and have entries determined by Eq. (7.13). For the general case Eq. (7.28) may be more compactly written in the form

$$\mathbf{Y}_{pr}\mathbf{V}_{pr} = \mathbf{I}_{pr} \tag{7.29}$$

where $\mathbf{V}_{pr}$ and $\mathbf{I}_{pr}$ are the respective column vectors of branch voltages and currents, while $\mathbf{Y}_{pr}$ represents the primitive admittance matrix of the network. The primitive equations do not tell how the branches are configured within the network. The geometrical configuration of the branches, called the *topology*, is provided by a *directed graph*, as shown in Fig. 7.13(a) in which each branch of the network of Fig. 7.11 is represented between its end nodes by a directed line

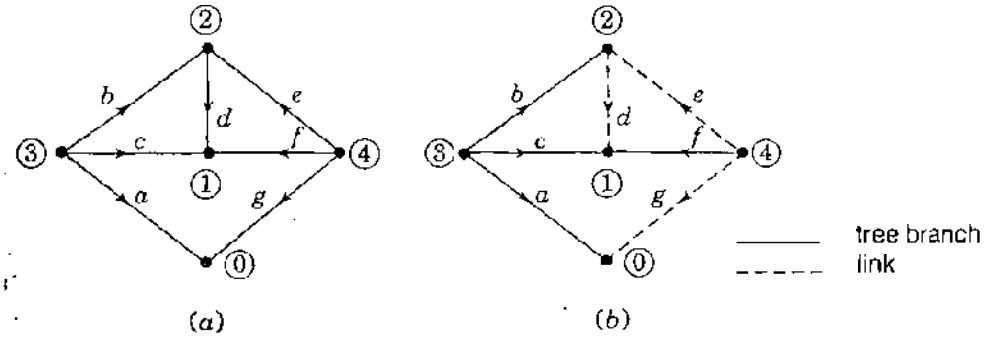


FIGURE 7.13
The linear graph for Fig. 7.11 showing: (a) directed-line segments for branches; (b) branches a, b, c, and f define a tree while branches d, e, and g are links.

segment with an arrow in the direction of the branch current. When a branch connects to a node, the branch and node are said to be *incident*. A *tree* of a graph is formed by those branches of the graph which interconnect or *span* all the nodes of the graph without forming any closed path. In general, there are many possible trees of a network since different combinations of branches can be chosen to span the nodes. Thus, for example, branches a, b, c, and f in Fig. 7.13(b) define a tree. The remaining branches d, e, and g are called *links*, and when a link is added to a tree, a closed path or *loop* is formed.

A graph may be described in terms of a *connection* or *incidence matrix*. Of particular interest is the *branch-to-node incidence matrix* $\hat{\mathbf{A}}$, which has one row for each branch and one column for each node with an entry a_{ij} in row i and column j according to the following rule:

$$a_{ij} = \begin{cases} 0 & \text{if branch } i \text{ is not connected to node } (j) \\ 1 & \text{if current in branch } i \text{ is directed away from node } (j) \\ -1 & \text{if current in branch } i \text{ is directed toward node } (j) \end{cases} \tag{7.30}$$

This rule formalizes for the network as a whole the procedure used to set up the coefficient matrices of Eqs. (7.6) and (7.15) for the individual branches. In network calculations we usually choose a reference node. The column corresponding to the reference node is then omitted from $\hat{\mathbf{A}}$ and the resultant matrix is denoted by $\mathbf{A}$. For example, choosing node (0) as the reference in Fig. 7.13 and invoking the rule of Eq. (7.30), we obtain the rectangular branch-to-node matrix

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccc} (1) & (2) & (3) & (4) \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{7.31}$$

The nonreference nodes of a network are often called *independent nodes* or *buses*, and when we say that the network has N buses, we generally mean that there are N independent nodes not including the reference node. The $\mathbf{A}$ matrix has the row-column dimension $B \times N$ for any network with B branches and N nodes excluding the reference. We note that each row of Eq. (7.31) has two nonzero entries which add to zero except for rows a and g, each of which has

only one nonzero entry. This is because branches a and g of Fig. 7.11 have one end connected to the reference node for which no column is shown.

The voltage across each branch may be expressed as the difference in its end-bus voltages measured with respect to the reference node. For example, in Fig. 7.11 the voltages at buses ①, ②, ③, and ④ with respect to reference node ⓪ are denoted by V_1, V_2, V_3, and V_4, respectively, and so the voltage drops across the branches are given by

$$\begin{aligned} V_a &= V_3 \\ V_b &= V_3 - V_2 \\ V_c &= V_3 - V_1 \\ V_d &= V_2 - V_1 \\ V_e &= V_4 - V_2 \\ V_f &= V_4 - V_1 \\ V_g &= V_4 \end{aligned} \quad \text{or} \quad \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \\ V_g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

in which the coefficient matrix is the **A** matrix of Eq. (7.31). This is one illustration of the general result for any N-bus network given by

$$\mathbf{V}_{pr} = \mathbf{AV} \tag{7.32}$$

where $\mathbf{V}_{pr}$ is the $B \times 1$ column vector of branch voltage drops and $\mathbf{V}$ is the $N \times 1$ column vector of bus voltages measured with respect to the chosen reference node. Equations (7.6) and (7.15) are particular applications of Eq. (7.32) to individual branches. We further note that Kirchhoff's current law at nodes ① to ④ of Fig. 7.11 yields

$$\begin{bmatrix} 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \\ I_e \\ I_f \\ I_g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_3 \\ I_4 \end{bmatrix}$$

where $I_3 = 1.00\angle -90^\circ$ and $I_4 = 0.68\angle -135^\circ$ are the external currents injected at nodes ③ and ④, respectively. The coefficient matrix in this equation is $\mathbf{A}^T$. Again, this is illustrative of a general result applicable to every electrical network since it simply states, in accordance with Kirchhoff's current law, that the sum of all the branch currents incident to a node of the network equals the

injected current at the node. Accordingly, we may write

$$\mathbf{A}^T \mathbf{I}_{pr} = \mathbf{I} \tag{7.33}$$

where $\mathbf{I}_{pr}$ is the $B \times 1$ column vector of branch currents and $\mathbf{I}$ is the $N \times 1$ column vector with a nonzero entry for each bus with an external current source. Equations (7.5) and (7.16) are particular examples of Eq. (7.33).

The A matrix fully describes the topology of the network and is independent of the particular values of the branch parameters. The latter are supplied by the primitive admittance matrix. Therefore, two different network configurations employing the same branches will have different A matrices but the same $\mathbf{Y}_{pr}$. On the other hand, if changes occur in branch parameters while maintaining the same network configuration, only $\mathbf{Y}_{pr}$ is altered but not A.

Multiplying Eq. (7.29) by $\mathbf{A}^T$, we obtain

$$\mathbf{A}^T \mathbf{Y}_{pr} \mathbf{V}_{pr} = \mathbf{A}^T \mathbf{I}_{pr} \tag{7.34}$$

The right-hand side of Eq. (7.34) equals $\mathbf{I}$ and substituting for $\mathbf{V}_{pr}$ from Eq. (7.32), we find

$$\left\{\mathbf{A}^T \mathbf{Y}_{pr} \mathbf{A}\right\} \mathbf{V} = \mathbf{I} \tag{7.35}$$

We may write Eq. (7.35) in the more concise form

$$\mathbf{Y}_{bus} \mathbf{V} = \mathbf{I} \tag{7.36}$$

where the $N \times N$ bus admittance matrix $\mathbf{Y}_{bus}$ for the system is given by

$$\underbrace{\mathbf{Y}_{bus}}_{N \times N} = \underbrace{\mathbf{A}^T}_{N \times B} \underbrace{\mathbf{Y}_{pr}}_{B \times B} \underbrace{\mathbf{A}}_{B \times N} \tag{7.37}$$

$\mathbf{Y}_{bus}$ has one row and one column for each of the N buses in the network, and so the standard form of the four independent equations of the example system of Fig. 7.11 is

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{7.38}$$

The four unknowns are the bus voltages V_1, V_2, V_3, and V_4 when the bus-injected currents I_1, I_2, I_3, and I_4 are specified. Generally, $\mathbf{Y}_{pr}$ is symmetrical, in which case taking the transpose of each side of Eq. (7.37) shows that $\mathbf{Y}_{bus}$ is also symmetrical.

