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ON SATURATED FUSION SYSTEMS AND BRAUER INDECOMPOSABLE MODULES

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ملخص رسالة بعنوان

أنظمة الاندماج المشبعة و حلقيات براور الغير قابلة للتحليل

إعداد طالبة الماجستير

هدايه طارق كاتب

هذه الرسالة تناولت عرض نوعين مهمين من انواع الحلقيات في الجبر وهي: حلقيات براور الغير قابلة للتحليل وحلقيات سكوت. ايضاً تناولت عرض نظام مميز على الزمر المتهية وهو نظام الاندماج المشبع حيث ان لهذا النظام وحلقيات براور الغير قابلة للتحليل علاقة بواسطة حلقيات سكوت. ثم قدمت اثبات للضرب التنسوري لحلقيات براور الغير قابلة للتحليل ولحلقيات سكوت ولأنظمة الاندماج المشبعة. تضمنت هذا الرسالة اربعة فصول وهيا كالتالي:

الفصل الاول: تعاريف اساسية استخدمت في هذه الرسالة.

الفصل الثاني: تعاريف ونتائج هامه لحلقيات براور الغير قابلة للتحليل وحلقيات سكوت.

الفصل الثالث: تعاريف ونتائج هامه لانظمة الاندماج وانظمة الاندماج المشبعة.

الفصل الرابع: نتائج هذه الرسالة واثباتاتها. ونتائج كالتالي:

النتيجة الاولى: الضرب التنسوري لاثنين من حلقيات براور الغير قابلة للتحليل هو حلقية براور غير قابلة للتحليل.

النتيجة الثانية: الضرب التنسوري لاثنين من حلقيات سكوت هو حلقية سكوت.

النتيجة الثالثة: الضرب التنسوري لنظامين اندماج هو نظام اندماج.

النتيجة الرابعة: الضرب التنسوري لنظامين اندماج مشبعة هو نظام اندماج مشبع.

Contents

Table 1: Notation we used in this work

G	A finite group.
p	A prime number.
K	A field of characteristic 0.
$\mathcal{O}$	A complete discrete valuation ring.
$\mathfrak{p}$	A maximal ideal of $\mathcal{O}$, with residue field $F = \mathcal{O}/\mathfrak{p}$.
F	An algebraically closed field of characteristic p, residue field of $\mathcal{O}$.
$\mathbb{Z}$	Integers.
$\mathbb{Z}_n$	Integers modulo n.
$\mathbb{N}$	Natural numbers = $\{\text{integers}\, n : n \geq 0\}$.
S_n	The symmetric group of degree n.
A_n	The alternating group of degree n.
V_4	The Klein's 4-group. $V_4 = \langle a, b \| a^2 = b^2 = (ab)^2 = 1 \rangle$.
D_{2n}	The dihedral group of order $2n$. $D_{2n} = \langle a, b \| a^n = b^2 = 1, aba = b \rangle$.
FG	A group algebra of a finite group G over a field F.
1_G	The identity element of a finite group G.
$a \equiv b \bmod n$	n divides $a - b$ where $a, b \in \mathbb{Z}$ with $n > 0$.
$\gcd(a, b)$	The greatest common divisor of integers a, b.
$\max(m, n)$	A maximal elements of m and n where $m, n \in \mathbb{Z}$.
$R[x_1, x_2, \cdots, x_n]$	The polynomial ring.
$\mathrm{Mat}_n(R)$	An algebra of $(n \times n)$-matrices with coefficients in a commutative ring R.
$R\Omega$	The permutation module of a finite group G over a ring R.
$\mathrm{Hom}_R(M, W)$	The set of all R-module homomorphism over a ring R.
$\mathrm{End}_F(V)$	The endomorphism algebra of a vector space V over a field F.
$\mathrm{Aut}_R(A)$	The automorphism group of a G-algebra A over a ring R.
$\mathrm{Iso}_G(Q, S)$	The set of all G-group isomorphism over a finite group G.
$\mathrm{Mor}_{\mathcal{F}}(Q, S)$	The morphism of a fusion system $\mathcal{F}$.
f_g	The conjugation map where g in a finite group G.
$\mathrm{Ker}(f)$	The kernel of a homomorphism f.
$\mathrm{Im}(f)$	The image of a homomorphism f.
$\psi\|_P$	The restriction of a mapping ψ.
$\|G\|$	The order of a finite group G.
G/H	The set of left cosets gH of a subgroup H in a finite group G where g in G.
$L \setminus G/H$	The set of double cosets LgH of subgroups L and H in a finite group G where g in G.
$[G/H]$	The set of representatives of left cosets gH in G where g in G .
$[L \setminus G/H]$	The set of representatives of double cosets LgH in G where g in G.

Table 2: Notation we used in this work

$[G : H]$	The index of a subgroup H in a finite group G.
$\sim_G$	The conjugate relation in a finite group G.
$\leq_G$	An order relation between subgroups of a finite group G.
$M_1 \cong M_2$	Modules M_1 and M_2 are isomorphic.
$G_1 \times G_2$	A cartesian product of sets G_1 and G_2 - A direct product of finite groups G_1 and G_2.
$M \oplus N$	An internal direct sum of modules M and N.
$\otimes_F$	A tensor product of F-modules.
H^g	A conjugate of a subgroup H of a finite group G by g in G.
$\mathrm{Cl}_G(H)$	The set of H-conjugacy classes of a finite group G.
$\widehat{C}$	The class sum of a conjugacy class C.
$\mathrm{C}_{FG}(H)$	The centralizer of a subgroup H of a finite group G in FG.
$\mathrm{C}_X(P)$	The set of fixed points of a basis X under a p-subgroup P of a finite group G.
$\mathrm{Stab}_G(x)$	The stabilizer of x in a finite group G.
$N_G(H)$	The normalizer of a subgroup H in a finite group G.
$Z(FG)$	The center of a group algebra FG.
$\mathrm{Inv}_H(M)$	A fixed elements of an FG-module M under the H-action.
M^H	A fixed elements of an FG-module M under the H-action.
$\mathrm{Ind}_H^G(W)$	The induction of an FH-module W from a subgroup H to a finite group G.
$\mathrm{Res}_H^G(M)$	The restriction of an FG-module M from a finite group G to a subgroup H of G.
Tr_H^G	The relative trace map from a subgroup H of a finite group G to G.
M_H^G	The image of the relative trace map of the module M.
$M(H)$	The Brauer quotient of module M with respect to a subgroup H of a finite group G.
Br_H^M	The Brauer homomorphism of module M.
$\mathrm{vx}(M)$	A vertex of an FG-module M.
$\mathrm{s}(M)$	A source of an FG-module M.
$\mathrm{Sc}(G, H)$	A Scott FG-module with respect to a subgroup H of a finite group G.
$s_P(M)$	The Scott coefficient of $\mathcal{O}G$-module M associated with a p-subgroup P of a finite group G.
$\mathrm{J}(A)$	The Jacobson radical of a G-algebra.
$U(A)$	A group of units of G-algebra A.
$I \trianglelefteq R$	An ideal I of a ring R.
$\dim_F(V)$	The dimension of the F-vector space V over a field F.
$\langle x \rangle$	A cyclic group generated by x.
$\mathcal{F}_P(G)$	The fusion system of a finite group G over a p-subgroup P of G.
$\mathrm{Ob}(\mathcal{F})$	The object of a fusion system $\mathcal{F}$.
$\mathrm{Syl}_p(H)$	The set of Sylow p-subgroups of H.
n_p	The number of Sylow p-subgroups of a finite group G.
$Q^{\mathcal{F}}$	The set of all $\mathcal{F}$-conjugate to a subgroup Q of a finite group G in $\mathcal{F}$.

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Abstract

This dissertation is about important types of modules in algebra, Brauer indecomposable module and Scott module and about system on finite group, saturated fusion system. More precisely, we will introduce important properties and examples about them. Then we study the tensor product of Brauer indecomposable module, Scott module and saturated fusion system.

Key-words

G-algebras, Brauer indecomposable modules, Scott modules, fusion systems, saturated fusion systems, tensor product.

Introduction

In this dissertation we shall assume that G is a finite group and p a prime number dividing the order of G. Let $\mathcal{O}$ be a complete discrete valuation ring with quotient field K of characteristic 0. We assume that the residue field $F = \mathcal{O}/\mathfrak{p}$ is an algebraically closed field which has characteristic p, where $\mathfrak{p}$ denotes the unique maximal ideal of $\mathcal{O}$. With this assumption we refer to the triple $(K, \mathcal{O}, F)$ as a p-modular system. We let R be $\mathcal{O}$ or F.

In (1956), Brauer introduced in case of the group algebra a surjective homomorphism between subalgebra of fixed points and Brauer quotient of algebra. This surjective homomorphism is called Brauer homomorphism. In (1968), Green introduced the notion of G-algebra. In (1980), the idea of defining such a homomorphism for an arbitrary G-algebra is due to Broué and Puig. The Brauer quotient of module is a factor of fixed points of module by the sum of the image of the relative trace map. The restriction of Brauer quotient is defined a special module called " Brauer indecomposable module." This module has an important relationship with saturated fusion system [24].

A Scott module is the unique summand of an induced module that contains the trivial module in its base: Scott proved that the trivial module is also in its top. This work was characteristically left unpublished by Scott; it was later rediscovered by Alperin who did not publish it either. The first mention of them in the literature was in (1982), by David Burry [6].

The beginning of fusion of finite groups is due to Burnside in the proof of normal p-complement theorem for a fixed prime number p. Burnside's book was published in (1911). Brauer's work in representation theory both ordinary and modular made the foundation for deep theory in this direction of research. Brauer's work spread from (1940) to (1970). Alperin contributed in this field for Sylow intersections and fusion in his paper in journal of algebra (1967). In modern language of this direction of research, Lluis Puig gave the unifying approach for G-fusion in a Sylow p-subgroup. In (1980), L.Puig created the notion of Frobenius category on a finite p-group. We note that L. Puig did not publish his work about fusion system for long time. Then in (2006), we have seen his first paper in this area of research. In (2009), we have seen L. Puig's book which contains deep theory and construction of fusion system. In fact, in the literature mathematicians call this science Puig theory [3] . The book of B. Kulshammer, which is published around (1991), contains the core work of Puig regarding the results about nilpotent blocks. The book by Thévenaz, which is published around (1996), contains comprehensive treatment of G-algebra and Puig theory. The book is very comprehensive treatment of modular representation theory. Then, we have seen in the book *Fusion Systems in Algebra and Topology*, by Michael Aschbacher, Radha Kessar, Bob Oliver, which is published around (2011). This book gives an excellent treatment of fusion systems and the relationship with topology. Also we see the book of David Caravan which contains some well organized theory in fusion system. There is a new book which is due to Marcus Linkelmann as well as many good paper of him. The title of that book is *The Block Theory of Finite Group Algebras*, Volume 1.

This dissertation is divided into four chapters. In Chapter one, we will introduce some notation and background materials that going to be used in the remainder of the research. In the first section, we introduce concepts of algebra, Jacobson radical. We give some examples about them. We introduce several equivalent condition to definition of local algebra. Then we introduce theorem about endomorphism module local algebra. In the second section of this chapter, we introduce definitions of a G -algebra and an interior G-algebra. We introduce the most important concepts about them. Then we show a relationship between G-algebra and interior G-algebra. In the third section, we introduce definition of G-conjugate relation and show this relation is equivalence relation. We introduce definition of conjugation map and show algebra structure about it. We introduce concept of automizer. We record some note about Sylow theorems. Then we introduce Burnside theorem.

In Chapter two, we introduce concepts of induced module and restriction module. Then we give us some properties about them. In the second section, we introduce concepts of a set of invariant elements and relative trace map on two algebra structures, modules structure and G-algebras structure. Then we gives some examples and properties about them. In the third section, We introduce concepts of Brauer quotient, Brauer homomorphism and important module of our research " Brauer indecomposable module". Then we give us some properties about them. In the fourth section, we introduce concepts of relative projective module, source module and vertex subgroup. Then we give us properties about them. In the fifth section, we introduce concept of a p-permutation module. We give us some examples and properties about it. In the last section, we introduce concept of an important module of our research "Scott module". We introduce concept of Scott coefficient. Then we give us some results and properties of Scott module and Scott coefficients.

In Chapter three, we introduce concepts of fused and control fused. Then we give some examples about them. We introduce concept of an important system in our research " fusion system". We give us types of subgroups in object of fusion system. Then we give us examples about them. We show some lemmas giving us the relationship between types of subgroups in object of fusion system. In the second section, we introduce concept of special case of fusion system " saturated fusion system" is important system in our research. Then we introduce Puig's theorem. We give us a relationship between subgroups of object in saturated fusion system. We introduce important theorem gives us equivalent condition to definition of saturated fusion system. Then we give us the relationship between Brauer indecomposable module and saturated fusion system.

