Chapter I: Infinite Series, Power Series

1. The Geometric Series

A series is an infinite ordered set of terms combined together by the addition operator. The term ***infinite series*** is used to confirm the fact that series contain an infinite number of terms.

In the geometric series (progression) we multiply each term by some fixed number to get the next term. For example,

a) 2, 4, 8, 16, 32, ….

b) $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{18}{81}, \ldots$ (1.1)

c) a, ar, ar^2, ar^3, ….

are geometric progressions. Let us consider the following expression

$$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\ldots. \tag{1.2}$$

This expression is an example of an infinite series, and we are asked to find its sum. Not all infinite series have sums.

Let us find the sum of *n* terms in (1.2). The formula for the sum of *n* terms of the geometric progression (1.1 c) is

$$S_n=\frac{a\left(1-r^n\right)}{1-r} \tag{1.3}$$

Using (1.3) in (1.2), we find

$$S_n=\frac{2}{3}+\frac{4}{9}+\ldots+\left(\frac{2}{3}\right)^n=\frac{\frac{2}{3}\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}}=2\left[1-\left(\frac{2}{3}\right)^n\right] \tag{1.4}$$

as n increases, $\left(\frac{2}{3}\right)^n$ decreases and approaches zero. Then the sum of n terms approaches 2 as n increases, and we say that the sum of the series is 2.

Series such as (1.2) whose terms form a geometric progression are called geometric series. We can write a geometric series in the form

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \qquad (1.5)$$

the sum of the geometric series is by definition

$$S = \lim_{n\to\infty} S_n \qquad (1.6)$$

The geometric series has a sum if and only if $|r| < 1$, and in this case the sum is

$$S = \frac{a}{1-r} \qquad (1.7)$$

Prove:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}$$

where a is the first term of the series, and r is the common ratio. We can derive this formula as follows:

Let $$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \qquad (1.8)$$

Then $$rs = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n \qquad (1.9)$$

Subtracting equations (8) and (9), we find

$$s - rs = a - ar^n$$

so $$s\left(1-r\right) = a\left(1-r^n\right)$$

$$\therefore s = \frac{a\left(1-r^n\right)}{\left(1-r\right)}$$

Repeating decimals

A repeating decimal can be thought of as a geometric series whose common ratio is a power of $\left(1/10\right)$. For example:

$$0.3333... = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

Use equation (1.7) to find convert the decimal to a fraction:

$$0.3333... = \frac{a}{1-r} = \frac{\left(\frac{3}{10}\right)}{1-\left(\frac{1}{10}\right)} = \left(\frac{3}{10}\right)\times\left(\frac{10}{9}\right) = \frac{1}{3}$$

Problems

(1) prove that

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}$$

(2) find the fractions that are equivalent to the following repeating decimals:

(1) 0.5555…	(2) 0.818181…	(3) 0.583333…
(4) 0.61111…	(5) 0.7777…	(6) 0.185185…
(7) 0.243243…	(8) 0.26666…	(9) 0.123412341234…
(10) 0.99999…	(11)0.090909…	(12)0.14381438…

2. Definitions and Notation

There are many other infinite series besides geometric series. Here are some examples:

$$\text{(a)}\ 1^2 + 2^2 + 3^2 + 4^2 + \dots$$

$$\text{(b)}\ \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^2} + \frac{4}{2^4} + \dots \qquad (2.1)$$

$$\text{(c)}\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

In general, an ***infinite series*** means an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \dots + a_n + \dots$$

Where a_n (one for each positive integer *n*) are numbers or functions given by some formula or rule. The three dots mean that the series never ends.

$$\text{(a)}\ 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 + \dots$$

$$\text{(b)}\ \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots + \frac{n}{2^n} + \dots$$

$$\text{(c)}\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1} x^n}{n} + \dots \qquad (2.2)$$

$$\text{(d)}\ x - x^2 + \frac{x^3}{2} + \dots + \frac{(-1)^{n-1}\ x^n}{(n-1)!} + \dots$$

We can write the series in a shorter abbreviated form using a summation sign Σ followed by the formula for the nth term. For example (2.1a) would be written

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots = \sum_{n=1}^{\infty} n^2$$

The series (2.1d) would be written

$$\mathbf{x - x^2 + \frac{x^3}{2} - \frac{x^3}{6} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\ x^n}{(n-1)!}}$$

Problems

1. Write out several terms of the following series:

(a) $\sum_{n=1}^{\infty} \frac{n}{2^2}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (c) $\sum_{n=1}^{\infty} \frac{n}{n+5}$

(d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ (e) $\sum_{n=1}^{\infty} \frac{2n(2n+1)}{3n+5}$ (f) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

2. Write the following series in the abbreviated Σ form.

(a) $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \ldots$

(b) $\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \ldots$

(c) $\frac{1}{7} + \frac{2}{9} + \frac{3}{11} + \frac{4}{13} + \ldots$

(d) $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \ldots$

(e) $\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \ldots$

(f) $\frac{1}{3} + \frac{2}{5} + \frac{4}{7} + \frac{8}{9} + \frac{16}{11} + \ldots$

3. Convergent and Divergent Series

We have been talking about series which have a finite sum. We have also seen that there are series which do not have finite sums, for example (2.1 a).

$$1^2 + 2^2 + 3^2 + 4^2 + \dots$$

If a series has a finite sum, it is called ***convergent***. Otherwise it is called ***divergent***.