Example 7.5. Determine the per-unit bus admittance matrix of the example system of Fig. 7.11 using the tree shown in Fig. 7.13 with reference node ⓪.

Solution. The primitive admittance matrix $\mathbf{Y}_{pr}$ describing the branch admittances is given by Eq. (7.28) and the branch-to-node incidence matrix $\mathbf{A}$ for the specified tree is given by Eq. (7.31). Therefore, performing the row-by-column multiplications for $\mathbf{A}^T\mathbf{Y}_{pr}$ indicated by

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} -j0.8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -j6.25 & j3.75 & \cdot & \cdot & \cdot & \cdot \\ \cdot & j3.75 & -j6.25 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -j8.0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -j5.0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -j2.5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -j0.8 \end{bmatrix}$$

we obtain the intermediate result

$$\mathbf{A}^T\mathbf{Y}_{pr} = \begin{bmatrix} 0 & -j3.75 & j6.25 & j8.0 & 0 & j2.5 & 0 \\ 0 & j6.25 & -j3.75 & -j8.0 & j5.0 & 0 & 0 \\ -j0.8 & -j2.5 & -j2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -j5.0 & -j2.5 & -j0.8 \end{bmatrix}$$

which we may now postmultiply by $\mathbf{A}$ to calculate

$$\mathbf{Y}_{bus} = \mathbf{A}^T\mathbf{Y}_{pr}\mathbf{A} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -j16.75 & j11.75 & j2.50 & j2.50 \\ j11.75 & -j19.25 & j2.50 & j5.00 \\ j2.50 & j2.50 & -j5.80 & 0 \\ j2.50 & j5.00 & 0 & -j8.30 \end{bmatrix} \end{array}$$

Since currents are injected at only buses ③ and ④, the nodal equations in matrix form are written

$$\begin{bmatrix} -j16.75 & j11.75 & j2.50 & j2.50 \\ j11.75 & -j19.25 & j2.50 & j5.00 \\ j2.50 & j2.50 & -j5.80 & 0 \\ j2.50 & j5.00 & 0 & -j8.30 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

Example 7.6. Solve the node equations of Example 7.5 to find the bus voltages by inverting the bus admittance matrix.

Solution. Premultiplying both sides of the matrix nodal equation by the inverse of the bus admittance matrix (determined by using a standard program on a computer

or calculator) yields

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} j0.73128 & j0.69140 & j0.61323 & j0.63677 \\ j0.69140 & j0.71966 & j0.60822 & j0.64178 \\ j0.61323 & j0.60822 & j0.69890 & j0.55110 \\ j0.63677 & j0.64178 & j0.55110 & j0.69890 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

Performing the indicated multiplications, we obtain the per-unit results

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.96903\angle -18.4189^\circ \\ 0.96734\angle -18.6028^\circ \\ 0.99964\angle -15.3718^\circ \\ 0.94866\angle -20.7466^\circ \end{bmatrix} = \begin{bmatrix} 0.91939 - j0.30618 \\ 0.91680 - j0.30859 \\ 0.96388 - j0.26499 \\ 0.88715 - j0.33605 \end{bmatrix}$$

7.6 THE METHOD OF SUCCESSIVE ELIMINATION.

In industry-based studies of power systems the networks being solved are geographically extensive and often encompass many hundreds of substations, generating plants and load centers. The $\mathbf{Y}_{bus}$ matrices for these large networks of thousands of nodes have associated systems of nodal equations to be solved for a correspondingly large number of unknown bus voltages. For such solutions computer-based numerical techniques are required to avoid direct matrix inversion, thereby minimizing computational effort and computer storage. The method of successive elimination, called *gaussian elimination*, underlies many of the numerical methods solving the equations of such large-scale power systems. We now describe this method using the nodal equations of the four-bus system

$$Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 + Y_{14}V_4 = I_1 \tag{7.39}$$

$$Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 + Y_{24}V_4 = I_2 \tag{7.40}$$

$$Y_{31}V_1 + Y_{32}V_2 + Y_{33}V_3 + Y_{34}V_4 = I_3 \tag{7.41}$$

$$Y_{41}V_1 + Y_{42}V_2 + Y_{43}V_3 + Y_{44}V_4 = I_4 \tag{7.42}$$

Gaussian elimination consists of reducing this system of four equations in the four unknowns V_1, V_2, V_3, and V_4 to a system of three equations in three unknowns and then to a reduced system of two equations in two unknowns, and so on until there remains only one equation in one unknown. The final equation yields a value for the corresponding unknown, which is then substituted back in the reduced sets of equations to calculate the remaining unknowns. The successive elimination of unknowns in the forward direction is called *forward elimination* and the substitution process using the latest calculated values is

called *back substitution*. Forward elimination begins by selecting one equation and eliminating from this equation one variable whose coefficient is called the *pivot*. We exemplify the overall procedure by first eliminating V_1 from Eqs. (7.39) through (7.42) as follows:

Step 1

1. Divide Eq. (7.39) by the pivot Y_{11} to obtain

$$V_1 + \frac{Y_{12}}{Y_{11}}V_2 + \frac{Y_{13}}{Y_{11}}V_3 + \frac{Y_{14}}{Y_{11}}V_4 = \frac{1}{Y_{11}}I_1 \tag{7.43}$$

2. Multiply Eq. (7.43) by Y_{21}, Y_{31}, and Y_{41} and subtract the results from Eqs. (7.40) through (7.42), respectively, to get

$$\left(Y_{22} - \frac{Y_{21}Y_{12}}{Y_{11}}\right)V_2 + \left(Y_{23} - \frac{Y_{21}Y_{13}}{Y_{11}}\right)V_3 + \left(Y_{24} - \frac{Y_{21}Y_{14}}{Y_{11}}\right)V_4 = I_2 - \frac{Y_{21}}{Y_{11}}I_1 \tag{7.44}$$

$$\left(Y_{32} - \frac{Y_{31}Y_{12}}{Y_{11}}\right)V_2 + \left(Y_{33} - \frac{Y_{31}Y_{13}}{Y_{11}}\right)V_3 + \left(Y_{34} - \frac{Y_{31}Y_{14}}{Y_{11}}\right)V_4 = I_3 - \frac{Y_{31}}{Y_{11}}I_1 \tag{7.45}$$

$$\left(Y_{42} - \frac{Y_{41}Y_{12}}{Y_{11}}\right)V_2 + \left(Y_{43} - \frac{Y_{41}Y_{13}}{Y_{11}}\right)V_3 + \left(Y_{44} - \frac{Y_{41}Y_{14}}{Y_{11}}\right)V_4 = I_4 - \frac{Y_{41}}{Y_{11}}I_1 \tag{7.46}$$

Equations (7.43) through (7.46) may be written more compactly in the form

$$V_1 + \frac{Y_{12}}{Y_{11}}V_2 + \frac{Y_{13}}{Y_{11}}V_3 + \frac{Y_{14}}{Y_{11}}V_4 = \frac{1}{Y_{11}}I_1 \tag{7.47}$$

$$Y_{22}^{(1)}V_2 + Y_{23}^{(1)}V_3 + Y_{24}^{(1)}V_4 = I_2^{(1)} \tag{7.48}$$

$$Y_{32}^{(1)}V_2 + Y_{33}^{(1)}V_3 + Y_{34}^{(1)}V_4 = I_3^{(1)} \tag{7.49}$$

$$Y_{42}^{(1)}V_2 + Y_{43}^{(1)}V_3 + Y_{44}^{(1)}V_4 = I_4^{(1)} \tag{7.50}$$

where the superscript denotes the Step 1 set of derived coefficients

$$Y_{jk}^{(1)} = Y_{jk} - \frac{Y_{j1}Y_{1k}}{Y_{11}} \qquad \text{for } j \text{ and } k = 2,3,4 \tag{7.51}$$

and the modified right-hand side expressions

$$I_j^{(1)} = I_j - \frac{Y_{j1}}{Y_{11}} I_1 \qquad \text{for } j = 2, 3, 4 \tag{7.52}$$

Note that Eqs. (7.48) through (7.50) may now be solved for V_2, V_3, and V_4 since V_1 has been eliminated; and the coefficients constitute a reduced 3×3 matrix, which can be interpreted as representing a reduced equivalent network with bus ① absent. In this three-bus equivalent the voltages V_2, V_3, and V_4 have exactly the same values as in the original four-bus system. Moreover, the effect of the current injection I_1 on the network is taken into account at buses ②, ③, and ④, as shown by Eq. (7.52). The current I_1 at bus ① is multiplied by the factor $-Y_{j1}/Y_{11}$ before being *distributed*, so to speak, to each bus (*j*) still in the network.