In Chapter four, we study that the tensor product of three algebra structures. In the first section, we study that the tensor product of Brauer indecomposable modules. In the second section, we study the tensor product of Scott modules. Finally, we study that the tensor product of fusion systems. The main theorems in this Chapter:

- **Theorem:** Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let M_i be Brauer indecomposable FG_i-module with $i = 1, 2$. Then
$$M_1 \otimes_F M_2$$
is a Brauer indecomposable $FG_1 \otimes_F FG_2$-module.

- **Theorem:** Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let H_i be a subgroup of G_i, with $i = 1, 2$. Let W_i for $i = 1, 2$ be FH_i-module. Then
$$\mathrm{Ind}_{H_1}^{G_1}(W_1) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(W_2) \cong \mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(W_1 \otimes_F W_2).$$

- **Theorem:** Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let H_i be subgroup of G_i, with $i = 1, 2$. Let $Sc(G_i, H_i)$ for $i = 1, 2$ be a Scott FG_i-module. Then
$$Sc(G_1, H_1) \otimes_F Sc(G_2, H_2)$$
is a Scott $FG_1 \otimes_F FG_2$- module.

- **Theorem:** Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let $\mathcal{F}_i$ be the fusion system of G_i over P_i. Let $F\mathcal{F}_i$ for $i = 1, 2$ be the finite dimensional algebra over F which is associated with the fusion system $\mathcal{F}_{P_i}(G_i)$. Then
$$F[\mathcal{F}_1 \times \mathcal{F}_2] \cong F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$$
as an algebra isomorphism.

- **Theorem:** Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. If $\mathcal{F}_i$ is a saturated fusion system for $i = 1, 2$ then the fusion system $F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$ is a saturated fusion system.

We would like to say that these theorems are written in three papers:

- Tensor product of Brauer indecomposable modules.
- Tensor product of Scott modules.
- Tensor product of fusion systems.

Chapter 1

Algebras

We shall introduce in this chapter, concepts of algebra, Jacobson radical and local algebra. We give some examples about them. Then we introduce theorem tells us condition of local algebra on endomorphism module is important for become module an indecomposable module. In second section, we introduce concepts of G-algebra and interior G-algebra. Then we give some important concepts and examples about them. We introduce a relationship between G-algebra and interior G-algebra. In third section, we introduce concept of conjugate relation and algebra structure about it. We introduce concept of automizer and important property about it. We record some note about Sylow theorems. Then we introduce Burnside theorem.

1.1 Local algebras

Let p be a fixed prime number. Let G be a finite group. Let $(K, \mathcal{O}, F)$ be a p-modular system. Let R be $\mathcal{O}$ or F. We followed references [2], [4], [10], [12], [16], [17] and [20].

Definition 1.1.1. [10] An algebra over R is an R-module A with a ring structure, which must satisfy $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and all $a, b \in A$.

We list some examples are about of algebra. The main example is the first one.

Example 1.1.1.

(a) Let G be a finite group. The group algebra $FG = \{\sum \alpha_i g_i : \alpha_i \in F,\ g_i \in G\}$ is an algebra over F where F is the trivial FG-module.

(b) Every ring is an algebra over $\mathbb{Z}$.

(c) Every field is an algebra over itself.

(d) The polynomial ring $R\,[x_1, x_2, \cdots, x_n]$ is an algebra over R.

(e) The matrix ring $(\mathrm{Mat}_n(R), +, \cdot)$ is an algebra over R.

(f) Consider V as a vector space over F, then the endomorphism ring $(\mathrm{End}_F(V), +, \circ)$ is an algebra over F.

Definition 1.1.2. [16] The Jacobson radical $\mathrm{J}(A)$ of A is intersection of all maximal left ideals I of A

$$\mathrm{J}(A) = \cap_{I \triangleleft A} I.$$

The following example is about of Jacobson radical.

Example 1.1.2.

(a) The Jacobson radical of $\mathbb{Z}_8$ is $\mathrm{J}(\mathbb{Z}_8) = \langle 2 \rangle$.

(b) The Jacobson radical of $\mathbb{Z}_6$ is $\mathrm{J}(\mathbb{Z}_6) = \langle 2 \rangle \cap \langle 3 \rangle = \{0\}$.

The ideal $\mathrm{J}(A)$ is important to study the concept of local algebra.

Lemma 1.1.1. [14] Let A be a finite dimension algebra over F. Then the following condition are equivalent:

(i) A contains exactly two idempotents.

(ii) Every element in A is either nilpotent or unit.

(iii) $A/J(A) \cong F$ as field.

(iv) A is the disjoint union of $J(A)$ and $U(A)$.

Definition 1.1.3. An algebra A over a field F is called local algebra if it satisfies any condition of Lemma 1.1.1.

Example 1.1.3. All field (and skew field) are local algebras over itself.

Theorem 1.1.1. *[2] Let M be a finite dimensional F-module. Then M is indecomposable F-module if and only if every endomorphism algebra of M is either unit or nilpotent.*

Proof. ($\Leftarrow$) to prove by contradiction. Suppose that every endomorphism of M is either unit or nilpotent and M is not indecomposable F-module. So, suppose $M = N_1 \oplus N_2$ where N_1 and N_2 are non-zero F-submodules of M. Also suppose $i_1 : N_1 \longrightarrow M$ is an inclusion map defined by $i_1(x) = (x, 0), \forall x \in N_1$ and $\pi_1 : M \longrightarrow N_1$ is a projection map defined by $\pi_1(x, y) = x$, $\forall x \in N_1$, $\forall y \in N_2$. Also suppose $i_2 : N_2 \longrightarrow M$ is an inclusion map defined by $i_2(y) = (0, y), \forall y \in N_2$ and $\pi_2 : M \longrightarrow N_2$ is a projection map defined by $\pi_2(x, y) = y$, $\forall x \in N_1$, $\forall y \in N_2$. Now note that, $i_1\pi_1$ and $i_2\pi_2$ are both endomorphisms of M with $i_1\pi_1(x, y) = i_1(\pi_1(x, y)) = i_1(x) = (x, 0)$. So $(i_1\pi_1)^2(x, y) = i_1\pi_1(i_1\pi_1(x, y)) = i_1\pi_1(x, 0) = i_1(\pi_1(x, 0)) = i_1(x) = (x, 0)$ by induction $(i_1\pi_1)^n(x, y) = (x, 0)$, $\forall x \in N_1$. This means $(i_1\pi_1)^n \neq 0$, $\forall n \in \mathbb{N}$. Hence $i_1\pi_1$ is not nilpotent. By assumption $i_1\pi_1$ is unit. In the same way $i_2\pi_2$ is not nilpotent, so $i_2\pi_2$ is also unit by assumption. But

$$(i_1\pi_1)(i_2\pi_2) = 0 \tag{1.1}$$

since $(i_1\pi_1)(i_2\pi_2)(x, y) = i_1\pi_1(0, y) = (0, 0)$. Also since $i_1\pi_1$ is unit then $(i_1\pi_1)^{-1}$ exists. By multiply the equation (1.1) from left by $(i_1\pi_1)^{-1}$, we have $i_2\pi_2 = 0$. This is contradict that $i_2\pi_2$ is unit. Hence M is indecomposable F-module.

($\Rightarrow$) Suppose that M is a finite dimensional indecomposable F-module and $f \in \mathrm{End}_F(M)$. Now we have
$$M \supset \mathrm{Im}f \supset \mathrm{Im}f^2 \supset \cdots$$
and
$$M \supset \ker f \supset \ker f^2 \supset \cdots$$
are two descending chains of F-submodules of M. But since M is a finite dimensional then there exist positive integer n such that:

$$\mathrm{Im}f^n = \mathrm{Im}f^{n+1} = \mathrm{Im}f^{n+2} = \cdots$$

and there exist positive integer m such that:

$$\ker f^m = \ker f^{m+1} = \ker f^{m+2} = \cdots.$$

If we take $k = \max(m, n)$, then it follows that:

$$\mathrm{Im}f^k = \mathrm{Im}f^{2k} \text{ and } \ker f^k = \ker f^{2k}. \tag{1.2}$$

Now if $x \in \ker f^k \cap \mathrm{Im}f^k$, then $x \in \ker f^k$ and $x \in \mathrm{Im}f^k$. So $x = f^k(y)$, $y \in M$ and $f^k(x) = 0$. Thus $f^k(f^k(y)) = f^k(x) = 0$ and $f^{2k}(y) = 0$ then $y \in \ker f^{2k}$. But from (1.2) we have $y \in \ker f^k$. Thus $f^k(y) = 0$, then $x = 0$. Hence $\ker f^k \cap \mathrm{Im}f^k = \{0\}$. Now suppose $x \in M$ then $f^k(x) = f^{2k}(y)$ for some $y \in M$. Since f^k is a homomorphism, then $f^k(x - f^k(y)) = f^k(x) - f^{2k}(y) = 0$. Hence $x - f^k(y) \in \ker f^k$, so $x \in \ker f^k + f^k(y) \subseteq \ker f^k + \mathrm{Im}f^k$. Then $M \subseteq \ker f^k + \mathrm{Im}f^k$. Also since $\ker f^k + \mathrm{Im}f^k \subseteq M$. Then $M = \ker f^k + \mathrm{Im}f^k$, hence $M = \ker f^k \oplus \mathrm{Im}f^k$. But since M is indecomposable F-module then either $\mathrm{Im}f^k = \{0\}$ or $\ker f^k = \{0\}$. If $\mathrm{Im}f^k = \{0\}$, then $f^k = 0$. Hence f is nilpotent. If $\ker f^k = \{0\}$, then f is one to one. Since f is endomorphism thus f is onto. Hence f is unit. □

Corollary 1.1.1. *A finite dimension F-module is indecomposable if and only if the endomorphism algebra of it is local algebra.*

1.2 G-algebras

Throughout this section, G denotes a finite group and p a prime number. Let $(K, \mathcal{O}, F)$ be a p-modular system. Let R be $\mathcal{O}$ or F. We followed references [3], [8],

[13], [14], [23] and [24].

Definition 1.2.1. [3] Let A be an algebra over R. A G-algebra over R is a group homomorphism $\psi : G \longrightarrow \mathrm{Aut}(A)$ defined by $\psi(g)(a) = a^g$ for all $a \in A$ and all $g \in G$. We denotes a^g for the image of a under the map $\psi(g) \in \mathrm{Aut}(A)$, for all $r \in R$, a, $b \in A$ and all $g, h \in G$ we have:

(i) $a^{1_G} = a$, $\forall a \in A$.

(ii) $(a^g)^h = a^{gh}$.

(iii) $(a + b)^g = a^g + b^g$.

(iv) $(ab)^g = a^g \ \ b^g$.

(v) $(ra)^g = ra^g$.

Then we called A a G-algebra over R.

Remark 1.2.1.

The G-algebra structure corresponds to the conjugation action $a^g = g^{-1}ag$ by $g \in G$.

The following examples are about of G-algebra.

Example 1.2.1.

The group algebra FG has a G-algebra structure over F by inclusion map of G in FG with conjugation $t^g = g^{-1}tg$ where $g \in G$ and $t \in FG$. Also if N is a normal subgroup of G then FN is a G-algebra over F by restriction of the action G on FG to FN.

Example 1.2.2.

Consider M as an FG-module. The endomorphism algebra of M over F has a G-algebra structure over F satisfies $(\phi)^g(m) = \phi(mg^{-1})g$ for all $\phi \in \mathrm{End}_F(M)$, $g \in G$ and all $m \in M$.

Lemma 1.2.1. Let G be a finite group. Let H be a subgroup of G. If A is a G-algebra over R then A is an H-algebra over R.

Proof. Since A is a G-algebra over R thus there is a group homomorphism $\psi : G \longrightarrow \mathrm{Aut}(A)$. The restriction of ψ to $\bar{\psi} : H \longrightarrow \mathrm{Aut}(A)$ is also group homomorphism. Hence A is an H-algebra over R. □

Definition 1.2.2. Let G be a finite group. Let A be a G-algebra over R. The G-subalgebra of A is a subalgebra B of A satisfying $b^g \in B$ for $g \in G$ and $b \in B$.

Definition 1.2.3. Let G be a finite group. Let A be a G-algebra over R. The G-ideal of A is an ideal I of A satisfying $x^g \in I$ for $g \in G$ and $x \in I$.

The following lemma gives us the algebra structure for factor of G-algebra by G-ideal of G-algebra.

Lemma 1.2.2. Let G be a finite group. Let A be a G-algebra over R. Let I be a G-ideal of A. Then A/I is a G-algebra over R.