If we have the series a_n

$$a_1 + a_2 + a_3 + a_4 + \dots + a_n + \dots$$

Now consider the sums S_n that we obtain by adding more and more terms of the series. We define

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

Each S_n is called a ***partial sum***; it is the sum of the first *n* terms of the series. as n increases, the partial sums may increase without any limit as in the series (2.1a). They may oscillate as in the series $1 - 2 + 3 - 4 + 5 - \dots$ (which has partial sums 1, -1, 2, -2, 3,…) or they may have some more complicated behavior.

One possibility is that the S_n's may, after a while, not change very much any more; the a_n's become very small, and the S_n's come closer and closer to some value S.

$$\lim_{n\to\infty} S_n = S \qquad (3.1)$$

It is understood that S is a finite number. If this happens, we make the following definitions:

(1) If the partial sums S_n of an infinite series tend to a limit S, the series is called ***convergent***. Otherwise it is called ***divergent***.

(2) The limiting value S is called the sum of the series.

(3) The difference $R_n = S-S_n$ is called the remainder. From equation (3.1), we see that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(S-S_n) = S-S = 0$$

4. Testing Series for Convergence; The Preliminary Test

Preliminary test.

If $\lim_{n\to\infty} a_n \neq 0$, the series is ***divergent***.

If $\lim_{n\to\infty} a_n = 0$, we must test further.

This is not a test for convergence; the preliminary test can never tell you that a series converges. For example, the harmonic series

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

The nth term certainly tends to zero, but we shall soon show that the series $\sum_{n=1}^{\infty}\left(1/n\right)$ is divergent. On the other hand, in the series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots+\sum_{n=1}^{\infty}\frac{n}{n+1}$$

The terms are tending to 1, so by the preliminary test, this series diverges and no further testing is needed.

Problems:

Use the preliminary test to decide whether the following series are divergent or require further testing.

(1) $\frac{1}{2}-\frac{4}{5}+\frac{9}{10}-\frac{16}{17}+\frac{25}{26}-\frac{36}{37}+\ldots$

(2) $\sqrt{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{4}}{3}+\frac{\sqrt{5}}{4}+\frac{\sqrt{6}}{5}+\ldots$

(3) $\sum_{n=1}^{\infty}\frac{n+3}{n^2+10n}$

(4) $\sum_{n=1}^{\infty}\frac{(-1)^2 n^2}{(n+1)^2}$

(5) $\sum_{n=1}^{\infty}\frac{n!}{n!+1}$

(6) $\sum_{n=1}^{\infty}\frac{n!}{(n+1)!}$

(7) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{\sqrt{n^3+1}}$

(8) $\sum_{n=1}^{\infty}\frac{\ln n}{n}$

(9) $\sum_{n=1}^{\infty}\frac{3^n}{2^n+3^n}$

(10) $\sum_{n=1}^{\infty}\left(1-\frac{1}{n^2}\right)$

5. Tests for Convergence of Series of Positive Terms; Absolute Convergence

A. The Comparison test:

The terms of the sequence a_n are compared to those of another sequence b_n. if,

For all n, $0 \le a_n \le b_n$, and $\sum_{n=1}^{\infty} b_n$ converges, then so $\sum_{n=1}^{\infty} a_n$ converges.

For all n, $a_n \ge b_n$, and $\sum_{n=1}^{\infty} b_n$ diverges, then so $\sum_{n=1}^{\infty} a_n$ diverges.

Example. Test $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$ for convergence.

As a comparison series, we choose the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Notice that we do not care about the first few terms in a series, because they can affect the sum of the series but not whether it converges.

In our example, the terms of $\sum_{n=1}^{\infty}\left(1/n!\right)$ are smaller than the corresponding terms of $\sum_{n=1}^{\infty}\left(1/2^n\right)$ for all $n>3$. We know that the geometric series converges because its ratio is $\left(\frac{1}{2}\right)$. Therefore $\sum_{n=1}^{\infty}\left(1/n!\right)$ converges also.