We next consider the elimination of the variable V_2.

Step 2

1. Divide Eq. (7.48) by the new pivot $Y_{22}^{(1)}$ to obtain

$$V_2 + \frac{Y_{23}^{(1)}}{Y_{22}^{(1)}} V_3 + \frac{Y_{24}^{(1)}}{Y_{22}^{(1)}} V_4 = \frac{1}{Y_{22}^{(1)}} I_2^{(1)} \tag{7.53}$$

2. Multiply Eq. (7.53) by $Y_{32}^{(1)}$ and $Y_{42}^{(1)}$ and subtract the results from Eqs. (7.49) and (7.50) to get

$$\left(Y_{33}^{(1)} - \frac{Y_{32}^{(1)} Y_{23}^{(1)}}{Y_{22}^{(1)}} \right) V_3 + \left(Y_{34}^{(1)} - \frac{Y_{32}^{(1)} Y_{24}^{(1)}}{Y_{22}^{(1)}} \right) V_4 = I_3^{(1)} - \frac{Y_{32}^{(1)}}{Y_{22}^{(1)}} I_2^{(1)} \tag{7.54}$$

$$\left(Y_{43}^{(1)} - \frac{Y_{42}^{(1)} Y_{23}^{(1)}}{Y_{22}^{(1)}} \right) V_3 + \left(Y_{44}^{(1)} - \frac{Y_{42}^{(1)} Y_{24}^{(1)}}{Y_{22}^{(1)}} \right) V_4 = I_4^{(1)} - \frac{Y_{42}^{(1)}}{Y_{22}^{(1)}} I_2^{(1)} \tag{7.55}$$

In a manner similar to that of Step 1, we rewrite Eqs. (7.53) through (7.55) in the form

$$V_2 + \frac{Y_{23}^{(1)}}{Y_{22}^{(1)}} V_3 + \frac{Y_{24}^{(1)}}{Y_{22}^{(1)}} V_4 = \frac{1}{Y_{22}^{(1)}} I_2^{(1)} \tag{7.56}$$

$$Y_{33}^{(2)} V_3 + Y_{34}^{(2)} V_4 = I_3^{(2)} \tag{7.57}$$

$$Y_{43}^{(2)} V_3 + Y_{44}^{(2)} V_4 = I_4^{(2)} \tag{7.58}$$

where the second set of calculated coefficients is given by

$$Y_{jk}^{(2)} = Y_{jk}^{(1)} - \frac{Y_{j2}^{(1)} Y_{2k}^{(1)}}{Y_{22}^{(1)}} \qquad \text{for } j \text{ and } k = 3, 4 \tag{7.59}$$

and the net currents injected at buses ③ and ④ are

$$I_j^{(2)} = I_j^{(1)} - \frac{Y_{j2}^{(1)}}{Y_{22}^{(1)}} I_2^{(1)} \qquad \text{for } j = 3, 4 \tag{7.60}$$

Equations (7.57) and (7.58) describe a further reduced equivalent network having only buses ③ and ④. Voltages V_3 and V_4 are exactly the same as those of the original four-bus network because the current injections $I_3^{(2)}$ and $I_4^{(2)}$ represent the effects of all the original current sources.

We now consider elimination of the variable V_3.

Step 3

1. Divide Eq. (7.57) by the pivot $Y_{33}^{(2)}$ to obtain

$$V_3 + \frac{Y_{34}^{(2)}}{Y_{33}^{(2)}} V_4 = \frac{1}{Y_{33}^{(2)}} I_3^{(2)} \tag{7.61}$$

2. Multiply Eq. (7.61) by $Y_{43}^{(2)}$ and subtract the result from Eq. (7.58) to obtain

$$Y_{44}^{(3)} V_4 = I_4^{(3)} \tag{7.62}$$

in which we have defined

$$Y_{44}^{(3)} = Y_{44}^{(2)} - \frac{Y_{43}^{(2)} Y_{34}^{(2)}}{Y_{33}^{(2)}} \qquad \text{and} \qquad I_4^{(3)} = I_4^{(2)} - \frac{Y_{43}^{(2)}}{Y_{33}^{(2)}} I_3^{(2)} \tag{7.63}$$

Equation (7.62) describes the single equivalent branch admittance $Y_{44}^{(3)}$ with voltage V_4 from bus ④ to reference caused by the equivalent injected current $I_4^{(3)}$.

The final step in the forward elimination process yields V_4.

Step 4

1. Divide Eq. (7.62) by $Y_{44}^{(3)}$ to obtain

$$V_4 = \frac{1}{Y_{44}^{(3)}} I_4^{(3)} \tag{7.64}$$

At this point we have found a value for bus voltage V_4 which can be substituted

back in Eq. (7.61) to obtain a value for V_3. Continuing this process of back substitution using the values of V_3 and V_4 in Eq. (7.56), we obtain V_2 and then solve for V_1 from Eq. (7.47).

Thus, the gaussian-elimination procedure demonstrated here for a four-bus system provides a systematic means of solving large systems of equations without having to invert the coefficient matrix. This is most desirable when a large-scale power system is being analyzed. The following example numerically illustrates the procedure.

Example 7.7. Using gaussian elimination, solve the nodal equations of Example 7.5 to find the bus voltages. At each step of the solution find the equivalent circuit of the reduced coefficient matrix.

Solution. In Example 7.5 the nodal admittance equations in matrix form are found to be

$$\begin{array}{c} \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{array}\begin{array}{c} \begin{array}{cccc} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{array} \\ \begin{bmatrix} -j16.75 & j11.75 & \boxed{j2.50} & j2.50 \\ \boxed{j11.75} & -j19.25 & \underline{j2.50} & j5.00 \\ j2.50 & j2.50 & -j5.80 & 0.00 \\ j2.50 & j5.00 & 0.00 & -j8.30 \end{bmatrix} \end{array}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

Step 1

To eliminate the variable V_1 from rows 2, 3, and 4, we first divide the first row by the pivot $-j16.75$ to obtain

$$V_1 - 0.70149V_2 - 0.14925V_3 - 0.14925V_4 = 0$$

We now use this equation to eliminate the $j11.75$ entry in (row 2, column 1) of $\mathbf{Y}_{\text{bus}}$, and in the process all the other elements of row 2 are modified. Equation (7.51) shows the procedure. For example, to modify the element $j2.50$ underlined in (row 2, column 3), subtract from it the product of the elements enclosed by rectangles divided by the pivot $-j16.75$; that is,

$$Y_{23}^{(1)} = Y_{23} - \frac{Y_{21}Y_{13}}{Y_{11}} = j2.50 - \frac{j11.75 \times j2.50}{-j16.75} = j4.25373 \text{ per unit}$$

Similarly, the other elements of new row 2 are

$$Y_{22}^{(1)} = -j19.25 - \frac{j11.75 \times j11.75}{-j16.75} = -j11.00746 \text{ per unit}$$

$$Y_{24}^{(1)} = j5.00 - \frac{j11.75 \times j2.50}{-j16.75} = j6.75373 \text{ per unit}$$