Proof. We will prove this map $\psi : G \longrightarrow \mathrm{Aut}(A/I)$ is define by $\psi(g)(a+I) = (a+I)^g = a^g + I$ is a group homomorphism for $g \in G$ and $a + I \in A/I$.
Suppose that $g_1, g_2 \in G$ and $a + I \in A/I$. From definition of ψ we have

$$\begin{array}{rcl} \psi(g_1g_2)(a+I) & = & (a+I)^{g_1g_2} \\ & = & (a^{g_1g_2}+I) \\ & = & ((a^{g_1})^{g_2}+I) \\ & = & (a^{g_1}+I)^{g_2} \\ & = & ((a+I)^{g_1})^{g_2} \\ & = & \psi(g_2)(a+I)^{g_1} \\ & = & \psi(g_2)\psi(g_1)(a+I) \end{array}$$

Hence ψ is a group homomorphism. Then A/I is a G-algebra over R. □

The following definition about the morphism that connect between G-algebras.

Definition 1.2.4. Let G be a finite group. Let A and B be two G-algebras over R. A homomorphism $\Psi : A \longrightarrow B$ is a homomorphism of G-algebras satisfying $(\Psi(a))^g = \Psi(a^g)$ for $g \in G$ and $a \in A$.

Now we will introduce definition of the spacial and important structure of G-algebra.

Definition 1.2.5. [3] Let G be a finite group. Let A be an algebra over R. An interior G-algebra over R is a group homomorphism $\varphi : G \longrightarrow U(A)$ satisfy $\varphi(g)a = g.a$ and $a\varphi(g) = a.g$ for all $a \in A$ and $g \in G$.

The following theorem gives us the relationship between G-algebra and interior G-algebra.

Theorem 1.2.1. *Every interior G-algebra A over R is a G-algebra over R by*

$$\psi(g)(a) = a^{\varphi(g)}, \quad \text{for } g \in G,\ a \in A.$$

Remark 1.2.2. The converse of the above theorem does not hold in general. For example, if we take $F = \mathbb{Z}_2$ is the field which has characteristic 2,
$G = V_4$ the Klein 4-group which has order 4 and
$H = \langle a \rangle$ is a normal subgroup of V_4 which has order 2.
The group algebra $\mathbb{Z}_2V_4 = \{0, 1, a, b, c, 1+a, 1+b, 1+c, a+b, a+c, b+c, 1+a+b, 1+a+c, 1+b+c, a+b+c, 1+a+b+c\}$ is a V_4-algebra over $\mathbb{Z}_2$.
Also $\mathbb{Z}_2V_4$ is interior V_4-algebra over $\mathbb{Z}_2$.
The subalgebra $\mathbb{Z}_2\langle a \rangle = \{0, 1, a, 1+a\}$ is a V_4-algebra over $\mathbb{Z}_2$ but it is not interior V_4-algebra over $\mathbb{Z}_2$. Because is has not group homomorphism from V_4 to $U(\mathbb{Z}_2\langle a \rangle) = \{1, a\}$.

The following examples are about of interior G-algebra.

Example 1.2.3. The group algebra FG is an interior G-algebra over F by the inclusion map of G in FG. If N is a normal subgroup of G then the subalgebra FN may be not has structure interior G-algebra see Remark 1.2.2.

Example 1.2.4. Consider M as an FG-module. The endomorphism algebra of M over F is an interior G-algebra by the group homomorphism $\varphi : G \longrightarrow \mathrm{Aut}_F(M)$ where $U(\mathrm{End}_F(M)) = \mathrm{Aut}_F(M)$.

The following lemma describes structure of interior G-algebra on subgroup of G.

Lemma 1.2.3. Let G be a finite group. Let H be a subgroup of G. If A is an interior G-algebra over R then A is an interior H-algebra over R.

Proof. Since A is an interior G-algebra over R thus there is a group homomorphism $\varphi : G \longrightarrow U(A)$. The restriction of φ to $\bar{\varphi} : H \longrightarrow U(A)$ is also group homomorphism. Hence A is an interior H-algebra over R. $\square$

The following definition of the morphism between the interior G-algebras.

Definition 1.2.6. Let G be a finite group. Let A and B be two interior G-algebras over R. A homomorphism $\Phi : A \longrightarrow B$ such that $\Phi(g.a) = g.\Phi(a)$ and $\Phi(a.g) = \Phi(a).g$ for all $g \in G$ and $a \in A$ is called a homomorphism of interior G-algebras.

1.3 Conjugation and Sylow theory

In this section, we shall introduce a deep theorem which is due to Burnside. That theorem is the beginning of the concept of control fusion in finite group theory.

However, we shall start by revise the concept of conjugation in group's elements as well as the conjugation of p-subgroups for fixed p prime number. Then we will mention the embbeding of the automizer of the automorphism group for p-local subgroup P of G. We followed references [7], [10], [12], [19] and [20].

Let G be a finite group. Then we can consider the action of G on itself. There are many ways to do such action. One of them is the action by conjugation.

Definition 1.3.1. For all $x, y \in G$, we say that x is G-conjugate to y and denoted $x \sim_G y$ if there exists an element $g \in G$ such that $x = g^{-1}yg = y^g$.

Lemma 1.3.1. The G-conjugate relation in above is an equivalence relation on G.

Remark 1.3.1.

- The equivalence classes of the G-conjugation relation as above are called conjugation classes of G. (orbits)
- These orbits are partition of G. Namely,

$$G = [x_1] \dot{\cup} [x_2] \cdots [x_t] \, .$$

- We can consider the conjugation action of $X = \{H : H \leq G\}$. Then we have G-conjugate classes of subgroups of G where their number is the index $[G : N_G(H)]$.

  In particular for fixed prime p. We can consider the set X of all p-subgroups of G as the following
$$X = \{P : P \leq G\}.$$
  Then G acts on X by conjugation.

We give the following map.

Definition 1.3.2. Let G be a finite group, we define the conjugation map for all $g \in G$ as the following
$$f_g : G \longrightarrow G$$
such that
$$f_g(x) = g^{-1}xg, \quad \forall x \in G.$$

Lemma 1.3.2. For all $g \in G$ we have f_g is an automorphism (inner automorphism) of G with $Ker(f_g) = C_G(g)$.

Proof. Suppose that $x, y \in G$ then
$$\begin{aligned} f_g(xy) &= g^{-1}xyg \\ &= (g^{-1}xg)(g^{-1}yg) \\ &= f_g(x)f_g(y). \end{aligned}$$
Hence f_g is a group homomorphism. Also,
$$\begin{aligned} \mathrm{Ker}f_g &= \{x \in G : f_g(x) = x\} \\ &= \{x \in G : g^{-1}xg = x\} \\ &= \{x \in G : xg = gx\} \\ &= C_G(g). \end{aligned}$$
Hence f_g is monomorphism. Moreover, f_g is epimorphism. Since that for all $y \in G$ there is $x = gyg^{-1} \in G$ such that $f_g(x) = f_g(gyg^{-1}) = y$. Hence f_g is automorphism. □

We remark that, now have a map from G to $\mathrm{Aut}(G)$. define as the following
$$\psi : G \longrightarrow \mathrm{Aut}(G)$$
such that
$$\psi(g) = f_g, \quad \forall g \in G.$$
In fact, this map is a group homomorphism.

Definition 1.3.3. Let G be a finite group. Let H be a subgroup of G. The quotient group $N_G(H)/C_G(H)$ is called the automizer of H in G.

proposition 1.3.1. *Let G be a finite group. Let H be a subgroup of G. Then the automizer group of H in G is isomorphic to a subgroup of the automorphism group of H. $N_G(H)/C_G(H) \lesssim \mathrm{Aut}_G(H)$.*

In particular, if H is normal in G, then $G/C_G(H) \lesssim \mathrm{Aut}_G(H)$.

Now we recorded some note of Sylow theorem, which is related to fusion system. For finite group G, we can fix a prime number p. Then the order of G, $|G|$ can be written as $|G| = p^m.r$ where $\gcd(p,r) = 1$ and $m, r \in \mathbb{N}$. It is well known that in finite group theory there exists a p-subgroup of G of order p^m and such p-subgroup is called Sylow p-subgroup of G. In particular, G contains a p-subgroup of order p^t for all $1 \leq t \leq m$. Each p-subgroup of G is contained in a Sylow p-subgroup of G. The number of Sylow p-subgroup of G is the index $[G : N_G(P)]$ for any particular Sylow p-subgroup of G. That number n_p of Sylow p-subgroups satisfies the congruence relation.

$$n_p \equiv 1 \bmod (p).$$

Now, we study Burnside theorem which explains the relationship between G-conjugate elements in P and $N_G(P)$-conjugate elements in P, where in this case P is an abelian Sylow p-subgroup of G.

Theorem 1.3.1. *(Burnside). Let p be a fixed prime number. Let G be a finite group. Let P be a Sylow p-subgroup of G. Assume that P is abelian. Then for all elements x and y in P, we have that x and y are G-conjugate if and only if x and y are $N_G(P)$-conjugate.*

Proof. ($\Rightarrow$) Suppose that x is G-conjugate to y then there exists $g \in G$ such that $x = g^{-1}yg$. Since P is abelian and $x, y \in P$ we have $P \leq C_G(x)$ and $P \leq C_G(y)$. Also, $P^g = g^{-1}Pg \leq C_G(x)$. Thus P and P^g are Sylow p-subgroups of $C_G(x)$. From Sylow's theorem we have there exists $z \in C_G(x)$ such that $P^{gz} = P$. Thus $gz = n \in N_G(P)$ and $x = z^{-1}xz = z^{-1}g^{-1}ygz = n^{-1}yn$. Hence x is $N_G(P)$-conjugate to y.
($\Leftarrow$) Suppose that x is $N_G(P)$-conjugate to y then there exists $g \in N_G(P)$ such that $x = g^{-1}yg$. Since $N_G(P) \leq G$ then $g \in G$. Hence x is G-conjugate to y. □

Remark 1.3.2.

- Burnside theorem is the starting point of the concept of fusion and control fusion.
- We remark that, we can consider non-abelian Sylow p-subgroup P and in this case Burnside theorem can be stated as following: If P is any Sylow p-subgroup of G, then two normal subsets " conjugate classes " of P are conjugate in G if and only if they are conjugate in the normalizer $N_G(P)$.

As corollary, we mention that two elements of the center of P, are conjugate in G if and only if they are conjugate in $N_G(P)$. This is a special case of Theorem 1.3.1.

Chapter 2

Brauer Indecomposable Modules and Scott Modules

In this chapter we introduce several concepts in important types of modules, Brauer indecomposable module and Scott module. We introduce concepts of induced module and restriction module. Then we give some examples and properties about them. In second section, we introduce concepts of the set of invariant elements and relative trace map in two algebra structure, modules structure and G-algebras structure. We give some examples and properties about them. In third section, we introduce concepts of Brauer quotient and Brauer homomorphism in modules structure and G-algebras structure. We give some properties about them. Also we introduce concept of Brauer indecomposable module. In fourth section, we introduce concept of relative projective module. We give some important properties about it. In fifth section, we introduce concept of p-permutation module in modules structure and G-algebras structure. Then we give some properties about it. In the last section, we introduce concepts of Scott module and Scott coefficient. Then we give important properties about them.

Throughout this chapter, G denotes a finite group and p a prime number. Let $(K, \mathcal{O}, F)$ be a p-modular system. Let R be $\mathcal{O}$ or F.

2.1 Induced modules

In this section, we consider how to construct module of a group from a module of its subgroup. We followed references [4], [5], [7], [16], [23] and [24].

The following definition of induced module and restriction module.

Definition 2.1.1. [16] Let G be a finite group. Let H be a subgroup of G. Let W be an RH-module. The induced module of W from H to G is an RG-module. We regarded it from the following form

$$\mathrm{Ind}_H^G(W) = W \otimes_{RH} RG = \bigoplus_{i=1}^{[G:H]} W \otimes_{RH} t_i$$

where $t_i \in [G/H]$. It is clear $W \subseteq \mathrm{Ind}_H^G(W)$. The dual notion to induction is the of restriction. Let M be an RG-module. Then the restriction module of M from G to H is an RH-module which is denoted by $\mathrm{Res}_H^G(M)$.

Remark 2.1.1.

Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be an RG-module. Then $\mathrm{Res}_P^G(M)$ is a direct sum of modules isomorphic to $\mathrm{Ind}_Q^P(R)$ where $\mathrm{Ind}_Q^P(R)$ is indecomposable RP-module and Q is a subgroup of P.

Now we introduce examples are about of induced module.

Example 2.1.1.

Let G be a finite group. The group algebra RG over R obtained by induced of the trivial RG-submodule R from identity subgroup 1_G to G has the form

$$\mathrm{Ind}_{1_G}^G(R) \cong R \otimes_{R1_G} RG \cong RG.$$

Example 2.1.2.