Problems

1. Show that $n! > 2^n$ for all $n > 3$.

2. Use the comparison test to prove the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{1}{n\,2^n}\right)$, and $\sum_{n=1}^{\infty}\left(\frac{1}{2^n+3^n}\right)$

B. The Integral test:

We can use this test when the terms of the series are positive and not increasing, that is, when $a_{n+1} \le a_n$. This test can still be used even if the condition $a_{n+1} \le a_n$ does not hold for a finite number of terms.

The ***integral test*** states that:

If $\int^{\infty} a_n\, dn$ is ***finite*** $\rightarrow \sum^{\infty} a_n$ is convergent series

If $\int^{\infty} a_n\, dn$ is ***infinite*** $\rightarrow \sum^{\infty} a_n$ is divergent series

Example. Test for convergence the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

Using the integral test, we evaluate

$$\int^{\infty} \frac{1}{n}\, dn = [\ln\, n]^{\infty} = \infty$$

Since the integral is infinite, the series diverges.

Problems

Use the integral test to find whether the following series converge or diverge.

(1) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$, where $\int^{\infty} \frac{\ln n}{n} dn = \frac{1}{2}(\ln n)^2$

(2) $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$, where $\int^{\infty} \frac{n\, dn}{n^2+4} = \frac{1}{2} \ln(n^2+4)$

(3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$, where $\int^{\infty} \frac{dn}{\sqrt{n+1}} = 2\sqrt{n+1}$

C. Ratio Test

The integral test depends on your being able to integrate $a_n\, dn$; this is not always easy. We consider another test which will handle many cases in which we cannot evaluate the integral.

Assume that for all n, $a_n > 0$. Suppose that there exists *r* such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r \qquad (5.1)$$

If $r < 1$, then the series ***converges***. If $r > 1$, then the series ***diverges***. If $r = 1$, the ratio is inconclusive, and the series may converge or diverge.

Example 1: Test for convergence the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} + \ldots$$

Using equation (5.1), we find

$$\left(\frac{1}{(n+1)!}\right)\div\left(\frac{1}{n!}\right)=\left(\frac{n!}{(n+1)!}\right)=\frac{n(n-1)\ ...3.2.1}{(n+1)(n)(n-1)...3.2.1}=\frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0$$

So, the series converges.

Example 2: Test for convergence the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}+...$$

We find

$$\left(\frac{1}{n+1}\right)\div\left(\frac{1}{n}\right)=\left(\frac{n}{n+1}\right)$$

$$\therefore \lim_{n\to\infty}\left(\frac{n}{n+1}\right)=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=1$$

Here the test tells us nothing and we must use some different test.

Problems:

Use the ratio test to find whether the following series converge or diverge:

(1) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$	(2) $\sum_{n=0}^{\infty}\frac{3^n}{2^{2n}}$	(3) $\sum_{n=0}^{\infty}\frac{n!}{(2n)!}$
(4) $\sum_{n=0}^{\infty}\frac{5^n\,(n!)^2}{(2n)!}$	(5) $\sum_{n=1}^{\infty}\frac{10^n}{(n!)^2}$	(6) $\sum_{n=1}^{\infty}\frac{n!}{100^n}$
(7) $\sum_{n=0}^{\infty}\frac{3^{2n}}{2^{3n}}$	(8) $\sum_{n=0}^{\infty}\frac{e^n}{\sqrt{n!}}$	(9) $\sum_{n=0}^{\infty}\frac{(n!)^3\,e^{3n}}{(3n)!}$
(10) $\sum_{n=0}^{\infty}\frac{100^n}{n^{200}}$	(11) $\sum_{n=0}^{\infty}\frac{n!(2n)!}{(3n)!}$	(12) $\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n!}$

D. A Special Comparison Test

This test has two parts: (a) a convergence test, and (b) a divergence test.

(a) If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and a_n / b_n tends to a finite limit, then $\sum_{n=1}^{\infty} a_n$ ***converges***.

(b) If $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms and $a_n \geq 0$ and a_n / d_n tends to a limit greater than 0 (or tends to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ ***diverges***.

Example 1. Test for convergence

$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$$

For large n, we find $2n^2 - 5n + 1$ is nearly $2n^2$ to quite high accuracy. Similarly, the denominator is nearly $4n^3$.

So we consider as a comparison series just

$$\sum_{n=3}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=3}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}\div\frac{1}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n^2\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{4-\frac{7}{n}+\frac{2}{n^2}}\right)$$

$$=\frac{\sqrt{2}}{4}$$

This is a finite limit, so the given series ***converges***.

Example 2. Test for convergence

$$\sum_{n=2}^{\infty}\frac{3^n-n^3}{n^5-5n^2}$$

Here we must decide which is the important term as $n\to\infty$; is it 3^n or n^3? We find out by comparing their logarithms since ln N and N increase or decrease together. We have $\ln 3^n = n\ln 3$, and $\ln n^3 = 3\ln n$. Now ln n is much smaller than n, so for large n we have $n\,\ln 3 > 3\,\ln n$, and $3^n > n^3$. Thus the comparison series is $\sum_{n=2}^{\infty}\left(\frac{3^n}{n^5}\right)$. It is clear that this series is divergent.