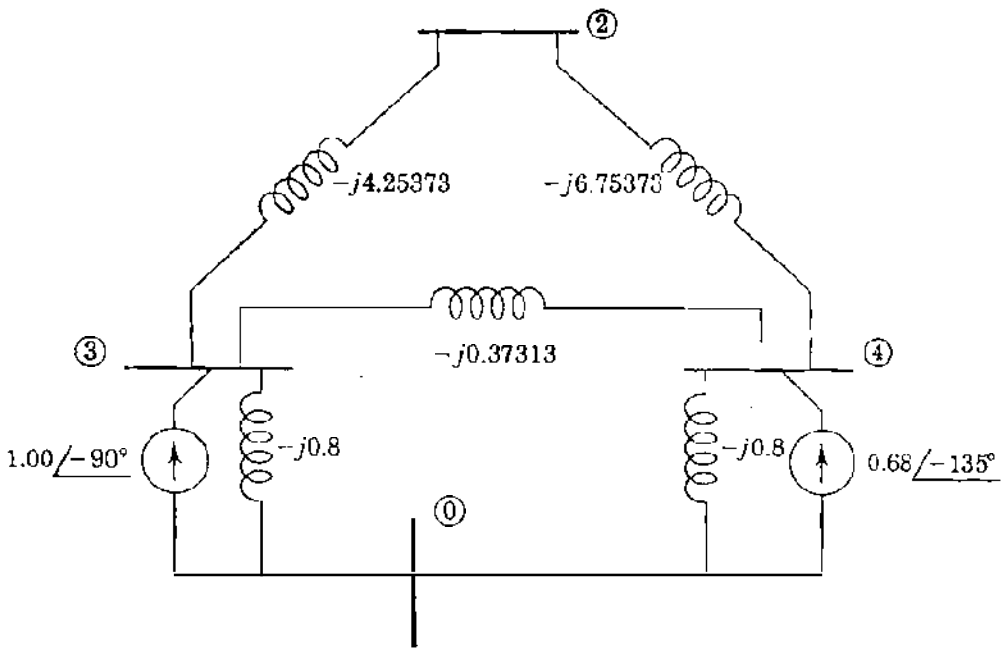


FIGURE 7.14
The equivalent three-bus network following Step 1 of Example 7.7

Modified elements of rows 3 and 4 are likewise found to yield

$$\left[\begin{array}{c|ccc} 1 & -0.70149 & -0.14925 & -0.14925 \\ \hline 0 & -j11.00746 & j4.25373 & j6.75373 \\ 0 & j4.25373 & -j5.42686 & j0.37313 \\ 0 & j6.75373 & j0.37313 & -j7.92686 \end{array}\right]\left[\begin{array}{c} V_1 \\ \hline V_2 \\ V_3 \\ V_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \\ 1.00/-90^\circ \\ 0.68/-135^\circ \end{array}\right]$$

Because $I_1 = 0$, no current is distributed from bus ① to the remaining buses ②, ③, and ④, and so the currents I_2^1, I_3^1, and I_4^1 in the right-hand-side vector have the same values before and after Step 1. The partitioned system of equations involving the unknown voltages V_2, V_3, and V_4 corresponds to the three-bus equivalent network constructed in Fig. 7.14 from the reduced coefficient matrix.

Step 2

Forward elimination applied to the partitioned 3×3 system of the last equation proceeds as in Step 1 to yield

$$\left[\begin{array}{cc|cc} 1 & -0.70149 & -0.14925 & -0.14925 \\ 0 & 1 & -0.38644 & -0.61356 \\ \hline 0 & 0 & -j3.78305 & j2.98305 \\ 0 & 0 & j2.98305 & -j3.78305 \end{array}\right]\left[\begin{array}{c} V_1 \\ V_2 \\ \hline V_3 \\ V_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1.00/-90^\circ \\ 0.68/-135^\circ \end{array}\right],$$

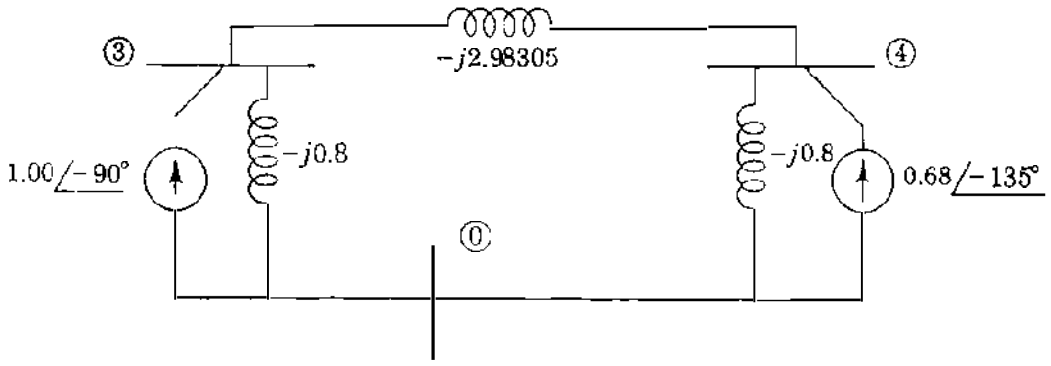


FIGURE 7.15
The equivalent two-bus network following Step 2 of Example 7.7.

in which the injected currents $I_3^{(2)}$ and $I_4^{(2)}$ of Step 2 are also unchanged because $I_2^{(1)} = I_2 = 0$. At this stage we have eliminated V_1 and V_2 from the original 4×4 system of equations and there remains the 2×2 system in the variables V_3 and V_4 corresponding to $\mathbf{Y}_{\text{bus}}$ of Fig. 7.15. Note that nodes ① and ② are eliminated.

Step 3

Continuing the forward elimination, we find

$$\left[\begin{array}{ccc|c} 1 & -0.70149 & -0.14925 & -0.14925 \\ 0 & 1 & -0.38644 & -0.61356 \\ 0 & 0 & 1 & -0.78853 \\ \hline 0 & 0 & 0 & -j1.43082 \end{array}\right]\left[\begin{array}{c} V_1 \\ V_2 \\ V_3 \\ \hline V_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0.26434\angle 0^\circ \\ \hline 1.35738\angle -110.7466^\circ \end{array}\right]$$

in which the entry for bus ③ in the right-hand-side vector is calculated to be

$$\frac{I_3^{(2)}}{Y_{33}^{(2)}} = \frac{1.00\angle -90^\circ}{3.78305\angle -90^\circ} = 0.26434\angle 0^\circ \text{ per unit}$$

and the modified current at bus ④ is given by

$$I_4^{(3)} = I_4^{(2)} - \frac{Y_{43}^{(2)}}{Y_{33}^{(2)}} I_3^{(2)}$$

$$= 0.68\angle -135^\circ - \frac{j2.98305}{-j3.78305} 1.00\angle -90^\circ$$

$$= 1.35738\angle -110.7466^\circ \text{ per unit}$$

Figure 7.16(*a*) shows the single admittance resulting from Step 3.

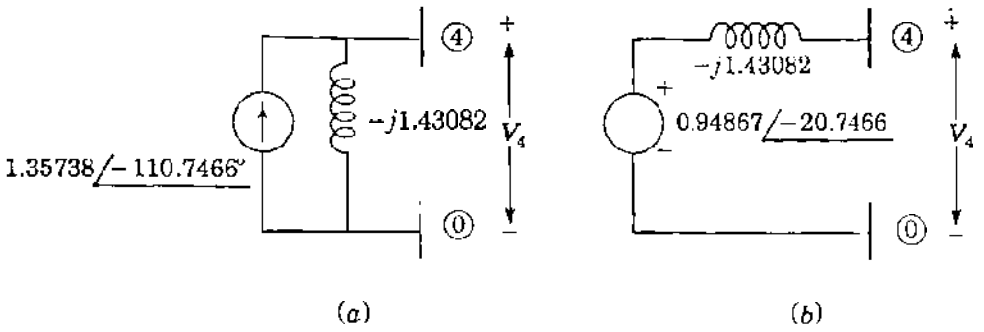


FIGURE 7.16
The equivalent circuits following (*a*) Step 3 and (*b*) Step 4 of Example 7.7.