Let $F = \mathbb{Z}_2$ be a field which has characteristic 2. Let $G = D_8$ be a dihedral group of degree 4. If we fixed the prime number $p = 2$ and we take $H = \langle b \rangle = \{1, b\}$ is a subgroup of D_8 which has order 2. The index of $\langle b \rangle$ in D_8 is $[G : H] = 4$. The set of left cosets representatives of $\langle b \rangle$ in D_8 is $[D_8/\langle b \rangle] = \{1, a, a^2, a^3\}$. Then induced of trivial $\mathbb{Z}_2 D_8$-module $\mathbb{Z}_2$ from $\langle b \rangle$ to D_8 as the form

$$\mathrm{Ind}_{\langle b \rangle}^{D_8}(\mathbb{Z}_2) \cong \mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} \mathbb{Z}_2 D_8 \cong \bigoplus_{i=1}^{4} \mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} t_i \quad \text{where} \quad t_i \in [D_8/\langle b \rangle].$$

Thus

$$\mathrm{Ind}_{\langle b \rangle}^{D_8}(\mathbb{Z}_2) \cong (\mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} 1) \oplus (\mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} a) \oplus (\mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} a^2) \oplus (\mathbb{Z}_2 \otimes_{\mathbb{Z}_2\langle b \rangle} a^3).$$

proposition 2.1.1. *Let G be a finite group. Let M be an RG-module isomorphic the direct sum of the R-submodules $\{gV | g \in G\}$ where V be an R-submodule of M. Let $H = \{g \in G | gV = V\}$. Then $M \cong \mathrm{Ind}_H^G(V)$.*

The following definition of permutation module. This module has a relationship with induced module.

Definition 2.1.2. [16] Let G be a finite group. Let Ω be a set with an action of G. Then $R\Omega \cong \bigoplus_{w\in\Omega} Rw$ is the permutation RG-module on Ω.

Remark 2.1.2. If G in above definition acts on Ω transitively then we said $R\Omega$ is a transitive permutation RG-module on Ω.

The following example describes when the permutation module isomorphic to the induced module of R.

Example 2.1.3.

Consider G is a finite group and Ω is a set with an action transitively of G by permutations. If we take $w \in \Omega$ and $H = \mathrm{Stab}_G(w)$ where H is a subgroup of G Then H is the stabilizer of Rw and the permutation module $R\Omega \cong \mathrm{Ind}_H^G(R)$.

The following lemmas give us some properties about of induced module.

Lemma 2.1.1. Let G be a finite group. Let H and L be two subgroups of G where $L \leq H$. Let W be an RL-module. Then

$$\mathrm{Ind}_H^G(\mathrm{Ind}_L^H(W)) \cong \mathrm{Ind}_L^G(W).$$

Proof. From Definition 2.1.1 we have

$$\mathrm{Ind}_H^G(\mathrm{Ind}_L^H(W)) \cong \mathrm{Ind}_H^G(W \otimes_{RL} RH) \cong (W \otimes_{RL} RH) \otimes_{RH} RG.$$

From the associativity of the tensor product we have

$$(W \otimes_{RL} RH) \otimes_{RH} RG \cong W \otimes_{RL} (RH \otimes_{RH} RG) \cong W \otimes_{RL} RG.$$

Then

$$\mathrm{Ind}_H^G(\mathrm{Ind}_L^H(W)) \cong W \otimes_{RL} RG.$$

Thus from Definition 2.1.1 we have

$$W \otimes_{RL} RG \cong \mathrm{Ind}_L^G(W).$$

Hence

$$\mathrm{Ind}_H^G(\mathrm{Ind}_L^H(W)) \cong \mathrm{Ind}_L^G(W).$$

□

Lemma 2.1.2. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Let W be an RH-module. Then

$$M \otimes_R \mathrm{Ind}_H^G(W) \simeq \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M) \otimes_R W).$$

Lemma 2.1.3. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Let W be an RH-module. Then

(i) $\mathrm{Ind}_H^G(\mathrm{Hom}_R(\mathrm{Res}_H^G(M), W) \simeq \mathrm{Hom}_R(M, \mathrm{Ind}_H^G(W))$ as RG-isomorphism.

(ii) $\mathrm{Ind}_H^G(\mathrm{Hom}_R(W, \mathrm{Res}_H^G(M)) \simeq \mathrm{Hom}_R(\mathrm{Ind}_H^G(W), M)$ as RG-isomorphism.

Lemma 2.1.4. Let p be a prime number. Let G be a finite group. Let H be a subgroup of G. Let W be an FH-module. Then for all $g \in G$ we have

$$\mathrm{Ind}_H^G(W \otimes g) \cong \mathrm{Ind}_H^G(W).$$

Lemma 2.1.5. Let G be a finite group. Let H be a subgroup of G. Let W be an RH-module. Let $g \in G$. Then

$$\mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_{H^g}^G(W^g).$$

Lemma 2.1.6. Let G be a finite group. Let H be a subgroup of G. Let W be a free RH-module. Then $\mathrm{Ind}_H^G(W)$ is a free RG-module.

Proof. Since W is a free RH-module thus it has a basis $X = \{x_1, \cdots, x_w\} \subseteq W$. then

$$\mathrm{Ind}_H^G(X) \cong X \otimes_{RH} RG \subseteq W \otimes_{RH} RG.$$

From Definition 2.1.1 we have

$$W \otimes_{RH} RG \cong \mathrm{Ind}_H^G(W).$$

Hence $\mathrm{Ind}_H^G(X) \subseteq \mathrm{Ind}_H^G(W)$. Also from Definition 2.1.1 we have

$$\mathrm{Ind}_H^G(X) \cong X \otimes_{RH} RG \cong \bigoplus_{i=1}^{r} X \otimes_{RH} t_i \quad \text{where} \quad t_i \in [G/H]\,.$$

Thus

$$\mathrm{Ind}_H^G(X) \cong x_1 \otimes_{RH} t_1 \oplus \cdots \oplus x_w \otimes_{RH} t_1 \oplus \cdots \oplus x_1 \otimes_{RH} t_r \oplus \cdots \oplus x_w \otimes_{RH} t_r.$$

Since that X is a linear independent and spans W then $\mathrm{Ind}_H^G(X)$ is also linear independent and spans $\mathrm{Ind}_H^G(W)$. Hence $\mathrm{Ind}_H^G(X)$ is a basis of $\mathrm{Ind}_H^G(W)$. Then $\mathrm{Ind}_H^G(W)$ is a free RG-module. □

2.2 Relative trace map

In this section, we introduce the concepts of the set of invariant elements and relative trace map in two algebra structures, modules structure and G-algebras structure. We followed references [3], [4], [5], [8], [14], [16], [23] and [24].

Now we begin by modules structure. We start by definition of the set of invariant elements of module.

Definition 2.2.1. [16] Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. The H-invariant elements (H-fixed points) of M defined by the following:

$$\mathrm{Inv}_H(M) = \{m \in M : mh = m, \forall h \in H\}.$$

Remark 2.2.1.

- We write M^H to denote $\mathrm{Inv}_H(M)$.
- Consider G a finite group, H and L two subgroups of G where $H \leq L$ and M is an RG-module. Then $M^L \subset M^H$. In particular $M^G \subset M^H$ then M^G is smallest submodule of M and the largest is M^{1_G} where 1_G is the trivial subgroup of G.

The following examples are about of the set of invariant elements of modules.

Example 2.2.1.

Let $F = \mathbb{Z}_2$ be a field which has characteristic 2. Let $G = S_3$ be the symmetric group of three letters which has order 6. The group algebra is $FG = \mathbb{Z}_2S_3$. If we take $H = A_3$ is the alternating subgroup of S_3 which has order 3 then $M = \mathbb{Z}_2A_3 = \{0, 1, (123), (132), 1+(123), 1+(132), (123)+(132), 1+(123)+(132)\}$ is $\mathbb{Z}_2S_3$-module. The H-invariant elements of M is $\mathrm{Inv}_{A_3}(\mathbb{Z}_2A_3) = \{0, 1 + (123) + (132)\}$.

Example 2.2.2.

Consider G a finite group and M, W are two RG-modules. The set of all R-module homomorphism $\mathrm{Hom}_R(M, W)$ is an R-module then

$$(\mathrm{Hom}_R(M, W))^G = \mathrm{Hom}_{RG}(M, W).$$

The following lemma gives us the relationship between sets of invariant elements of module and G-conjugate subgroups of G.

Lemma 2.2.1. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Let $g \in G$. Then

$$M^H g = M^{H^g}.$$

The set of invariant elements of module has special algebra structures. We show it in the following lemmas.

Lemma 2.2.2. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M^H is an RG-submodule of M.

Proof. Since $0_M \in M$ and $0_M h = 0_M$ for all $h \in H$, then $0_M \in M^H$. Hence from Definition 2.2.1 we have clearly $\varnothing \neq M^H \subseteq M$. Now suppose that $m_1, m_2 \in M^H$ thus $m_1 h = m_1$ and $m_2 h = m_2$ for all $h \in H$. Since that M is an RG-module we have

$$\begin{aligned}(m_1 - m_2)h &= (m_1 + (-m_2))h\\ &= m_1h + (-m_2)h\\ &= m_1h + (-(m_2h))\\ &= m_1 + (-m_2)\\ &= m_1 - m_2.\end{aligned}$$

Hence $m_1 - m_2 \in M^H$. Also suppose that $r \in RG$ and $m \in M^H$ thus $mh = m$ for all $h \in H$. Since that M is an RG-module we have

$$\begin{aligned}(rm)h &= r(mh)\\ &= rm.\end{aligned}$$

Hence $rm \in M^H$. So M^H is an RG-submodule of M. □

Lemma 2.2.3. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M^H is an $FN_G(H)/H$-module.

Proof. From Definition 2.2.1 clearly we have H acts trivial on M^H. Also from Lemma 2.2.1 for $g \in G$ we have $M^H g = M^{H^g}$. If $g \in N_G(H)$ then $H^g = H$. Thus $M^H g = M^H$. Hence we can consider M^H as an $FN_G(H)$-module. Also we can consider M^H as an $FN_G(H)/H$-module. □

Now we will introduce definition of relative trace map.

Definition 2.2.2. [16] Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. A relative trace map from M^H to M^G is defined by the following:

$$\mathrm{Tr}_H^G : M^H \longrightarrow M^G$$

$$\mathrm{Tr}_H^G(m) = \sum_{t \in [G/H]} mt, \quad \forall m \in M^H.$$

The inclusion map $\mathrm{Res}_H^G : M^G \longrightarrow M^H$ is the restriction map.

Remark 2.2.2.

We write M_H^G to denote the image of $\mathrm{Tr}_H^G(M^H)$.

The following lemma tells us when the relative trace map is surjective.

Lemma 2.2.4. Let G be a finite group. Let H be a subgroup of G. Let Q be a Sylow p-subgroup of H. Let M be an RG-module. Then $M^H = M_Q^H$.

Lemma 2.2.5. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. The relative trace map $\mathrm{Tr}_H^G : M^H \longrightarrow M^G$ has the following properties:

(i) Tr_H^G is well-defined.

(ii) Tr_H^G is an R-linear map.

(iii) Tr_H^H, Res_H^H are identity maps for all $H \leq G$.

(iv) (Transitivity) If $H \le L \le G$ then $\mathrm{Tr}_H^G = \mathrm{Tr}_L^G \circ \mathrm{Tr}_H^L$.

(v) $\mathrm{Tr}_H^G(M) = \mathrm{Tr}_{H^g}^G(M)$.

(vi) If $H \le_G L$ then $\mathrm{Tr}_H^G(M) \subset \mathrm{Tr}_L^G(M)$.

Theorem 2.2.1. *Let G be a finite group. Let H and L be two subgroups of G. Let M be an RG-module. The relative trace map $\mathrm{Tr}_H^G : M^H \longrightarrow M^G$ has the following statements hold:*

(i) *(Mackey decomposition)* $\mathrm{Tr}_H^G(m) = \sum_{t\in[H\backslash G/L]} \mathrm{Tr}_{H^t\cap L}^L(mt)$ *for* $m \in M^H$.

(ii) $\mathrm{Tr}_H^G(M) \subset \sum_{t\in[H\backslash G/L]} \mathrm{Tr}_{H^t\cap L}^L(M)$.

(iii) *(Mackey formula)* $\mathrm{Res}_H^G \mathrm{Tr}_L^G = \sum_{t\in[H\backslash G/L]} \mathrm{Tr}_{L^t\cap H}^H \mathrm{Res}_{L^t\cap H}^{L^t} f_t$.

Lemma 2.2.6. Let G be a finite group. Let H and L be two subgroups of G. Let W be an RH-module. Then there is an isomorphism :

$$\mathrm{Tr}_H^G : W^H \longrightarrow (\mathrm{Ind}_H^G(W))^G$$

such that

$$\mathrm{Tr}_H^G(w) = \sum_{t\in[G/H]} w \otimes_{RH} t, \quad \forall w \in W^H.$$

Now we begin by G-algebra structure.

Definition 2.2.3. [24] Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. The H-invariant elements (H-fixed points) of A, namely

$$\mathrm{Inv}_H(A) = \{a \in A : a^h = a, \ \forall h \in H\}.$$

Remark 2.2.3.