Now by test (b)

$$\lim_{n\to\infty}\left(\frac{3^n-n^3}{n^5-5n^2}\div\frac{3^n}{n^5}\right)=\lim_{n\to\infty}\frac{1-\frac{n^3}{3^n}}{1-\frac{5}{n^3}}=1$$

Which is greater than zero, so the given series ***diverges***.

Problems

Use the special comparison test to find whether the following series converge or diverge.

(1) $\sum_{n=9}^{\infty}\frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$ (2) $\sum_{n=0}^{\infty}\frac{n(n+1)}{(n+2)^2(n+3)}$ (3) $\sum_{n=5}^{\infty}\frac{1}{2^n-n^2}$

(4) $\sum_{n=1}^{\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-3}$ (5) $\sum_{n=3}^{\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}$ (6) $\sum_{n=1}^{\infty}\frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}$

6. Alternating Series

An ***alternating series*** is an infinite series of the form

$$\sum_{n=0}^{\infty}(-1)^{n}\,a_{n} \qquad (6.1)$$

with $a_n \geq 0$ (or $a_n \leq 0$) for all n. Its terms alternate between positive and negative. Like any series, an alternating series converges if and only if the associated sequence of partial sums ***converges***.

Alternating series test

We ask two questions about an alternating series. Does it converge? Does it converge absolutely (that is, when we make all signs positive)? Let us consider the second question first. The series of absolute values

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

is the harmonic series, which diverges. We say the series (6.1) is ***not absolutely convergent***. Next we must ask whether (6.1) converges as it stands. If it had turned out to be absolutely convergent, we would not have to ask this question since an absolutely convergent series is also convergent. However, a series which is not absolutely convergent may converge or it may diverge. For alternating series the test is very simple:

<u>Test for alternating series</u>:-

An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $\left|a_{n+1}\right| \leq \left|a_{n}\right|$ and $\lim_{n\to\infty} a_n = 0$

In our example $\frac{1}{n+1} \leq \frac{1}{n}$, and $\lim_{n\to\infty}\frac{1}{n}=0$, so series (6.1) ***converges***.

The series which is ***convergent absolutely***, so the alternating series is convergent

The series which is ***convergent conditionally***, it's not absolute convergent and the alternating series is convergent.

Problems

Test the following series for **convergence**.

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (2) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ (3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ (4) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

(5) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ (6) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+5}$ (7) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{1+n^2}$ (8) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{10 n}}{n+2}$

7. Useful facts about series

We state the following facts:

1. The convergence or divergence of a series is not affected by multiplying every term of the series by the same constant. Neither is it affected by changing a finite number of terms (for example, omitting the first few terms).

2. Two convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ may be added (or subtracted) term by term (a_n + b_n). The resulting series is convergent, and its sum is obtained by adding (subtracting) the sums of the two given series.

3. The terms of an absolutely convergent series may be rearranged in any order without affecting either the convergence or the sum.

8. Power Series; Interval of Convergence

Series with their general term given as $u_n(x)=a_n x^n$ are called ***power series***;

$$f(x)=a_0+a_1 x+a_2 x^2+\ldots=\sum_{n=0}^{\infty} a_n x^n$$

Where the coefficients a_n are independent of x. To use the ratio test we write

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right|=\left|\frac{a_{n+1}}{a_n}\right||x|$$

and find the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{R}$$

Hence the condition for the convergence of a power series is obtained as

$$|\ x\ |<R \Rightarrow -R<x<R$$

where R is called the radius of convergence. At the end points the ratio test fails; hence these points must be analyzed separately.

Example 1: For the power series

$$1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots+\frac{x^n}{n}+\ldots$$

We use the ratio test

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\left(\frac{x^{n+1}}{n+1}\right)\div\left(\frac{x^n}{n}\right)\right|=\left|\frac{n}{n+1}\right||x|$$

$$\lim_{n\to\infty}\left|\frac{n}{n+1}\right|=1$$

$$R=\lim_{n\to\infty}\left|\frac{n+1}{n}\right|=\lim_{n\to\infty}\left|1+\frac{1}{n}\right|=1$$

So the radius of convergence R is 1; thus the series converges in the interval $-1 < x < 1$. On the other hand, at the end point x=1 it is divergent, while at the other end point, x=-1, it is convergent. So the interval of convergence is $-1 \le x < 1$

Example 2: For the power series

$$1 + x + 2!x^2 + 3!x^3 + \dots + n!x^n + \dots$$

The ratio

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} = (n+1)$$

gives

$$\lim_{n\to\infty} (n+1) = \frac{1}{R} \to \infty$$

Thus the radius of convergence is zero. Note that this series converges only for x=0.