Step 4

The forward-elimination process terminates with the calculation

$$V_4 = \frac{I_4^{(3)}}{Y_{44}^{(3)}} = \frac{1.35738\angle -110.7466°}{1.43082\angle -90°} = 0.94867\angle -20.7466° \text{ per unit}$$

corresponding to the source transformation of Fig. 7.16(*b*). Therefore, forward elimination leads to the triangular coefficient matrix given by

$$\begin{bmatrix} 1 & -0.70149 & -0.14925 & -0.14925 \\ 0 & 1 & -0.38644 & -0.61356 \\ 0 & 0 & 1 & -0.78853 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.26434\angle 0° \\ 0.94867\angle -20.7466° \end{bmatrix}$$

Since $V_4 = 0.94867\angle -20.7466°$, we now begin the back-substitution process to determine V_3 using the third row entries as follows:

$$V_3 - 0.78853V_4 = V_3 - 0.74805\angle -20.7466° = 0.26434\angle 0°$$

which yields

$$V_3 = 0.99965\angle -15.3716° \text{ per unit}$$

Back substitution for V_3 and V_4 in the second-row equation

$$V_2 - 0.38644V_3 - 0.61356V_4 = 0$$

yields

$$V_2 = 0.96734\angle -18.6030° \text{ per unit}$$

When the calculated values of V_2, V_3, and V_4 are substituted in the first-row equation

$$V_1 - 0.70149V_2 - 0.14925V_3 - 0.14925V_4 = 0$$

we obtain

$$V_1 = 0.96903\angle -18.4189^\circ \text{ per unit}$$

and so the per-unit bus voltages are

$$V_1 = 0.96903\angle -18.4189^\circ = 0.91939 - j0.30618$$

$$V_2 = 0.96734\angle -18.6030^\circ = 0.91680 - j0.30859$$

$$V_3 = 0.99964\angle -15.3716^\circ = 0.96388 - j0.26499$$

$$V_4 = 0.94867\angle -20.7466^\circ = 0.88715 - j0.33605$$

which agree almost exactly with the results found in Example 7.6.

7.7 NODE ELIMINATION (KRON REDUCTION)

Section 7.6 shows that gaussian elimination removes the need for matrix inversion when solving the nodal equations of a large-scale power system. At the same time it is also shown that elimination of variables is identical to network reduction since it leads to a sequence of reduced-order network equivalents by node elimination at each step. This may be important in analyzing a large interconnected power system if there is special interest in the voltages at only some of the buses of the overall system. For instance, one electric utility company with interconnections to other companies may wish to confine its study of voltage levels to those substations within its own service territory. By selective numbering of the system buses, we may apply gaussian elimination so as to reduce the $\mathbf{Y}_{bus}$ equations of the overall system to a set which contains only those bus voltages of special interest. The coefficient matrix in the reduced-order set of equations then represents the $\mathbf{Y}_{bus}$ for an equivalent network containing only those buses which are to be retained. All other buses are eliminated in the mathematical sense that their bus voltages and current injections do not appear explicitly. Such reduction in size of the equation set leads to efficiency of computation and helps to focus more directly on that portion of the overall network which is of primary interest.

In gaussian elimination one bus-voltage variable at a time is sequentially removed from the original system of N equations in N unknowns. Following Step 1 of the procedure, the variable V_1 does not explicitly appear in the

resultant $(N-1)\times(N-1)$ system, which fully represents the original network if the actual value of the voltage V_1 at bus ① is not of direct interest. If knowledge of V_2 is also not of prime interest, we can interpret the $(N-2)\times(N-2)$ system of equations resulting from Step 2 of the procedure as replacing the actual network by an $(N-2)$ bus equivalent with buses ① and ② removed, and so on. Consequently, in our network calculations if it is advantageous to do so, we may eliminate k nodes from the network representation by employing the first k steps of the gaussian-elimination procedure. Of course, the current injections (if any) at the eliminated nodes are taken into account at the remaining $(N-k)$ nodes by successive application of expressions such as that in Eq. (7.54).

Current injection is always zero at those buses of the network to which there is no external load or generating source connected. At such buses it is usually not necessary to calculate the voltages explicitly, and so we may eliminate them from our representation. For example, when $I_1 = 0$ in the four-bus system, we may write nodal admittance equations in the form

$$\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array}\begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{7.65}$$

and following elimination of node ①, we obtain the 3×3 system

$$\begin{array}{c} ② \\ ③ \\ ④ \end{array}\begin{array}{c} \begin{array}{ccc} ② & ③ & ④ \end{array} \\ \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & Y_{24}^{(1)} \\ Y_{32}^{(1)} & Y_{33}^{(1)} & Y_{34}^{(1)} \\ Y_{42}^{(1)} & Y_{43}^{(1)} & Y_{44}^{(1)} \end{bmatrix} \end{array} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{7.66}$$

in which the superscripted elements of the reduced coefficient matrix are calculated as before. Systems in which those nodes with zero current injections are eliminated are said to be *Kron*[1] *reduced*. Hence, the system having the particular form of Eq. (7.65) is Kron reduced to Eq. (7.66), and for such systems node elimination and Kron reduction are synonymous terms.

Of course, regardless of which node has the zero current injection, a system can be Kron reduced without having to rearrange the equations as in Eq. (7.65). For example, if $I_p = 0$ in the nodal equations of the N-bus system, we

[1]After Dr. Gabriel Kron (1901–1968) of General Electric Company, Schenectady, NY, who contributed greatly to power system analysis.

may directly calculate the elements of the new, reduced bus admittance matrix by choosing Y_{pp} as the pivot and by eliminating bus p using the formula

$$Y_{jk(\text{new})} = Y_{jk} - \frac{Y_{jp}Y_{pk}}{Y_{pp}} \tag{7.67}$$

where j and k take on all the integer values from 1 to N except p since row p and column p are to be eliminated. The subscript (new) distinguishes the elements in the new $\mathbf{Y}_{\text{bus}}$ of dimension $(N-1) \times (N-1)$ from the elements in the original $\mathbf{Y}_{\text{bus}}$.

Example 7.8. Using Y_{22} as the initial pivot, eliminate node ② and the corresponding voltage V_2 from the 4×4 system of Example 7.7

Solution. The pivot Y_{22} equals $-j19.25$. With p set equal to 2 in Eq. (7.67), we may eliminate row 2 and column 2 from $\mathbf{Y}_{\text{bus}}$ of Example 7.7 to obtain the new row 1 elements

$$Y_{11(\text{new})} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22}} = -j16.75 - \frac{(j11.75)(j11.75)}{-j19.25} = -j9.57792$$

$$Y_{13(\text{new})} = Y_{13} - \frac{Y_{12}Y_{23}}{Y_{22}} = -j2.50 - \frac{(j11.75)(j2.50)}{-j19.25} = j4.02597$$

$$Y_{14(\text{new})} = Y_{14} - \frac{Y_{12}Y_{24}}{Y_{22}} = j2.50 - \frac{(j11.75)(j5.00)}{-j19.25} = j5.55195$$

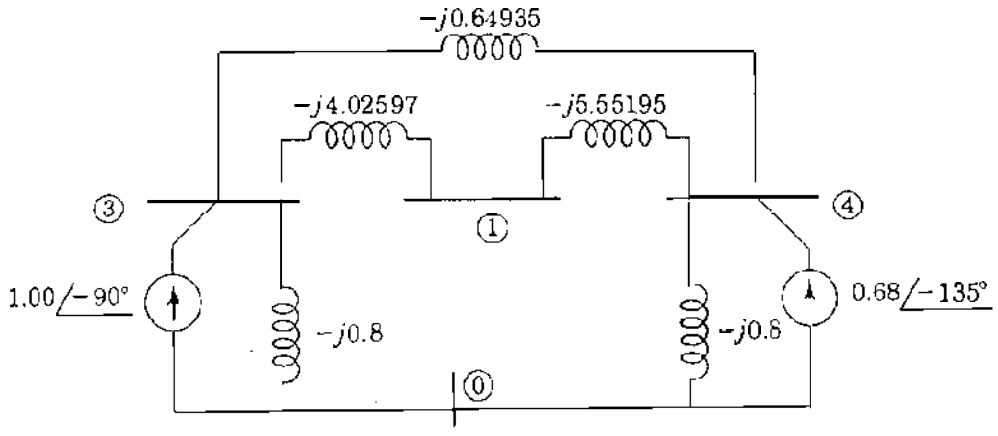


FIGURE 7.17
The Kron-reduced network of Example 7.8.