- We write A^H to denote $\mathrm{Inv}_H(A)$.
- Consider G as a finite group, H and L are two subgroups of G where $H \le L$ and A is a G-algebra over R. Then $A^L \le A^H$. In particular $A^G \le A^H$ thus A^G is smallest and the largest is A^{1_G} where 1_G is the trivial subgroup of G.
- Let G be a finite group, H be a subgroup of G and $A = RG$ be a G-algebra over R. Then the G-invariant elements of A is $A^G = Z(RG) = \bigoplus_{i=1}^n R\widehat{C}_i$ where $\widehat{C}_i = \sum_{x\in C_i} x$ and C_i is a G-conjugacy class of G. The set of all G-conjugacy class of G denoted by $Cl(G)$ then $\{\widehat{C}\}_{C\in Cl(G)}$ is a basis of Z(RG). The H-invariant elements of A is $A^H = C_{RG}(H)$ where H acts on G by conjugation. The H-orbits are H-conjugacy classes of G. The set of all H-conjugacy classe of G denoted by $Cl_H(G)$ where $\{\widehat{D}\}_{D\in Cl_H(G)}$ is a basis of $C_{RG}(H)$.

The following examples are about of set of invariant elements.

Example 2.2.3.

Let $F = \mathbb{Z}_2$ be a field which has characteristic 2. Let $G = V_4$ be the Kelin 4-group which has order 4. The group algebra $A = FG = \mathbb{Z}_2V_4 = \{0, 1, a, b, c, 1+a, 1+b, 1+c, a+b, a+c, b+c, 1+a+b, 1+a+c, 1+b+c, a+b+c, 1+a+b+c\}$ is a V_4-algebra over $\mathbb{Z}_2$. The V_4-invariant elements of $\mathbb{Z}_2V_4$ is $A^G = Z(\mathbb{Z}_2V_4) = \mathbb{Z}_2V_4$. If we take $H = \langle a \rangle$ is a subgroup of V_4 which has order 2. Then the $\langle a \rangle$-invariant elements of $\mathbb{Z}_2V_4$ is $A^H = C_{\mathbb{Z}_2V_4}(\langle a \rangle) = \mathbb{Z}_2V_4$.

Example 2.2.4.

Let $F = \mathbb{Z}_3$ be a field which has characteristic 3. Let $G = S_2$ be the symmetric group of two letters which has order 2. The group algebra $A = FG = \mathbb{Z}_3S_2 = \{0, 1, 2, (1), (12), 1+(1), 1+(12), 2+(1), 2+(12)\}$ is a S_2-algebra over $\mathbb{Z}_3$. The S_2-invariant elements of $\mathbb{Z}_3S_2$ is $A^G = Z(\mathbb{Z}_3S_2) = \mathbb{Z}_3S_2$. If we take $H = \langle 1_{S_2} \rangle$ is the trivial subgroup of S_2. Then the $\langle 1_{S_2} \rangle$-invariant elements of $\mathbb{Z}_3S_2$ is $A^H = C_{\mathbb{Z}_3S_2}(\langle 1_{S_2} \rangle) = \mathbb{Z}_3S_2$.

Example 2.2.5.

Let G be a finite group, H be a subgroup of G and M be an RG-module. The endomorphism algebra $\mathrm{End}_R(M)$ is a G-algebra over R. Then $(\mathrm{End}_R(M))^G = \mathrm{End}_{RG}(M)$ and $(\mathrm{End}_R(M))^H = \mathrm{End}_{RH}(M)$.

The following lemma describes a relationship between the sets of invariant elements of G-algebra by G-conjugate subgroups of G.

Lemma 2.2.7. *Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. Let $g \in G$. Then*

$$A^{H^g} = (A^H)^g.$$

The following lemmas describe algebra structures of the set of invariant elements of G-algebra.

Lemma 2.2.8. *Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. Then A^H is an R-subalgebra of A and it has the same unity element of A.*

Proof. From definition of A^H clearly we have $A^H \subseteq A$. Also since $0_A \in A$ and satisfy $0_A^h = 0_A$ for all $h \in H$ then $0_A \in A^H$. Hence $\varnothing \neq A^H \subseteq A$. Now suppose that $a_1, a_2 \in A^H$ then $a_1^h = a_1$ and $a_2^h = a_2$ for all $h \in H$. Since that A is a G-algebra over R then we have

$$\begin{aligned}
(a_1 - a_2)^h &= (a_1 + (-a_2))^h \\
&= a_1^h + (-a_2)^h \\
&= a_1^h + (-a_2^h) \\
&= a_1 + (-a_2) \\
&= a_1 - a_2.
\end{aligned}$$

Hence $a_1 - a_2 \in A^H$. Also since that $a_1, a_2 \in A$ and A is a G-algebra over R then we have

$$\begin{aligned} (a_1a_2)^h &= a_1^h a_2^h \\ &= a_1a_2. \end{aligned}$$

Hence $a_1a_2 \in A^H$. Also suppose that $r \in R$ and $a \in A^H$ thus $a^h = a$ for all $h \in H$. Since that A is a G-algebra over R then we have

$$\begin{aligned} (ra)^h &= ra^h \\ &= ra. \end{aligned}$$

Hence $ra \in A^H$. Hence A^H is an R-subalgebra of A. If A has unit element 1_A then we have $1_A^h = 1_A$ for all $h \in H$. Thus $1_A \in A^H$. Hence A^H has the same unit element of A. □

Lemma 2.2.9. Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. Then A^H is an $N_G(H)/H$-algebra over R.

Proof. From Definition 2.2.3 clearly we have H acts trivial on A^H. Also since for $g \in G$ we have $(A^H)^g = A^{H^g}$. If $g \in N_G(H)$ then $H^g = H$. Thus $(A^H)^g = A^H$. Hence we can consider A^H as an $N_G(H)$-algebra over R. Also we can consider A^H as an $N_G(H)/H$-algebra over R. □

The following definition of a homomorphism of $N_G(H)/H$-algebra.

Definition 2.2.4. Let G be a finite group. Let H be a subgroup of G. Let A and B be two G-algebras over R. Let $\Psi : A \longrightarrow B$ be a homomorphism of G-algebras. Then $\Psi(A^H) \subseteq B^H$ and the restriction $\Psi^H : A^H \longrightarrow B^H$ is a homomorphism of $N_G(H)/H$-algebra.

Definition 2.2.5. [24] Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. The relative trace map from A^H to A^G is defined by:

$$\mathrm{Tr}_H^G : A^H \longrightarrow A^G$$

$$\mathrm{Tr}_H^G(a) = \sum_{t \in [G/H]} a^t, \quad \forall a \in A^H.$$

The inclusion map $\mathrm{Res}_H^G : A^G \longrightarrow A^H$ is the restriction map.

Remark 2.2.4.

We write A_H^G to denote the image of $\mathrm{Tr}_H^G(A^H)$.

The following lemma tells us when the relative trace map is surjective.

Lemma 2.2.10. Let G be a finite group. Let H be a subgroup of G. Let Q be a Sylow p-subgroup of H. Let A be a G-algebra over R. Then $A^H = A_Q^H$.

Lemma 2.2.11. Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. The relative trace map and the restriction map have the following properties:

(i) Tr_H^H, Res_H^H are identity maps for all $H \leq G$.

(ii) (Transitivity) $\mathrm{Tr}_H^G \mathrm{Tr}_L^H = \mathrm{Tr}_L^G$ and $\mathrm{Res}_L^H \mathrm{Res}_H^G = \mathrm{Res}_L^G$ for all $L \leq H \leq G$.

(iii) *(Frobenius relations)* $\mathrm{Tr}_L^H(ab) = a\mathrm{Tr}_L^H(b)$ and $\mathrm{Tr}_L^H(ba) = \mathrm{Tr}_L^H(b)a$ for all $L \leq H \leq G, a \in A^H$ and all $b \in A^L$. In particular if I is an ideal of A^L then $\mathrm{Tr}_L^H(I)$ is an ideal of A^H.

(iv) For $a \in A^H$ and $b \in A^L$ we have $\mathrm{Tr}_H^G(a)\mathrm{Tr}_L^G(b) = \sum_{t \in [H\backslash G/L]} \mathrm{Tr}_{H^t \cap L}^G(a^t b)$.

(v) If $L \leq H$, $a \in A^L$ and $g \in G$ we have $(\mathrm{Tr}_L^H(a))^g = \mathrm{Tr}_{L^g}^{H^g}(a^g)$.

(vi) *(Mackey decomposition formal)* If $N, L \leq H$ and $a \in A^N$ then

$$\mathrm{Res}_L^H \mathrm{Tr}_N^H(a) = \sum_{h \in [L\backslash H/N]} \mathrm{Tr}_{L \bigcap N^h}^L \mathrm{Res}_{L \bigcap N^h}^{N^h}(a^h).$$

(vii) For all $L \leq H \leq G$ and all $a \in A^H$ we have $\mathrm{Tr}_L^H(a) = [H : L]\,.a$

(viii) For $L \leq H \leq G$ and $a \in A^H$ we have $\mathrm{Tr}_L^H \mathrm{Res}_L^H(a) = [H : L]\,.a$

Lemma 2.2.12. Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. The relative trace map $Tr_H^G : A^H \longrightarrow A^G$ has the following properties:

(i) The relative trace map is independent of the choice of coset representatives.

(ii) $\mathrm{Tr}_H^G(a) \in A^G$.

(iii) $\mathrm{Im}(\mathrm{Tr}_H^G) \trianglelefteq A^G$.

(iv) The relative trace map is a linear map.

(v) The relative trace map is not an algebra homomorphism in general.

Proof. (i) Suppose that

$$T = \{t_1, t_2, \ldots, t_n\} \quad \text{and} \quad M = \{m_1, m_2, \ldots, m_n\}$$

are two sets of cosets representatives of H in G. Then for any $t_i \in T$ there is $m_j \in M$ such that $t_i \in Hm_j$. So $t_i = hm_j$ where $h \in H$. Thus for $a \in A^H$ we have

$$\begin{array}{rcl}\mathrm{Tr}_H^G(a) &=& \sum_{t_i\in T} a^{t_i}\\ &=& \sum_{hm_j\in T} a^{hm_j}\\ &=& \sum_{hm_j\in T}(a^h)^{m_j}\\ &=& \sum_{m_j\in M} a^{m_j}.\end{array}$$

Hence Tr_H^G is independent of sets of cosets representatives of H in G.

(ii) Suppose that $[G:H]=n$ and $T=\{1,t_2,t_3,....,t_n.\}$ is a set of cosets representatives of H in G. For $g\in G$ we have $Tg=\{tg : t\in T\}$ is a set of cosets representatives of H in G. For $a\in A^H$ we have

$$\begin{array}{rcl}(\mathrm{Tr}_H^G(a))^g &=& (\sum_{t\in T} a^t\ )^g\\ &=& \sum_{tg\in Tg} a^{tg}\\ &=& \mathrm{Tr}_H^G(a).\end{array}$$

Hence $\mathrm{Tr}_H^G(a)\in A^G$.

(iii) $\mathrm{Im}(\mathrm{Tr}_H^G)=\{a\in A^G : \exists b\in A^H$ such that: $\mathrm{Tr}_H^G(b)=a\}\subseteq A^G$. Since that $0_A\in A^G$ and $0_A=0_A^t=\sum_{t\in T}0_A^t=\mathrm{Tr}_H^G(0_A)$ where T is a set of cosets representatives of H in G. Hence $0_A\in\mathrm{Im}(\mathrm{Tr}_H^G)$ thus $\varnothing\neq\mathrm{Im}(\mathrm{Tr}_H^G)\subseteq A^G$. Now suppose that $a_1,a_2\in\mathrm{Im}(\mathrm{Tr}_H^G)$ thus there are $b_1,b_2\in A^H$ such that $\mathrm{Tr}_H^G(b_1)=a_1$ and $\mathrm{Tr}_H^G(b_2)=a_2$. Then

$$\begin{array}{rcl}a_1-a_2 &=& \mathrm{Tr}_H^G(b_1)-\mathrm{Tr}_H^G(b_2)\\ &=& \sum_{t\in T} b_1^t-\sum_{t\in T} b_2^t\\ &=& \sum_{t\in T}(b_1^t-b_2^t)\\ &=& \sum_{t\in T}(b_1-b_2)^t\\ &=& \mathrm{Tr}_H^G(b_1-b_2).\end{array}$$

Hence $a_1-a_2\in\mathrm{Im}(\mathrm{Tr}_H^G)$. Also suppose that $x\in A^G$ and $a\in\mathrm{Im}(\mathrm{Tr}_H^G)$ thus there is $b\in A^H$ such that $\mathrm{Tr}_H^G(b)=a$. Then

$$\begin{array}{rcl}x.a &=& x.\mathrm{Tr}_H^G(b)\\ &=& x.\sum_{t\in T} b^t\\ &=& \sum_{t\in T} x.b^t\\ &=& \sum_{t\in T}(x.b)^t\\ &=& \mathrm{Tr}_H^G(xb).\end{array}$$

So $x.a\in\mathrm{Im}(\mathrm{Tr}_H^G)$. Similar for $a.x$ we have $a.x\in\mathrm{Im}(\mathrm{Tr}_H^G)$. Hence $\mathrm{Im}(\mathrm{Tr}_H^G)\trianglelefteq A^G$.