Example 3: For the power series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

we find

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{(n+1)}$$

and

$$\lim_{n\to\infty} \frac{1}{n+1} = \frac{1}{R} \to 0$$

Hence the radius of convergence is infinity. This series converges for all x values.

Example 4: For power series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^{n+1}\,x^{2n-1}}{(2n-1)!}+\ldots$$

The ratio

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{(2n+1)!}\div\frac{x^{2n-1}}{(2n-1)!}\right|=\left|\frac{x^2}{(2n+1)(2n)}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n)}\right|=0$$

So the radius of convergence is infinity. This series converges for all x values.

Example 5: For the power series

$$1+\frac{(x+2)}{\sqrt{2}}+\frac{(x+2)^2}{\sqrt{3}}+\ldots+\frac{(x+2)^n}{\sqrt{n+1}}+\ldots$$

The ratio

$$\frac{u_{n+1}}{u_n}=\left|\frac{(x+2)^{n+1}}{\sqrt{n+2}}\div\frac{(x+2)^n}{\sqrt{n+1}}\right|=\left|(x+2)\,\frac{\sqrt{n+1}}{\sqrt{n+2}}\right|$$

and

$$\lim_{n\to\infty}\left[(x+2)\frac{\sqrt{n+1}}{\sqrt{n+2}}\right]=|x+2|$$

The series converges for $|x+2|<1$; that is, for $-1<x+2<1$, or $-3<x<-1$.

If **x = -3**, the series is

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$$

This is convergent by the alternating series test.

For **x = -1**, the series is

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$$

This is divergent by the integral test. Thus the series converges for $-3\leq x<-1$

Problems

Find the interval of convergence of each of the following power series:

(1) $\sum_{n=0}^{\infty}(-1)^n x^n$ (2) $\sum_{n=0}^{\infty}\frac{(2x)^n}{3^n}$ (3) $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n(n+1)}$

(4) $\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n n^2}$ (5) $\sum_{n=1}^{\infty}\frac{x^{3n}}{n}$ (6) $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x}{5}\right)^n$

(7) $\sum_{n=1}^{\infty}\frac{n(-x)^n}{n^2+1}$ (8) $\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n}$ (9) $\sum_{n=1}^{\infty}n(-2x)^n$

9. Theorems about Power Series

- A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series within the same interval of convergence as the original series (that is, not necessarily at the endpoints of the interval).

- Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence. You may divide two series if the denominator series is not zero at x=0. The resulting series will have some interval of convergence.

- One series may be substituted in another provided that the values of the substituted series are in the interval of convergence of the other series.

- The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

1. Addition and subtraction

When two functions *f* and *g* are decomposes into power series around the same center c, the power series of the sum or difference of the functions can be obtained by termwise addition and subtraction. That is, if

$$f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n$$

$$g(x)=\sum_{n=0}^{\infty} b_n (x-c)^n$$

Then

$$f(x)\pm g(x)=\sum_{n=0}^{\infty} (a_n \pm b_n)(x-c)^n$$

2. Differentiation and Integration

Once a function is given as a power series, it is differentiable on the interior of the domain of convergence. It can be differentiated and integrated quite easily, by treating every term separately:

$$f'(x)=\sum_{n=1}^{\infty} a_n\, n\,(x-c)^{n-1}=\sum_{n=0}^{\infty} a_{n+1}\,(n+1)(x-c)^{n}$$

$$\int f(x)dx=\sum_{n=0}^{\infty}\frac{a_n\,(x-c)^{n+1}}{(n+1)}+k=\sum_{n=1}^{\infty}\frac{a_{n-1}\,(x-c)^{n}}{n}+k$$

Both of these series have the same radius of convergence as the original one.

10. Expanding Functions in Power Series

It is useful to find power series that represent given functions. We illustrate one method of obtaining such series by finding the series for $\sin x$. In this method we assume that there is such a series and set out to find what the coefficients in the series must be. Thus we write

$$\sin x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \qquad (10.1)$$

and try to find numerical values of the coefficients a_n to make (10.1) an identity (within the interval of convergence of the series). Since the interval of convergence of a power series contains the origin, (10.1) must hold when x=0. If we substitute x=0 into (10.1), we get $a_0 = 0$. Then to make (10.1) valid at x=0, we must have $a_0 = 0$.