Similar calculations yield the other elements of the Kron-reduced matrix

$$\begin{array}{c} \\ ① \\ ③ \\ ④ \end{array}\begin{array}{c} \begin{array}{ccc} ① & ③ & ④ \end{array} \\ \begin{bmatrix} -j9.57791 & j4.02597 & j5.55195 \\ j4.02597 & -j5.47532 & j0.64935 \\ j5.55195 & j0.64935 & -j7.00130 \end{bmatrix} \end{array} \begin{bmatrix} V_1 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.00\angle -90^\circ \\ 0.68\angle -135^\circ \end{bmatrix}$$

Because the coefficient matrix is symmetrical, the equivalent circuit of Fig. 7.17 applies. Further use of Eq. (7.67) to eliminate node ① from Fig. 7.17 leads to the Kron-reduced equivalent circuit shown in Fig. 7.15.

7.8 TRIANGULAR FACTORIZATION

In practical studies the nodal admittance equations of a given large-scale power system are solved under different operating conditions. Often in such studies the network configuration and parameters are fixed and the operating conditions differ only because of changes made to the external sources connected at the system buses. In all such cases the same $\mathbf{Y}_{bus}$ applies and the problem then is to solve repeatedly for the voltages corresponding to different sets of current injections. In finding repeat solutions considerable computational effort is avoided if all the calculations in the forward phase of the gaussian-elimination procedure do not have to be repeated. This may be accomplished by expressing $\mathbf{Y}_{bus}$ as the product of two matrices $\mathbf{L}$ and $\mathbf{U}$ defined for the four-bus system by

$$\mathbf{L} = \begin{bmatrix} Y_{11} & \cdot & \cdot & \cdot \\ Y_{21} & Y_{22}^{(1)} & \cdot & \cdot \\ Y_{31} & Y_{32}^{(1)} & Y_{33}^{(2)} & \cdot \\ Y_{41} & Y_{42}^{(1)} & Y_{43}^{(2)} & Y_{44}^{(3)} \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 1 & \dfrac{Y_{12}}{Y_{11}} & \dfrac{Y_{13}}{Y_{11}} & \dfrac{Y_{14}}{Y_{11}} \\ \cdot & 1 & \dfrac{Y_{23}^{(1)}}{Y_{22}^{(1)}} & \dfrac{Y_{24}^{(1)}}{Y_{22}^{(1)}} \\ \cdot & \cdot & 1 & \dfrac{Y_{34}^{(2)}}{Y_{33}^{(2)}} \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{7.68}$$

The matrices $\mathbf{L}$ and $\mathbf{U}$ are called the *lower-* and *upper-triangular factors* of $\mathbf{Y}_{bus}$ because they have zero elements above and below the respective principal diagonals. These two matrices have the remarkably convenient property that their product equals $\mathbf{Y}_{bus}$ (Problem 7.13). Thus, we can write

$$\mathbf{LU} = \mathbf{Y}_{bus} \tag{7.69}$$

The process of developing the triangular matrices $\mathbf{L}$ and $\mathbf{U}$ from $\mathbf{Y}_{bus}$ is called *triangular factorization* since $\mathbf{Y}_{bus}$ is factored into the product $\mathbf{LU}$. Once $\mathbf{Y}_{bus}$ is so factored, the calculations in the forward-elimination phase of gaussian

elimination do not have to be repeated since both **L** and **U** are unique and do not change for a given $\mathbf{Y}_{bus}$. The entries in **L** and **U** are formed by systematically recording the outcome of the calculations at each step of a single pass through the forward-elimination process. Thus, in forming **L** and **U**, no new calculations are involved.

This is now demonstrated for the four-bus system with coefficient matrix

$$\mathbf{Y}_{bus} = \begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array} \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \end{array} \tag{7.70}$$

When gaussian elimination is applied to the four nodal equations corresponding to this $\mathbf{Y}_{bus}$, we note the following.

Step 1 yields results given by Eqs. (7.47) through (7.50) in which:

1. The coefficients Y_{11}, Y_{21}, Y_{31}, and Y_{41} are eliminated from the *first column* of the original coefficient matrix of Eq. (7.70).
2. New coefficients 1, Y_{12}/Y_{11}, Y_{13}/Y_{11}, and Y_{14}/Y_{11} are generated to replace those in the *first row* of Eq. (7.70).

The coefficients in the other rows and columns are also altered, but we keep a separate record of only those specified in (1) and (2) since these are the only results from Step 1 which are neither used nor altered in Step 2 or subsequent steps of the forward-elimination process. Column 1 of L and row 1 of U in Eqs. (7.68) show the recorded coefficients.

Step 2 yields results given by Eqs. (7.56) through (7.58) in which:

1. Coefficients $Y_{22}^{(1)}$, $Y_{32}^{(1)}$, and $Y_{42}^{(1)}$ are eliminated from the *second column* of the reduced coefficient matrix corresponding to Eqs. (7.47) through (7.50).
2. New coefficients 1, $Y_{23}^{(1)}/Y_{22}^{(1)}$, and $Y_{24}^{(1)}/Y_{22}^{(1)}$ are generated in Eq. (7.56) for the *second row*.

These coefficients are not needed in the remaining steps of forward elimination, and so we record them as column 2 of **L** and row 2 of **U** to show the Step 2 record. Continuing this record-keeping procedure, we form columns 3 and 4 of L and rows 3 and 4 of **U** using the results from Steps 3 and 4 of Sec. 7.6.

Therefore, matrix **L** is simply a record of those columns which are successively eliminated and matrix **U** records those row entries which are successively generated at each step of the forward stage of gaussian elimination.

We may use the triangular factors to solve the original system of equations by substituting the product **LU** for $\mathbf{Y}_{bus}$ in Eq. (7.38) to obtain

$$\underbrace{\mathbf{L}}_{N\times N}\ \underbrace{\mathbf{U}}_{N\times N}\ \underbrace{\mathbf{V}}_{N\times 1} = \underbrace{\mathbf{I}}_{N\times 1} \tag{7.71}$$

As an intermediate step in the solution of Eq. (7.71), we may replace the product UV by a new voltage vector $\mathbf{V}'$ such that

$$\underbrace{\mathbf{L}\ \ \mathbf{V}'}_{N\times N\ \ N\times 1} = \underbrace{\mathbf{I}}_{N\times 1} \quad \text{and} \quad \underbrace{\mathbf{U}\ \ \mathbf{V}}_{N\times N\ \ N\times 1} = \underbrace{\mathbf{V}'}_{N\times 1} \tag{7.72}$$

Expressing Eq. (7.72) in full format shows that the original system of Eq. (7.38) is now replaced by two triangular systems given by

$$\begin{bmatrix} Y_{11} & \cdot & \cdot & \cdot \\ Y_{21} & Y_{22}^{(1)} & \cdot & \cdot \\ Y_{31} & Y_{32}^{(1)} & Y_{33}^{(2)} & \cdot \\ Y_{41} & Y_{42}^{(1)} & Y_{43}^{(2)} & Y_{44}^{(3)} \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \tag{7.73}$$

and

$$\begin{bmatrix} 1 & \dfrac{Y_{12}}{Y_{11}} & \dfrac{Y_{13}}{Y_{11}} & \dfrac{Y_{14}}{Y_{11}} \\ \cdot & 1 & \dfrac{Y_{23}^{(1)}}{Y_{22}^{(1)}} & \dfrac{Y_{24}^{(1)}}{Y_{22}^{(1)}} \\ \cdot & \cdot & 1 & \dfrac{Y_{34}^{(2)}}{Y_{33}^{(2)}} \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} \tag{7.74}$$

The lower triangular system of Eq. (7.73) is readily solved by forward substitution beginning with V_1'. We then use the calculated values of V_1', V_2', V_3', and V_4' to solve Eq. (7.74) by back substitution for the actual unknowns V_1, V_2, V_3, and V_4.

Therefore, when changes are made in the current vector **I**, the solution vector **V** is found in two sequential steps; the first involves forward substitution using **L** and the second employs back substitution using **U**.

Example 7.9. Using the triangular factors of $\mathbf{Y}_{bus}$, determine the voltage at bus ③ of Fig. 7.11 when the current source at bus ④ is changed to $I_4 = 0.60\angle -120°$ per unit. All other conditions of Fig. 7.11 are unchanged.