(iv) Suppose that $a_1,a_2\in A^H$. Thus

$$\begin{array}{rcl}\mathrm{Tr}_H^G(a_1+a_2) &=& \sum_{t\in T}(a_1+a_2)^t\\ &=& \sum_{t\in T}(a_1^t+a_2^t)\\ &=& \sum_{t\in T} a_1^t+\sum_{t\in T} a_2^t\\ &=& \mathrm{Tr}_H^G(a_1)+\mathrm{Tr}_H^G(a_2).\end{array}$$

Also suppose that $r\in R$ and $a\in A^H$. Thus

$$
\begin{array}{rcl}
\mathrm{Tr}_H^G(ra) & = & \sum_{t\in T}(ra)^t \\
& = & \sum_{t\in T} ra^t \\
& = & r\sum_{t\in T} a^t \\
& = & r\mathrm{Tr}_H^G(a).
\end{array}
$$

Hence relative trace map is a linear map.

(v) If we take $a \in A^H$ and $b \in A^G$ then

$$\mathrm{Tr}_H^G(ab) = \sum_{t\in T}(ab)^t$$

where T is a set of cosets representatives of H in G. From (*Frobenius relations*) we have

$$
\begin{array}{rcl}
\mathrm{Tr}_H^G(ab) & = & \sum_{t\in T}(ab)^t \\
& = & \sum_{t\in T} a^t b \\
& = & (\sum_{t\in T} a^t)b \\
& = & \mathrm{Tr}_H^G(a)b \neq \mathrm{Tr}_H^G(a)\mathrm{Tr}_H^G(b).
\end{array}
$$

Hence relative trace map is not algebra homomorphism in general. □

2.3 Brauer indecomposable modules

In this section, we introduce the concepts and properties of the Brauer quotient and Brauer homomorphism in two algebra structures, modules structure and G-algebras structure. We introduce concept of Brauer indecomposable module. We followed references [1], [3], [5], [11], [14], [22], [23] and [24].

Now we begin by modules structure.

Definition 2.3.1. [5] Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. A Brauer quotient (Brauer construction) of M with respect to H is defined as the following:

$$M(H) = M^H/I_H(M) \quad \text{where} \quad I_H(M) = \sum_{L<H} M_L^H.$$

Lemma 2.3.1. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then $M(H)$ is an $FN_G(H)/H$-module.

Proof. From Definition 2.2.1 clearly we have H acts trivial on M^H. Also since that for $g \in G$ we have $(M^H)g = M^{H^g}$. If $g \in N_G(H)$ then $H = H^g$. Thus $(M^H)g = M^H$. Hence M^H is an $FN_G(H)/H$-module. Also since that

$$\begin{array}{rcl}(I_H(M))^g &=& (\sum_{L<H} M_L^H)^g \\ &=& (\sum_{L<H} Tr_L^H(M^L))^g \\ &=& \sum_{L<H} Tr_{L^g}^{H^g}(M^L)^g \\ &=& \sum_{L<H} Tr_{L^g}^{H^g}(M^{L^g}) \\ &=& \sum_{L<H} M_{L^g}^{H^g} \\ &=& I_{H^g}(M).\end{array}$$

If $g \in N_G(H)$, then $(I_H(M))^g = I_H(M)$. Hence $I_H(M)$ is an $FN_G(H)/H$-submodule of M^H. Then $M(H) = M^H/I_H(M)$ is an $FN_G(H)/H$-module. □

Lemma 2.3.2. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. If $M(H) \neq 0$ then H is a p-group.

Definition 2.3.2. [5] Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. A Brauer homomorphism Br_H^M is the canonical surjective map which is defined as the following:

$$\mathrm{Br}_H^M : M^H \twoheadrightarrow M(H).$$

Remark 2.3.1.

A kernel of Brauer homomorphism is

$$\mathrm{Ker}(\mathrm{Br}_H^M) = \{m \in M^H : \mathrm{Br}_H^M(m) = 0_{M(H)}\}.$$

From the first isomorphism theorem and since that Br_H^M is the canonical surjective map we have $\mathrm{Ker}(\mathrm{Br}_H^M) = I_H(M)$.

The following lemmas give us some properties of Brauer homomorphism.

Lemma 2.3.3. Let G be a finite group. Let H and L be two subgroups of G. Let M be an RG-module. If $\mathrm{Br}_H^M(\mathrm{Tr}_L^G(m)) \neq 0$ for some $m \in M^L$ then H is a G-conjugate to a subgroup of L.

Lemma 2.3.4. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be an RG-module. Then

$$\mathrm{Tr}_{1_G}^{N_G(P)/P} \circ \mathrm{Br}_P^M = \mathrm{Br}_P^M \circ \mathrm{Tr}_P^G.$$

In particular

$$\mathrm{Br}_P^M(M_P^G) = (M(P))_{1_G}^{N_G(P)/P}.$$

Lemma 2.3.5. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M_1, M_2 and M_3 be RG-modules. Let $f : M_1 \times M_2 \longrightarrow M_3$ be a bi-linear map stable under G-action. Then f induces a bi-linear map $f_P : M_1(P) \times M_2(P) \longrightarrow M_3(P)$ stable under $N_G(P)/P$-action such that

$$f_P(\mathrm{Br}_P^{M_1}(m_1), \mathrm{Br}_P^{M_2}(m_2)) = \mathrm{Br}_P^{M_3}(f(m_1, m_2))$$

for all $m_1 \in M_1$ and $m_2 \in M_2$.

The following definition of important module in our research.

Definition 2.3.3. [11] Let G be a finite group. Let H be a subgroup of G. Let M be an FG-module. Then we said that M is a Brauer indecomposable FG-module if $M(H)$ is indecomposable or zero as an $FHC_G(H)$-module.

Now we will begin by G-algebra structure.

Definition 2.3.4. [24] Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. A Brauer quotient of A with respect to H is defined as the following:

$$A(H) = A^H / I_H(A) \quad \text{where} \quad I_H(A) = \sum_{L<H} A_L^H.$$

Lemma 2.3.6. Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. Then $A(H)$ is an $N_G(H)/H$-algebra over F.

Proof. From Definition 2.2.3 clearly we have H acts trivial on A^H. Also since that for $g \in G$ we have $(A^H)^g = A^{H^g}$. If $g \in N_G(H)$ then $H = H^g$. Thus $(A^H)^g = A^H$. Hence A^H is an $N_G(H)/H$-algebra over F. Also since that

$$\begin{array}{rcl} (I_H(A))^g & = & (\sum_{L<H} A_L^H)^g \\ & = & (\sum_{L<H} Tr_L^H(A^L))^g \\ & = & \sum_{L<H} Tr_{L^g}^{H^g}(A^L)^g \\ & = & \sum_{L<H} Tr_{L^g}^{H^g}(A^{L^g}) \\ & = & \sum_{L<H} A_{L^g}^{H^g} \\ & = & I_{H^g}(A). \end{array}$$

If $g \in N_G(H)$, then $(I_H(A))^g = I_H(A)$. Hence $I_H(A)$ is an $N_G(H)/H$-ideal in A^H. Then $A(H) = A^H/I_H(A)$ is an $N_G(H)/H$-algebra over F.

□

Lemma 2.3.7. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. If $A(H) \neq 0$ then H is a p-group.

Remark 2.3.2.

Consider G is a finite group, H a subgroup of G and A is a G-algebra over K. Then $A(H) \neq 0$ if $H = 1$.

Definition 2.3.5. [24] Let G be a finite group. Let H be a subgroup of G. Let A be a G-algebra over R. A Brauer homomorphism Br_H^A is the canonical surjective map which is defined as the following:

$$\mathrm{Br}_H^A : A^H \twoheadrightarrow A(H).$$

Remark 2.3.3.

A kernel of Brauer homomorphism is

$$\mathrm{Ker}(\mathrm{Br}_H^A) = \{a \in A^H : \mathrm{Br}_H^A(a) = 0_{A(H)}\}.$$

From the first isomorphism theorem and since that Br_H^A is the canonical surjective map we have $\mathrm{Ker}(\mathrm{Br}_H^A) = I_H(A)$.

The following definition of a homomorphism of Brauer quotient for $N_G(H)/H$-algebra.

Definition 2.3.6. Let G be a finite group. Let H be a subgroup of G. Let A and B be two G-algebras over R. A homomorphism of $N_G(H)/H$-algebra over F is

$$\Psi(H) : A(H) \longrightarrow B(H)$$

$$\Psi(\mathrm{Br}_H^A(a)) \longrightarrow \mathrm{Br}_H^B(\Psi(a)), \quad \forall a \in A^H$$

where $\Psi : A \longrightarrow B$ is a homomorphism of G-algebras. In fact Ψ is an induce of $\Psi(H)$.

The following lemma gives us the relationship between Brauer homomorphism and relative trace map.

Lemma 2.3.8. Let G be a finite group. Let H and L be two subgroups of G where $L \leq H$. Let A be a G-algebra over R . Then for $a \in A^L$ we have

$$\mathrm{Br}_L^A(\mathrm{Tr}_L^H(a)) = \mathrm{Tr}_{1_G}^{N_H(L)/L}(\mathrm{Br}_L^A(a)).$$

2.4 Relative projective modules

In this section, we have just generalized the notion of free modules. We shall now do the same for projective modules. This will allow us to establish the first connection between FG-modules and modules for p-subgroups. We will describe the theory of vertices and sources of indecomposable modules. We followed references [5], [7], [16], [23] and [24].

The following definition of an H-projective module.

Definition 2.4.1. [16] Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. If there is an RH-module N such that:

$$M \mid \mathrm{Ind}_H^G(N).$$

This means $L \oplus M \cong \mathrm{Ind}_H^G(N)$ where L is an RG-module. Then M is said to be an H-projective RG-module or projective RG-module relative to H.

Remark 2.4.1.

- If P is a minimum p-subgroup of a finite group G and M is a P-projective RG-module
$$M \mid \mathrm{Ind}_P^G(N)$$
where N is an RP-module. Then we called P is a vertex of M denoted by $\mathrm{vx}(M) = P$. Also we called N a source of M which is denoted by $\mathrm{s}(M) = N$.

- A source of M in above is satisfies the following three conditions:
$$\text{(i) } M \mid \mathrm{Ind}_P^G(N) \quad \text{(ii) } N \mid \mathrm{Res}_P^G(M) \quad \text{(iii) } \mathrm{vx}(N) = P.$$

- If a source of M in above is equal to trivial RP-submodule R. Then we called N is a trivial source of M and we said M has a trivial source module.

- If M is an RG-module with vertex P and source RP-module N, then for any $g \in G$ and from Lemma 2.1.4 we have $g^{-1}Pg$ is a vertex of M and $N \otimes g$ is a source of M.

The following lemma gives us property of vertex.

Lemma 2.4.1. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Let N be an RH-module. Then if M is an H-projective RG-module, then $\mathrm{vx}(M) \leq_G \mathrm{vx}(N)$.

Proof. Suppose that $\mathrm{vx}(M) = P$, thus $M \mid \mathrm{Ind}_P^G(L)$ where L is an RP-module. Also, suppose that $\mathrm{vx}(N) = Q$, thus $N \mid \mathrm{Ind}_Q^H(T)$ where T is an RQ-module. Then from properties of induced module and since that $M \mid \mathrm{Ind}_H^G(N)$ we have
$$M \mid \mathrm{Ind}_H^G(N) \mid \mathrm{Ind}_H^G(\mathrm{Ind}_Q^H(T)) \cong \mathrm{Ind}_Q^G(T).$$
Hence $M \mid \mathrm{Ind}_Q^G(T)$. Thus M is an Q-projective RG-module. Hence $\mathrm{vx}(M) = P \leq_G Q = \mathrm{vx}(N)$. $\square$

The following example is about of relative projective module.

Example 2.4.1.

Consider $F = \mathbb{Z}_2$ is a field which has characteristic 2 and $G = \mathbb{V}_4$ the Kelin 4-group which has order 4. The group algebra $M = \mathbb{Z}_2\mathbb{V}_4$ is $\mathbb{V}_4$-projective module
$$\mathbb{Z}_2\mathbb{V}_4 \cong \mathrm{Ind}_{\mathbb{V}_4}^{\mathbb{V}_4}(\mathbb{Z}_2\mathbb{V}_4).$$

The following lemmas give us properties of relative projective module.