Next we differentiate (10.1) term by term to get

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \qquad (10.2)$$

Again putting x=0, we get $a_1 = 1$. We differentiate again, and put x=0 to get

$$-\sin x = 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + \ldots \qquad (10.3)$$

$$0 = 2a_2$$

Continuing the process of taking successive derivatives of (10.1) and putting x=0, we get

$$-\cos x = (3)(2)a_3 + (4)(3)(2)a_4 x + \ldots \qquad (10.4)$$

$$-1 = 3!a_3 \Rightarrow a_3 = \frac{-1}{3!}$$

$$\sin x = (4)(3)(2)a_4 + (5)(4)(3)(2)a_5 x + \ldots$$

$$0 = a_4$$

$$\cos x = (5)(4)(3)(2)a_5 + (6)(5)(4)(3)(2)a_6 x + \ldots$$

$$1 = 5!a_5 \Rightarrow a_5 = \frac{1}{5!}$$

We substitute these values back into (10.1) and get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots \qquad (10.5)$$

The $\sin x$ series converges for all x. series obtained in this way are called ***Maclaurin series*** or ***Taylor series*** about the origin.

Taylor series means a series of powers of (x-a), where *a* is constant. It is found by writing (x-a) instead of x on the right hand side of equation (10.1), differentiating just as we have done, but substituting $x = a$ instead of $x = 0$ at each step.

We assume that there is a Taylor series for $f(x)$, and write

$$f(x)=a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+...+a_n(x-a)^n+...$$

$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+...+na_n(x-a)^{n-1}+...$$

$$f''(x)=2a_2+(3)(2)a_3(x-a)+(4)(3)a_4(x-a)^2\ ..+(n)(n-1)a_n(x-a)^{n-2}+...$$

$$f'''(x)=(3)(2)a_3+(4)(3)(2)a_4(x-a)..+(n)(n-1)(n-2)a_n(x-a)^{n-3}+...$$

$$f'''(x)=3!a_3+(4)(3)(2)a_4(x-a)..+(n)(n-1)(n-2)a_n(x-a)^{n-3}+...$$

$$f^{(n)}(x)=n(n-1)(n-2)...1\ a_n+\text{ terms containing powers of (x-a)} \qquad (10.6)$$

The symbol $f^{(n)}(x)$ means the nth derivative of f(x). we now put x=a in each equation and obtain

$$\begin{aligned} &f(a)=a_0\ ,\ f'(a)=a_1\ ,\ f''(a)=2a_2\ ,\ f'''(a)=3!a_3, \\ &f''''(a)=4!a_4\ ,\ .\ \ ...,\ f^{(n)}(a)=n!a_n \end{aligned} \qquad (10.7)$$

We can then write the Taylor series for f(x) about x = a:

$$f(x)=f(a)+(x-a)f'(a)+\frac{1}{2!}(x-a)^2f''(a)+\frac{1}{3!}(x-a)^3f'''(a)+...+\frac{1}{n!}(x-a)^nf^{(n)}(a)+...$$

(10.8)

The ***Maclaurin series*** for f(x) is the Taylor series about the origin. Putting a = 0 in (10.8), we obtain the Maclaurin series for f(x):

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...+\frac{x^n}{n!}f^{(n)}(0)+... \quad (10.9)$$

11. Techniques for Obtaining Power Series Expansions

There are often simpler ways for finding the power series of a function than the successive differentiation process. Theorem 4 in section 10 tells us that for a given function there is just one power series, that is, series of the form $\sum_{n=0}^{\infty} a_n x^n$.

(11.1) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ Convergent for all x

(11.2) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ Convergent for all x

(11.3) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ Convergent for all x

(11.4) $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$ Convergent for $-1 < x \leq 1$

(11.5) $$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots$$

Binomial series; p is any real number, positive or negative

Convergent for $|x| < 1$

We give examples of various useful methods of obtaining series expansions.

A. Multiplication of a Series by a Polynomial or by another series

Example 1: To find the series for $(x+1)\sin x$

we multiply (x+1) times the series (11.1) and collect terms,

$$(x+1)\sin x = (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

$$(x+1)\sin x \ = x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \frac{x^5}{5!} + \frac{x^6}{5!} - \ldots$$

You can see that this is easier to do than taking the successive derivatives of the product $(x+1)\sin x$, and Theorem 4 assures us that the result are the same.

Example 2: To find the series for $e^x \cos x$, we multiply (11.2) by (11.3):

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$$

$$e^x \cos x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!2!} - \ldots + \frac{x^4}{4!} + \ldots$$

$$e^x \cos x = 1 + x + 0x^2 - \frac{x^3}{3} - \frac{x^4}{6} \ldots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} \ldots$$

B. Division of Two Series or of a series by a Polynomial

Example 1: To find the series for $\left(\frac{1}{x}\right)\ln(1+x)$ we divide (11.4) by x:

$$\frac{1}{x}\ln(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots$$

Example 2: To find the series for **tan x**, we divide the series for sin x by the series for cos x by long division:

$$\begin{array}{r|l} & x+\frac{x^3}{3}+\frac{2x^5}{15}\dots \\ \hline 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots & x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots \\ & \underline{x-\frac{x^3}{2!}+\frac{x^5}{4!}} \\ & \quad \frac{x^3}{3}-\frac{x^5}{30}\dots \\ & \quad \underline{\frac{x^3}{3}-\frac{x^5}{6}\dots} \\ & \qquad \frac{2x^5}{15}\dots \text{ etc} \end{array}$$