Solution. The $\mathbf{Y}_{bus}$ for the network of Fig. 7.11 is given in Example 7.3. The corresponding matrix L may be assembled column by column from Example 7.7 simply by recording the column which is eliminated from the coefficient matrix at each step of the forward-elimination procedure. Then, substituting for L and the new current vector I in the equation $\mathbf{LV}' = \mathbf{I}$, we obtain

$$\begin{bmatrix} -j16.75 & \cdot & \cdot & \cdot \\ j11.75 & -j11.00746 & \cdot & \cdot \\ j2.50 & j4.25373 & -j3.78305 & \cdot \\ j2.50 & j6.75373 & j2.98305 & -j1.43082 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90° \\ 0.60\angle -120° \end{bmatrix}$$

Solution by forward substitution beginning with V_1' yields

$$V_1' = V_2' = 0; \qquad V_3' = \frac{1.00\angle -90°}{3.78305\angle -90°} = 0.26434$$

$$V_4' = \frac{0.60\angle -120° - (j2.98305)V_3'}{-j1.43082} = 0.93800\angle -12.9163°$$

The matrix U is shown directly following Step 4 of the forward elimination in Example 7.7. Substituting U and the calculated entries of $\mathbf{V}'$ in the equation $\mathbf{UV} = \mathbf{V}'$ gives

$$\begin{bmatrix} 1 & -0.70149 & -0.14925 & -0.14925 \\ \cdot & 1 & -0.38644 & -0.61356 \\ \cdot & \cdot & 1 & -0.78853 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.26434 \\ 0.93800\angle -12.9163° \end{bmatrix}$$

which we may solve by back substitution to obtain

$$V_4 = V_4' = 0.93800\angle -12.9163° \text{ per unit}$$

$$V_3 = 0.26434 - (-0.78853)V_4 = 0.99904\angle -9.5257° \text{ per unit}$$

If desired, we may continue the back substitution using the values for V_3 and V_4 to evaluate $V_2 = 0.96118\angle -11.5551°$ per unit and $V_1 = 0.96324\angle -11.4388°$ per unit.

When the coefficient matrix $\mathbf{Y}_{bus}$ is symmetrical, which is almost always the case, an important simplification results. As can be seen by inspection of Eq. (7.68), when the first column of **L** is divided by Y_{11}, we obtain the first row of U; when the second column of **L** is divided by $Y_{22}^{(1)}$, we obtain the second row of U; and so on for the other columns and rows of Eq. (7.68), provided $Y_{ij} = Y_{ji}$. Therefore, dividing the entries in each column of **L** by the principal diagonal element in that column yields $\mathbf{U}^T$ whenever $\mathbf{Y}_{bus}$ is symmetrical. We can then write

$$\mathbf{L} = \mathbf{U}^T\mathbf{D} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \dfrac{Y_{21}}{Y_{11}} & 1 & \cdot & \cdot \\ \dfrac{Y_{31}}{Y_{11}} & \dfrac{Y_{32}^{(1)}}{Y_{22}^{(1)}} & 1 & \cdot \\ \dfrac{Y_{41}}{Y_{11}} & \dfrac{Y_{42}^{(1)}}{Y_{22}^{(1)}} & \dfrac{Y_{43}^{(2)}}{Y_{33}^{(2)}} & 1 \end{bmatrix} \begin{bmatrix} Y_{11} & \cdot & \cdot & \cdot \\ \cdot & Y_{22}^{(1)} & \cdot & \cdot \\ \cdot & \cdot & Y_{33}^{(2)} & \cdot \\ \cdot & \cdot & \cdot & Y_{44}^{(3)} \end{bmatrix} \tag{7.75}$$

where diagonal matrix **D** contains the diagonal elements of **L**. Substituting in Eq. (7.71) for **L** from Eq. (7.75), we obtain the nodal admittance equations in the form

$$\mathbf{Y}_{bus}\mathbf{V} = \mathbf{U}^T\mathbf{D}\mathbf{U}\mathbf{V} = \mathbf{I} \tag{7.76}$$

Equation (7.76) may be solved for the unknown voltages **V** in three consecutive steps as follows:

$$\mathbf{U}^T\mathbf{V}'' = \mathbf{I} \tag{7.77}$$

$$\mathbf{D}\mathbf{V}' = \mathbf{V}'' \tag{7.78}$$

$$\mathbf{U}\mathbf{V} = \mathbf{V}' \tag{7.79}$$

These equations will be recognized as an extension of Eqs. (7.72). The intermediate result $\mathbf{V}''$ is first found from Eq. (7.77) by forward substitution. Next, each entry in $\mathbf{V}'$ is calculated from Eq. (7.78) by dividing the corresponding element of $\mathbf{V}''$ by the appropriate diagonal element of **D**. Finally, the solution **V** is obtained from Eq. (7.79) by back substitution as demonstrated in Example 7.9.

Example 7.10. Using Eqs. (7.77) through (7.79), determine the solution vector **V** of unknown voltages for the system and operating conditions of Example 7.9.

Solution. Substituting in Eq. (7.77) from the current vector I and matrix U of Example 7.9, we obtain

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ -0.70149 & 1 & \cdot & \cdot \\ -0.14925 & -0.38644 & 1 & \cdot \\ -0.14925 & -0.61356 & -0.78853 & 1 \end{bmatrix} \begin{bmatrix} V_1'' \\ V_2'' \\ V_3'' \\ V_4'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 0.60\angle -120^\circ \end{bmatrix}$$

Straightforward solution of this system of equations yields

$$V_1'' = V_2'' = 0 \qquad V_3'' = 1.00\angle -90^\circ \text{ per unit}$$

$$V_4'' = 0.60\angle -120^\circ + 0.78853V_3'' = 1.34210\angle -102.9164^\circ \text{ per unit}$$

Substituting for V'' in Eq. (7.78) leads to the diagonal system

$$\begin{bmatrix} -j16.75 & \cdot & \cdot & \cdot \\ \cdot & -j11.00746 & \cdot & \cdot \\ \cdot & \cdot & -j3.78305 & \cdot \\ \cdot & \cdot & \cdot & -j1.43082 \end{bmatrix} \begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.00\angle -90^\circ \\ 1.34210\angle -102.9164^\circ \end{bmatrix}$$

the solution of which is exactly equal to V' of Example 7.9, and so the remaining steps of this example match those of Example 7.9.

7.9 SPARSITY AND NEAR-OPTIMAL ORDERING

Large-scale power systems have only a small number of transmission lines connected to each bulk-power substation. In the network graph for such systems the ratio of the number of branches to the number of nodes is about 1.5 and the corresponding $\mathbf{Y}_{bus}$ has mainly zero elements. In fact, if there are 750 branches in a 500-node network (excluding the reference node), then since each node has an associated diagonal element and each branch gives rise to two symmetrically placed off-diagonal elements, the total number of nonzero elements is $(500 + 2 \times 750) = 2000$. This compares to a total of 250,000 elements in $\mathbf{Y}_{bus}$; that is, only 0.8% of the elements in $\mathbf{Y}_{bus}$ are nonzero. Because of the small number of nonzero elements, such matrices are said to be *sparse*. From the viewpoint of computational speed, accuracy, and storage it is desirable to process only the nonzero entries in $\mathbf{Y}_{bus}$ and to avoid *filling-in* new nonzero elements in the course of gaussian elimination and triangular factorization. Ordering refers to the sequence in which the equations of a system are processed. When a sparse

matrix is triangularized, the order in which the unknown variables are eliminated affects the accumulation of new nonzero entries, called *fill-ins*, in the triangular matrices **L** and **U**. To help minimize such accumulations, we may use ordering schemes as described in Sec. B.1 of the Appendix.

7.10 SUMMARY

Nodal representation of the power transmission network is developed in this chapter. The essential background for understanding the bus admittance matrix and its formation is provided. Incorporation of mutually coupled branches into $\mathbf{Y}_{bus}$ can be handled by the building-block approach described here. Modifications to $\mathbf{Y}_{bus}$ to reflect network changes are thereby facilitated.

Gaussian elimination offers an alternative to matrix inversion for solving large-scale power systems, and triangular factorization of $\mathbf{Y}_{bus}$ enhances computational efficiency and reduces computer memory requirements, especially when the network matrices are symmetric.

These modeling and numerical procedures underlie the solution approaches now being used in daily practice by the electric power industry for power-flow and system analysis.