Lemma 2.4.2. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M is an H-projective RG-module if and only if there exists RH-module W which satisfies the following three conditions:
$$\text{(i) } M \mid \mathrm{Ind}_H^G(W) \quad \text{(ii) } W \mid \mathrm{Res}_H^G(M) \quad \text{(iii) } \mathrm{vx}(M) =_G \mathrm{vx}(W).$$

Lemma 2.4.3. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M is an H-projective RG-module if and only if $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.

Proof. ($\Rightarrow$) Suppose that M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. If we take $W = \mathrm{Res}_H^G(M)$, then $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.
($\Leftarrow$) Suppose that $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$. If we take $W = \mathrm{Res}_H^G(M)$ is an RH-module, then $M \mid \mathrm{Ind}_H^G(W)$. Hence M is an H-projective RG-module. $\square$

Lemma 2.4.4. Let G be a finite group. Let H and L be two subgroups of G where $H \leq L$. Let M be an H-projective RG-module. Then M is an L-projective RG-module.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. Since $H \leq L$, then from properties of induced module we have

$$\mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_L^G(\mathrm{Ind}_H^L(W)) \cong \mathrm{Ind}_L^G(N)$$

where $N = \mathrm{Ind}_H^L(W)$. Thus $M \mid \mathrm{Ind}_L^G(N)$. Hence M is an L-projective RG-module. $\square$

Lemma 2.4.5. Let G be a finite group. Let H be a subgroup of G. Let M be an H-projective RG-module. Then for some $g \in G$ we have M is H^g-projective RG-module.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. From properties of induced module and for some $g \in G$ we have

$$\mathrm{Ind}_H^G(W) \cong \mathrm{Ind}_{H^g}^G(W^g).$$

Thus $M \mid \mathrm{Ind}_{H^g}^G(W^g)$. Hence M is an H^g-projective RG-module. $\square$

Lemma 2.4.6. Let G be a finite group. Let H be a subgroup of G. Let $M \cong W_1 \oplus W_2$ be an H-projective RG-module where W_1 and W_2 be two RG-submodules of M. Then W_1 and W_2 are H-projective RG-modules.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. Also, since $M \cong W_1 \oplus W_2$, then

$$W_1 \oplus W_2 \mid \mathrm{Ind}_H^G(W).$$

Hence W_1 and W_2 are H-projective RG-modules. $\square$

Lemma 2.4.7. Let G be a finite group. Let H be a subgroup of G. Let M be an H-projective RG-module. Let N be an RG-module. Then $M \otimes_R N$ is an H-projective RG-module.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. Thus

$$M \otimes_R N \mid \mathrm{Ind}_H^G(W) \otimes_R N.$$

From properties of induced module we have

$$\mathrm{Ind}_H^G(W) \otimes_R N \cong \mathrm{Ind}_H^G(W \otimes_R \mathrm{Res}_H^G(N)).$$

Then

$$M \otimes_R N \mid \mathrm{Ind}_H^G(W \otimes_R \mathrm{Res}_H^G(N)).$$

Hence $M \otimes_R N$ is an H-projective RG-module. □

Lemma 2.4.8. Let G be a finite group. Let H be a subgroup of G. Let M be an H-projective RG-module. Let N be an RG-module. Then $\mathrm{Hom}_R(M, N)$ is an H-projective RG-module.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. Thus

$$\mathrm{Hom}_R(M, N) \mid \mathrm{Hom}_R(\mathrm{Ind}_H^G(W), N).$$

From properties of induced module we have

$$\mathrm{Hom}_R(\mathrm{Ind}_H^G(W), N) \cong \mathrm{Ind}_H^G(\mathrm{Hom}_R(W, \mathrm{Res}_H^G(N)).$$

Thus

$$\mathrm{Hom}_R(M, N) \mid \mathrm{Ind}_H^G(\mathrm{Hom}_R(W, \mathrm{Res}_H^G(N)).$$

Hence $\mathrm{Hom}_R(M, N)$ is an H-projective RG-module. □

Lemma 2.4.9. Let G be a finite group. Let H be a subgroup of G. Let M be an H-projective RG-module. Let N be an RG-module. Then $\mathrm{Hom}_R(N, M)$ is an H-projective RG-module.

Proof. Since M is an H-projective RG-module, then from Definition 2.4.1 we have

$$M \mid \mathrm{Ind}_H^G(W)$$

where W is an RH-module. Thus

$$\mathrm{Hom}_R(N, M) \mid \mathrm{Hom}_R(N, \mathrm{Ind}_H^G(W)).$$

From properties of induced module we have

$$\mathrm{Hom}_R(N, \mathrm{Ind}_H^G(W)) \cong \mathrm{Ind}_H^G(\mathrm{Hom}_R(\mathrm{Res}_H^G(N), W).$$

Thus

$$\mathrm{Hom}_R(N, M) \mid \mathrm{Ind}_H^G(\mathrm{Hom}_R(\mathrm{Res}_H^G(N), W).$$

Hence $\mathrm{Hom}_R(N, M)$ is an H-projective RG-module. $\square$

Lemma 2.4.10. *(Higman's criterion).* Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M is an H-projective RG-module if and only if 1_M lies in the image of $\mathrm{Tr}_H^G : \mathrm{End}_{RH}(M) \longrightarrow \mathrm{End}_{RG}(M)$.

Lemma 2.4.11. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G where $[G : H]$ is invertible. Then every RG-module M is H-projective RG-module. In particular, M is P-projective RG-module if $P \in Syl_p(G)$.

Proof. Since $[G : H]$ is invertible of R, then for any RG-module M we have

$$1_M = [G : H]^{-1} \mathrm{Tr}_H^G(1_M).$$

Hence from Lemma 2.4.10 we have M is H-projective RG-module. $\square$

Lemma 2.4.12. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and Q be a Sylow p-subgroup of H. If M is an H-projective RG-module, then M is a Q-projective RG-module.

Proof. Since M is an H-projective RG-module, thus there is an RH-module N such that $M \mid \mathrm{Ind}_H^G(N)$. If we take $N = \mathrm{Ind}_Q^H(W)$ where W is an RQ-module, then

$$M \mid \mathrm{Ind}_H^G(\mathrm{Ind}_Q^H(W)).$$

From properties of induced module we have

$$\mathrm{Ind}_H^G(\mathrm{Ind}_Q^H(W)) \cong \mathrm{Ind}_Q^G(W).$$

Thus

$$M \mid \mathrm{Ind}_Q^G(W).$$

Hence M is a Q-projective RG-module. $\square$

Lemma 2.4.13. Let G be a finite group. Let H be a subgroup of G. Let M be an RG-module. Then M is projective RG-module if and only if M is $\{1\}$-projective RG-module.

Proof. ($\Rightarrow$) Suppose that M is projective RG-module, then $M \mid RG$. Since $RG \cong \mathrm{Ind}_{1_G}^G(R)$, then $M \mid \mathrm{Ind}_{1_G}^G(R)$. Hence M is $\{1\}$-projective RG-module.
($\Leftarrow$) Suppose that M is $\{1\}$-projective RG-module. Thus from Lemma 2.4.3 we have $M \mid \mathrm{Ind}_{1_G}^G(\mathrm{Res}_{1_G}^G(M))$. Thus if M is free as an R-module, then $M \mid \mathrm{Ind}_{1_G}^G(R) \cong RG$. Hence M is projective RG-module. $\square$

Lemma 2.4.14. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and let P be a p-subgroup of G. Let M be an H-projective RG-module. If $M(P) = 0$ then P is not G-conjugate to a subgroup of H. In particular if W is any RH-module and $(\mathrm{Ind}_H^G(W))(P) = 0$ then P is not G-conjugate to a subgroup of H.

2.5 p-permutation modules

The aim of this section is to introduce concept of a p-permutation module and we give us the main properties about it. We followed references [3], [5] and [24].

Definition 2.5.1. [24] Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Then the p-permutation $\mathcal{O}G$-module M is a free $\mathcal{O}G$-module where the basis of it stabilized by P.

Remark 2.5.1.

- If P is a p-subgroup of G and stabilizer the basis of $\mathcal{O}G$-module M then P^g is also stabilizer the basis of M where $g \in G$.

The following examples are about of p-permutation modules.

Example 2.5.1.

Let $F = \mathbb{Z}_2$ be a field which has characteristic $p = 2$. Let $G = S_3$ be the symmetric group of three letters which has order 6. The group algebra $A = \mathbb{Z}_2 S_3$. If $H = A_3$ the alternating subgroup of S_3 which has order 3. The cosets of A_3 in S_3 are $S_3/A_3 = \{A_3, (12) \circ A_3\}$. The stabilized of S_3/A_3 is $\langle(1)\rangle$. The permutation module given by this cosets is $M = \bigoplus_{i=1}^{2} \mathbb{Z}_2 g_i H = \mathbb{Z}_2 A_3 \oplus \mathbb{Z}_2(12) \circ A_3$ which is a 1-permutation $\mathbb{Z}_2 S_3$-module.

Example 2.5.2.

Let $F = \mathbb{Z}_3$ be a field which has characteristic $p = 3$. Let $G = V_4$ be the Klein 4-group which has order 4. The group algebra $A = \mathbb{Z}_3\mathbb{V}_4$. If $H = \langle a\rangle$ the subgroup of $\mathbb{V}_4$ which has order 2. The cosets of $\langle a\rangle$ in $\mathbb{V}_4$ are $\mathbb{V}_4/\langle a\rangle = \{\langle a\rangle, b\langle a\rangle\}$. The stabilized of $\mathbb{V}_4/\langle a\rangle$ is $\langle e\rangle$. The permutation module given by this cosets is $M = \bigoplus_{i=1}^{2} \mathbb{Z}_3 g_i H = \mathbb{Z}_3\langle a\rangle \oplus \mathbb{Z}_3 b\langle a\rangle$ which is a 1-permutation $\mathbb{Z}_3\mathbb{V}_4$-module.

Example 2.5.3.

Consider $F = \mathbb{Z}_2$ is a field which has characteristic 2. Let $G = D_8$ be the dihedral group of degree 4. The group algebra $A = \mathbb{Z}_2 D_8$. If $H = \langle a\rangle$ is a subgroup of D_8 which has order 4. The cosets of $\langle a\rangle$ in D_8 are $D_8/\langle a\rangle = \{\langle a\rangle, b\langle a\rangle\}$. The stabilized of $D_8/\langle a\rangle$ is $\langle a^2\rangle$. The permutation module given by this cosets $M = \bigoplus_{i=1}^{2} \mathbb{Z}_2 g_i H = \mathbb{Z}_2\langle a\rangle \oplus \mathbb{Z}_2 b\langle a\rangle$ is a 2-permutation $\mathbb{Z}_2 D_8$-module.

Now we will introduced lemmas that give us the most important properties of the p-permutation module.

Lemma 2.5.1. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and P be a p-subgroup of G. If W is a p-permutation $\mathcal{O}H$-module. Then $\mathrm{Ind}_H^G(W)$ is a p-permutation $\mathcal{O}G$-module.

Proof. Since W is a p-permutation $\mathcal{O}H$-module then it has a basis stabilized by P. Suppose that X is a basis of W and stabilized by P. Then from properties of induced module we have $\mathrm{Ind}_H^G(X)$ is a basis for $\mathrm{Ind}_H^G(W)$ and stabilized by P. Hence $\mathrm{Ind}_H^G(W)$ is a p-permutation $\mathcal{O}G$-module. $\square$

Lemma 2.5.2. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and P be a p-subgroup of G. If M is a p-permutation $\mathcal{O}G$-module. Then $\mathrm{Res}_H^G(M)$ is a p-permutation $\mathcal{O}H$-module.

Proof. Since M is a p-permutation $\mathcal{O}G$ -module then it has a basis stabilized by P. Consider X is a basis of M and stabilized by P. Then $\mathrm{Res}_H^G(X)$ is a basis for $\mathrm{Res}_H^G(M)$ and stabilized by P. Hence $\mathrm{Res}_H^G(M)$ is a p-permutation $\mathcal{O}H$-module. $\square$

Lemma 2.5.3. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M_1 and M_2 be two p-permutation $\mathcal{O}G$ -modules. Then $M_1 \oplus M_2$ is a p -permutation $\mathcal{O}G$-module.

Proof. Since M_1 is a p-permutation $\mathcal{O}G$-module then it has a basis stabilized by P. Consider X_1 is a basis of M_1 and stabilized by P. Also since M_2 is a p-permutation $\mathcal{O}G$-module then it has a basis stabilized by P. Consider X_2 is a basis of M_2 stabilized by P. Then $X_1 \cup X_2$ is a basis for $M_1 \oplus M_2$ and stabilized by P. Hence $M_1 \oplus M_2$ is a p-permutation $\mathcal{O}G$ -module. $\square$

Lemma 2.5.4. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be an $\mathcal{O}G$-module such that $M \cong M_1 \oplus M_2 \oplus M_3 \oplus ...... \oplus M_r$ where M_i be indecomposable $\mathcal{O}G$-submodules of M. Then M is a p-permutation $\mathcal{O}G$-module if and only if each M_i are p-permutation $\mathcal{O}G$-modules.