Example 3: To find the series for $(1/1+x)$ we do the long division:

$$\begin{array}{r|l} & 1-x+x^2-x^3\dots \\ \hline 1+x & 1 \\ & \underline{1+x} \\ & \quad -x \\ & \quad \underline{-x-x^2} \\ & \qquad x^2 \\ & \qquad \underline{x^2+x^3} \\ & \qquad\quad -x^3 \end{array}$$

C. Binomial Series

Series (11.5) looks like the beginning of the binomial theorem for the expansion $(a+b)^n$ if we put a=1, b=x, and n=p. The difference here is that we allow p to be negative or fractional, and in these cases the expansion is an infinite series. The series converges for $|x|<1$ as you can verify by the ratio test.

Example 1. We again find the series of $(1/1+x)$ by using the binomial series (11.5):

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots$$
$$= 1 - x + x^2 - x^3 + \ldots$$

D. Substitution of a Polynomial or a series for the variable in another series

Example 1: Find the series for e^{-x^2}. Since we know the series (11.3) for e^x, we simply replace the x there by $-x^2$ to get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + ..$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$$

Example 2: Find the series for $\sin x^2$. We must replace the **x** in (11.1) by $\mathbf{x^2}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

E. Combination of Methods

Example 1: Find the series for $\arctan x$, where $\arctan x = \tan^{-1} x$. Since

$$\int_0^x \frac{dt}{1+t^2} = \left[\arctan t\right]_0^x = \arctan x$$

We first write out (as a binomial series) $\left(1+t^2\right)^{-1}$ and then integrate term by term:

$$\left(1+t^2\right)^{-1} = 1 - t^2 + t^4 - t^6 + \ldots$$

$$\int_0^x \frac{dt}{1+t^2} = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right]_0^x$$

Thus, we have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

F. Taylor Series using the Basic Maclaurin series

Example 1: Find the first terms of the Taylor series for $\ln x$ about x=1. (This means a series of powers of (x-1) rather than powers of x)

$$\ln x = \ln\left(1 + (x-1)\right)$$

and use (11.4) with replaced by (x-1):

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\ln x = \ln\left(1+(x-1)\right) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$$

Example 2: Expand $\cos x$ about $x=(3\pi/2)$. We write

$$\cos x=\cos\left[\frac{3\pi}{2}+\left(x-\frac{3\pi}{2}\right)\right]=\sin\left(x-\frac{3\pi}{2}\right)$$
$$=\left(x-\frac{3\pi}{2}\right)-\frac{1}{3!}\left(x-\frac{3\pi}{2}\right)^3+\frac{1}{5!}\left(x-\frac{3\pi}{2}\right)^5-\ldots$$

Problems:

Find few terms of the ***Maclaurin series*** for each of the following functions:

(1) $x^2\ln(1-x)$

(2) $e^x\sin x$

(3) $\tan^2 x$

(4) $x\sqrt{1+x}$

(5) $\frac{1}{x}\sin x$

(6) $\frac{e^x}{1-x}$

(7) $\frac{1}{1+x+x^2}$

(8) $\sec x=\frac{1}{\cos x}$

(9) $\frac{1}{\sqrt{1-x^2}}$

(10) $\cos x^2$

(11) $e^{\sin x}$

(12) $\cosh x=\frac{e^x+e^{-x}}{2}$

(13) $\frac{x}{\sin x}$

(14) $\int_0^x\frac{\sin t\,dt}{t}$

(15) $\arcsin x=\int_0^x\frac{dt}{\sqrt{1-t^2}}$

12. Some Uses of Series

A. Numerical Computation: Let us do some numerical problems to illustrate computation using series.

Example 1: Evaluate $\ln\sqrt{\frac{(1+x)}{(1-x)}} - \tan x$ at x=0.0015.

First, we find $\ln\sqrt{\frac{1+x}{1-x}} = \int_0^x \frac{dt}{1-t^2}$

$$\int_0^x \frac{dt}{1-t^2} = \int_0^x (1-t^2)^{-1}\,dt = \int_0^x \left[1+(-1)(-t^2)+\frac{(-1)(-2)}{2!}(-t^2)^2+\frac{(-1)(-2)(-3)}{3!}(-t^2)^3+\ldots\right]dt$$

$$\ln\sqrt{\frac{1+x}{1-x}} = \left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right)$$

Thus

$$\left[\ln\sqrt{\frac{1+x}{1-x}} - \tan x\right]_{x=0.0015} = \left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right)-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots\right)$$

$$=\left[\frac{x^5}{15}+\frac{4x^7}{45}+\ldots\right]_{x=0.0015} = 5.06\times 10^{-16}$$