PROBLEMS

7.1. Using the building-block procedure described in Sec. 7.1, determine $\mathbf{Y}_{bus}$ for the circuit of Fig. 7.18. Assume there is *no* mutual coupling between any of the branches.

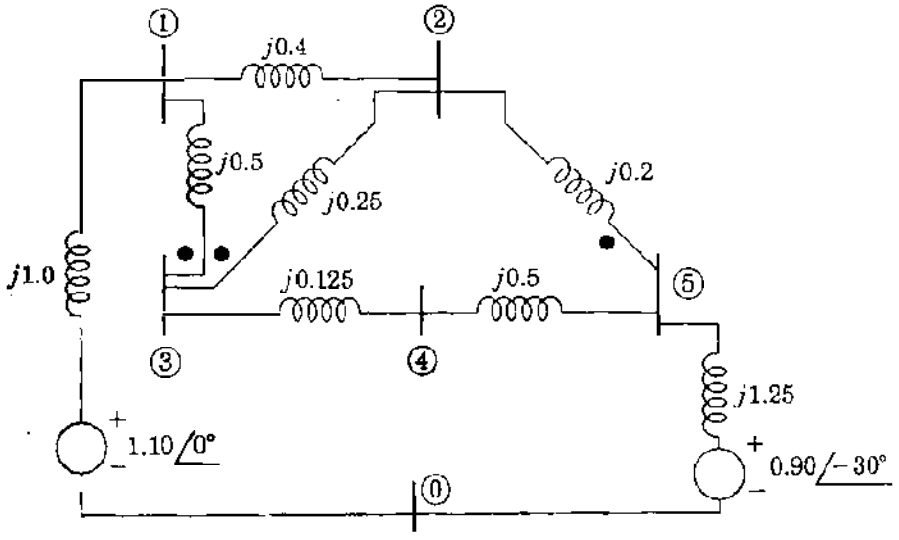


FIGURE 7.18
Values shown are voltages and impedances in per unit. Dots represent mutual coupling between branches unless otherwise stated in problems.

7.2. Using the $\mathbf{Y}_{bus}$ modification procedure described in Sec. 7.4 and assuming no mutual coupling between branches, modify the $\mathbf{Y}_{bus}$ obtained in Prob. 7.1 to reflect removal of the two branches ①–③ and ②–⑤ from the circuit of Fig. 7.18.

7.3. The circuit of Fig. 7.18 has the linear graph shown in Fig. 7.19, with arrows indicating directions assumed for the branches a to h. Disregarding all mutual coupling between branches:
   (a) Determine the branch-to-node incidence matrix **A** for the circuit with node ⓪ as reference.
   (b) Find the circuit $\mathbf{Y}_{bus}$ using Eq. (7.37).

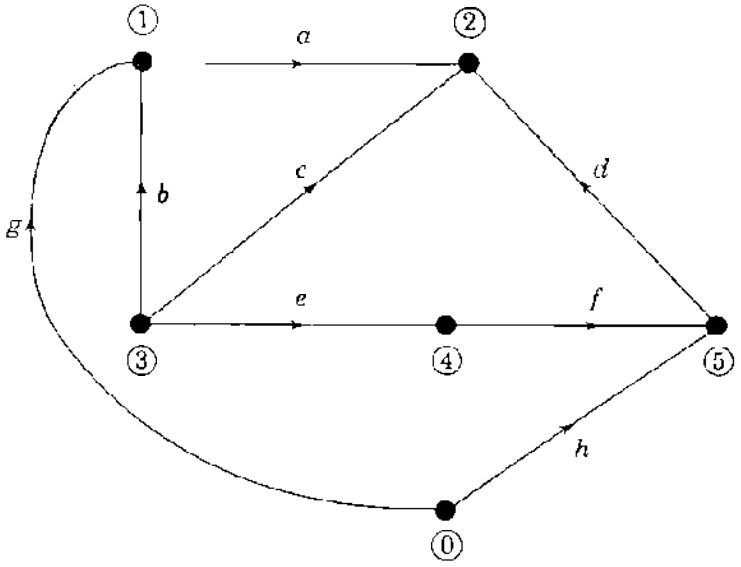


FIGURE 7.19
Linear graph for Prob. 7.3.

7.4. Consider that *only* the two branches ①–③ and ②–③ in the circuit of Fig. 7.18 are mutually coupled as indicated by the dots beside them and that their mutual impedance is $j0.15$ per unit (that is, ignore the dot on branch ②–⑤). Determine the circuit $\mathbf{Y}_{bus}$ by the procedure described in Sec. 7.2.

7.5. Solve Prob. 7.4 using Eq. (7.37). Determine the branch-to-node incidence matrix **A** from the linear graph of Fig. 7.19 with node ⓪ as reference.

7.6. Using the modification procedure of Sec. 7.4, modify the $\mathbf{Y}_{bus}$ solution of Prob. 7.4 (or Prob. 7.5) to reflect removal of the branch ②–③ from the circuit.

7.7. Modify the $\mathbf{Y}_{bus}$ determined in Example 7.3 to reflect removal of the mutually coupled branch ①–③ from the circuit of Fig. 7.11. Use the modification procedure of Sec. 7.4.

7.8. A new branch having a self-impedance of $j0.2$ per unit is added between nodes ② and ③ in the circuit of Fig. 7.11. Mutual impedance of $j0.1$ per unit couples this new branch to the branch already existing between nodes ② and ③. Modify the $\mathbf{Y}_{bus}$ obtained in Example 7.3 to account for the addition of the new branch.

7.9. Suppose that mutual coupling exists pairwise between branches ①–③ and ②–③, and *also* between branches ②–③ and ②–⑤ of Fig. 7.18, as shown by the dots in that figure. The mutual impedance between the former pair of

branches is $j0.15$ per unit (the same as in Prob. 7.4) and between the latter pair is $j0.1$ per unit. Use the procedure of Sec. 7.2 to find $\mathbf{Y}_{bus}$ for the overall circuit including the *three* mutually coupled branches.

7.10. Solve for the $\mathbf{Y}_{bus}$ of Prob. 7.9 using Eq. (7.37). Use the linear graph of Fig. 7.19 with reference node ⓪ to determine the branch-to-node incidence matrix **A**.

7.11. Suppose that the direction of branch d in Fig. 7.19 is reversed so that it is now directed from node ② to node ⑤. Find the branch-to-node incidence matrix **A** of this modified graph and then solve for the $\mathbf{Y}_{bus}$ of Prob. 7.9 using Eq. (7.37).

7.12. Using the $\mathbf{Y}_{bus}$ modification procedure described in Sec. 7.4, remove branch ②–③ from the $\mathbf{Y}_{bus}$ solution obtained in Prob. 7.9 (or Prob. 7.10 or Prob. 7.11).

7.13. Write nodal admittance equations for the circuit of Fig. 7.18 disregarding all mutual coupling. Solve the resultant equations for the bus voltages by the method of gaussian elimination.

7.14. Prove Eq. (7.69) based on Eq. (7.68).

7.15. Using the gaussian-elimination calculations of Prob. 7.13, find the triangular factors of $\mathbf{Y}_{bus}$ for the circuit of Fig. 7.18.

7.16. Use the triangular factors obtained in Prob. 7.15 to calculate new bus voltages for Fig. 7.18 when the voltage source at bus ⑤ is changed to $1.0\angle -45°$ per unit. Follow the procedure of Example 7.9.

7.17. Using the triangular factors obtained in Example 7.9, find the voltage at bus ③ of the circuit of Fig. 7.11 when an *additional* current of $0.2\angle -120°$ per unit is injected at bus ②. All other conditions of Fig. 7.11 are unchanged.

7.18. (*a*) Kron reduce $\mathbf{Y}_{bus}$ of the circuit of Fig. 7.18 to reflect elimination of node ②.
(*b*) Use the $\mathbf{Y} - \Delta$ transformation of Table 1.2 to eliminate node ② from the circuit of Fig. 7.18 and find $\mathbf{Y}_{bus}$ for the resulting reduced network. Compare results of parts (*a*) and (*b*).

7.19. Find the **L** and **U** triangular factors of the symmetric matrix

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$$

Verify the result using Eq. (7.75).