Example 2: Evaluate

$$\left[\frac{d^4}{dx^4}\left(\frac{1}{x}\sin x^2\right)\right]_{x=0.1}$$

First we find the four derivatives and then compute the result. However, it is easier to write out the series for $\sin x^2$, divide by x, and then differentiate:

$$\frac{1}{x}\sin x^2 = \frac{1}{x}\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\ldots\right) = x-\frac{x^5}{3!}+\frac{x^9}{5!}-\ldots$$

We differentiate this four times and evaluate at x=0.1:

$$f' = 1 - \frac{5x^4}{3!} + \frac{9x^8}{5!} - ...$$

$$f'' = -\frac{(5)(4)\,x^3}{3!} + \frac{(9)(8)\,x^7}{5!} - ....$$

$$f''' = -\frac{(5)(4)(3)\,x^2}{3!} + \frac{(9)(8)(7)\,x^6}{5!} - ....$$

$$f'''' = -\frac{(5)(4)(3)(2)\,x}{3!} + \frac{(9)(8)(7)(6)\,x^5}{5!} - ....$$

$$= -2 + 0.000252 - ...$$

B. Summing Series: If you can recognize a numerical series as the series of some function for a particular value of x, then you can find the sum of the series.

Example 1: Find the sum of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Start with the series (11.4),

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

and put x=1. Then we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Example 2: Use the series you know to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$$

Start with the series of **arc tanx**,

$$\arc\tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

and put x=1. Then we get

$$\arc\tan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Example 3: Use series you know to show that:

$$\ln 3 + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \dots = 2$$

We start with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and

$$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Then we put x=ln 3, we find

$$e^{\ln 3} - 1 = \ln 3 + \frac{(\ln 3)^2}{2!} + \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} + \dots = 2$$

Example 4: Use the series you know to show that

$$\frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \dots = 1$$

We begin with the series of sin x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

And put x=π, we find

$$\sin \pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots$$

Divide the previous eq. by π, we get

$$\frac{\sin \pi}{\pi} = 1 - \frac{\pi^2}{3!} + \frac{\pi^4}{5!} - \frac{\pi^6}{7!} + \ldots$$

and

$$1 - \frac{\sin \pi}{\pi} = \frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \ldots = 1$$

C. Evaluation of Definite Integrals:

Example 1: The Fresnel integrals (integrals of $\sin x^2$ and $\cos x^2$) occur in the problem of Fresnel diffraction in optics. We find

$$\int_0^1 \sin x^2\, dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7.\,3!} + \frac{x^{11}}{11.\,5!} - \frac{x^{15}}{15.\,7!} + \ldots \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{7.\,3!} + \frac{1}{11.5!} - \frac{1}{15.\,7!} + \ldots$$

$$= 0.33333 - 0.02381 + 0.00076 - 0.000013 + \ldots$$

$$= 0.31027$$

Example 2: Find the integral

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1}\left(1+(-x^2)+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\frac{(-x^2)^4}{4!}+\ldots\right)dx$$

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1}\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}+\ldots\right)dx$$

$$\int_0^{0.1} e^{-x^2}\,dx = \left[x-\frac{x^3}{3}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\frac{x^9}{9.4!}+..\right]_0^{0.1} = \left[0.1-\frac{0.001}{3}+\frac{0.00001}{10}\right] = 0.1$$

D. Evaluation of Indeterminate forms

Suppose we want to find

$$\lim_{x\to 0}\frac{1-e^x}{x}$$

If we try to substitute x=0, we get $(0/0)$. Expressions that lead us to such meaningless results when we substitute are called indeterminate forms. Many times they can be evaluated by using series. For example

$$\lim_{x\to 0}\frac{1-e^x}{x} = \lim_{x\to 0}\frac{1-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)}{x}$$

$$= \lim_{x\to 0}\left(\frac{-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\ldots}{x}\right)$$

$$= \lim_{x\to 0}\left(-1-\frac{x}{2!}-\frac{x^2}{3!}-\ldots\right) = -1$$

Example 1: Use Maclaurin series to evaluate the limits

$$\lim_{x\to 0}\frac{\sin x - x}{x^3}$$

First, we write the series

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) - x}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}{x^3}$$

$$= \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \ldots\right)$$

$$= -\frac{1}{6} = -0.166666$$

Example 2: Find the limits

$$\lim_{x\to 0}\frac{\ln(1-x)}{x}$$

First, we write the series

$$\lim_{x\to 0}\frac{\ln(1-x)}{x} = \lim_{x\to 0}\frac{\left((-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \ldots\right)}{x}$$

$$= \lim_{x\to 0}\frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)}{x}$$

$$= \lim_{x\to 0}\left(-1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)$$

$$= -1$$