Proof. ($\Leftarrow$) Suppose that M_i are p-permutation $\mathcal{O}G$-modules. Since $M \cong M_1 \oplus M_2 \oplus M_3 \oplus ...... \oplus M_r$ then from Lemma 2.5.3 we have M is a p-permutation $\mathcal{O}G$-module. ($\Rightarrow$) Suppose that M is a p- permutation $\mathcal{O}G$-module then it has a basis stabilized by P. Consider $X = X_1 \cup X_2 \cup \cdots \cup X_r$ is a basis of M stabilized by P where X_1 is a basis of M_1 stabilized by P, X_2 is a basis of M_2 stabilized by P, $\cdots$ and X_r is a basis of M_r stabilized by P. Hence M_1, M_2, $\cdots$ and M_r are p-permutation $\mathcal{O}G$-modules. $\square$

Lemma 2.5.5. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M_1 and M_2 be two p-permutation $\mathcal{O}G$-modules. Then $M_1 \otimes M_2$ is a p-permutation $\mathcal{O}G$-module.

Proof. Since M_1 is a p-permutation $\mathcal{O}G$-module then it has a basis stabilized by P. Consider X_1 is a basis of M_1 stabilized by P. Also since M_2 is a p-permutation $\mathcal{O}G$-module then it has a basis stabilized by P. Consider X_2 is a basis of M_2 stabilized by P. Then we have $X_1 \otimes X_2$ is a basis of $M_1 \otimes M_2$ stabilized by P. Hence $M_1 \otimes M_2$ is a p -permutation $\mathcal{O}G$-module. $\square$

An article Broué [5] gives us an equivalent condition to definition of p-permutation module. We show it in the following theorem.

Theorem 2.5.1. *Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and P be a p-subgroup of G. Let M be an indecomposable $\mathcal{O}G$-module. Then M is a p-permutation $\mathcal{O}G$-module if and only if one of the following hold:*

(i) *M isomorphic to a summand of* $\mathrm{Ind}_H^G(\mathcal{O})$.

(ii) *M has trivial source.*

Proof. ($\Rightarrow$) Suppose that M is a p-permutation $\mathcal{O}G$-module and P-projective $\mathcal{O}G$-module. Thus M is a summand of $\mathrm{Ind}_P^G(\mathrm{Res}_P^G(M))$. But from definition of $\mathrm{Res}_P^G(M)$, there exists a subgroup Q of P such that $\mathrm{Res}_P^G(M) \cong \mathrm{Ind}_Q^P(\mathcal{O})$. Thus M is a summand of $\mathrm{Ind}_P^G(\mathrm{Ind}_Q^P(\mathcal{O}))$. From properties of induced module we have $\mathrm{Ind}_P^G(\mathrm{Ind}_Q^P(\mathcal{O})) \cong \mathrm{Ind}_Q^G(\mathcal{O})$. Hence M is a summand of $\mathrm{Ind}_Q^G(\mathcal{O})$. If P is a vertex of M then $Q = P$. Thus $M \cong \mathrm{Ind}_P^G(\mathcal{O})$. Hence M has a trivial source.
($\Leftarrow$) Suppose that M isomorphic to a summand of $\mathrm{Ind}_H^G(\mathcal{O})$. Since $\mathrm{Ind}_H^G(\mathcal{O})$ is a p-permutation $\mathcal{O}G$-module and from Lemma 2.5.4 we have every summand of $\mathrm{Ind}_H^G(\mathcal{O})$ is p-permutation $\mathcal{O}G$-module. Since M isomorphic to a summand of $\mathrm{Ind}_H^G(\mathcal{O})$ then M is a p-permutation $\mathcal{O}G$-module. $\square$

Lemma 2.5.6. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be a p-permutation $\mathcal{O}G$-module. Then $M(P)$ the Brauer quotient of M with respect to P is a p-permutation $FN_G(P)/P$-module.

Definition 2.5.2. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be a p-permutation $\mathcal{O}G$-module. Then $M(P)$ has for F-basis the set $Br_P^M(C_X(P))$ Where X is a basis of M stabilized by P.

The following lemma describes vertex of p-permutation module.

Lemma 2.5.7. Let p be a fixed prime number. Let G be a finite group. Let M be an indecomposable p-permutation $\mathcal{O}G$-module. The vertex of M is a maximal p-subgroup P of G such that $M(P) \neq 0$.

The following lemma gives us equivalent condition for vertex of p-permutation module.

Lemma 2.5.8. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be a p-permutation $\mathcal{O}G$-module. Then M has a vertex P if and only if $M(P)$ is nontrivial and a projective $FN_G(P)/P$-module.

The following definition of p-permutation G-algebra.

Definition 2.5.3. Let G be a finite group. Let A be a G-algebra over F. Then A is a p-permutation G-algebra if the basis of A stabilized by P where P is any p-subgroup of G.

Example 2.5.4. The group algebra FG is a p-permutation G-algebra. Since it basis are elements of G and stabilized by any subgroup P of G. Similarly for group algebra FN where N is a normal subgroup of G.

Lemma 2.5.9. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let A be a G-algebra over F. Then $A(P)$ is a p-permutation $N_G(P)/P$-algebra over F such that $A(P) \neq 0$.

The following definition of a Brauer homomorphism of p-permutation G-algebra.

Definition 2.5.4. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let A be a p-permutation G-algebra over F. A Brauer homomorphism define as the following

$$\mathrm{Br}_P^A : A^P \longrightarrow A(P).$$

Remark 2.5.2.

Let p be a fixed prime number. Let G be a finite group. Let P and Q be two p-subgroups of G where $Q \leq P$. Let A be a p-permutation G-algebra over F. Then $\mathrm{Ker}(\mathrm{Br}_Q^A) \cap A^P \subseteq \mathrm{Ker}(\mathrm{Br}_P^A)$.

Definition 2.5.5. Let p be a fixed prime number. Let G be a finite group. Let P and Q be two p-subgroups of G where $Q \leq P$. Let A be a p-permutation G-algebra over F. A homomorphism define as the following

$$\mathrm{Br}_{P,Q}^A : \mathrm{Br}_Q^A(A^P) \longrightarrow A(P)$$

such that

$$\mathrm{Br}_{P,Q}^A(\mathrm{Br}_Q^A(a)) = \mathrm{Br}_P^A(a), \quad \forall a \in A^P.$$

is a surjective map.

Lemma 2.5.10. Let p be a fixed prime number. Let G be a finite group. Let P and Q be two p-subgroups of G where $Q \trianglelefteq P$. Let A be a p-permutation G-algebra over F. Then

$$A(Q)^P = \mathrm{Br}_Q^A(A^P) \text{ and } \mathrm{Ker}(\mathrm{Br}_{P,Q}^A) = \mathrm{Ker}(\mathrm{Br}_P^{A(Q)}).$$

2.6 Scott modules

In this section, we introduce concept of the important type of indecomposable p-permutation FG-module which called Scott module. We introduce concept of Scott coefficient. Then we give us important properties about them. We followed references [5], [11] and [16].

The following definition of important module in our research.

Definition 2.6.1. [11] Let G be a finite group. Let H be a subgroup of G. Let M be an FG-module. Then M is called a Scott FG-module with respect to H and is denoted by $\mathrm{Sc}(G, H)$ if M is the unique indecomposable summand of $\mathrm{Ind}_H^G(F)$ which contains F.

Remark 2.6.1.

- If P is a p-subgroup of G, then we called $\mathrm{Sc}(G, P)$ is a Scott module with vertex P.
- A Scott FG-module $\mathrm{Sc}(G, H)$ is an H-projective FG-module.
- From Definition 2.6.1. and from Theorem 2.5.1. we have $\mathrm{Sc}(G, P)$ is a p-permutation FG-module.
- There is another definition of a Scott module in [**?**] , Let p be a fixed prime number, Let G be a finite group. Let P be a p-subgroup of G. There exists an indecomposable p-permutation FG-module with vertex P denoted by $S_P(G, F)$, uniquely determined up to isomorphism by one of the following properties:

(i) F is isomorphic to a submodule of $S_P(G, F)$.

(ii) F is isomorphic to a quotient of $S_P(G, F)$.

The module $S_P(G, F)$ is called a Scott module of G associated to P.

The following theorem gives us some conditions of Scott module.

Theorem 2.6.1. *Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. The Scott module $M = Sc(G, P)$ satisfies one of the following :*

(i) $\mathrm{Hom}_{\mathcal{O}G}(M, \mathcal{O}) \neq 0$,

(ii) $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}, M) \neq 0$,

and we have $\mathrm{Hom}_{\mathcal{O}G}(\mathcal{O}, M) \simeq \mathrm{Hom}_{\mathcal{O}G}(M, \mathcal{O}) \simeq \mathcal{O}$.

The following lemma gives us condition of isomorphic between Scott modules.

Lemma 2.6.1. Let p be a fixed prime number. Let G be a finite group. Let H_i be subgroup of G, with $i = 1, 2$. Let P_i for $i = 1, 2$ be Sylow p-subgroup of H_i. Then

$$Sc(G, H_1) \simeq Sc(G, H_2) \Leftrightarrow P_1 =_G P_2.$$

In particular, $Sc(G, H_i) \simeq Sc(G, P_i)$.

The following definition of a Scott coefficient.

Definition 2.6.2. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be a p-permutation $\mathcal{O}G$-module. A Scott coefficient of M associated with P is the integer number of factors isomorphic to $\mathrm{Sc}(G, P)$ in a decomposition of $M/\mathfrak{p}M$ into direct sum of indecomposable modules. We denote it by $s_P(M)$ and regarded it by the following form

$$s_P(M) = \dim_F(\mathrm{Br}_P^M(M_P^G)).$$

Remark 2.6.2.

From Lemma 2.3.4 we have

$$s_P(M) = \dim_F((M(P))_{1_G}^{N_G(P)/P}).$$

The following lemmas gives us properties of the Scott coefficient.

Lemma 2.6.2. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M and L be two p-permutation $\mathcal{O}G$-modules. Then $s_P(M \oplus L) = s_P(M) + s_P(L)$.

Proof.

$$\begin{aligned} s_P(M \oplus L) &= \dim_F(((M \oplus L)(P))_{1_G}^{N_G(P)/P}) \\ &= \dim_F(((M)(P) \oplus L(P))_{1_G}^{N_G(P)/P}) \\ &= \dim_F((M(P))_{1_G}^{N_G(P)/P} \oplus (L(P))_{1_G}^{N_G(P)/P}) \\ &= \dim_F((M(P))_{1_G}^{N_G(P)/P}) + \dim_F((L(P))_{1_G}^{N_G(P)/P}) \\ &= s_P(M) + s_P(L). \end{aligned}$$

□

Lemma 2.6.3. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be a p-permutation $\mathcal{O}G$-module. Then

$$s_P(M) = s_P(\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})).$$

Proof. Since that M is a p-permutation $\mathcal{O}G$-module then it is has a basis stabilized by P. Consider X is an $\mathcal{O}$-basis of M stabilized by P. Then also since that $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ is a p-permutation $\mathcal{O}G$-module then it is has a basis stabilized by P. Consider Y is an $\mathcal{O}$-basis of $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ stabilized by P where the operations of P on X and Y are isomorphic. From Lemma 2.3.5 we have $M \times \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}) \longrightarrow \mathcal{O}$ induces to $M(P) \times \mathrm{Hom}_{\mathcal{O}}(M(P), \mathcal{O}) \longrightarrow F$. Thus it induces an $FN_G(P)/P$-homomorphism $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})(P) \longrightarrow \mathrm{Hom}_{\mathcal{O}}(M(P), \mathcal{O})$ and sends the basis $\mathrm{Br}_P^{\mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})}(C_Y(P))$ on to the basis of $Br_P^M(C_X(P))$. Hence it is an isomorphism. Then $s_P(M) = s_P(\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}))$. □

Lemma 2.6.4. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and let P be a p-subgroup of G. If P is not G-conjugate to a Sylow p-subgroup of H then $s_P(\mathrm{Ind}_H^G(\mathcal{O})) = 0$, in which case $s_P(\mathrm{Ind}_H^G(\mathcal{O})) = 1$.

Proof. Suppose that P is not G-conjugate to a Sylow p-subgroup of H then from Lemma 2.4.14 we have $(\mathrm{Ind}_H^G(\mathcal{O}))(P) = 0$. Thus

$$\begin{aligned} s_P(\mathrm{Ind}_H^G(\mathcal{O})) &= \dim_F(((\mathrm{Ind}_H^G(\mathcal{O}))(P))_{1_G}^{N_G(P)/P}) \\ &= \dim_F(0) = 0. \end{aligned}$$

□

The following lemma describes the Scott coefficient of Scott module.

Lemma 2.6.5. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Then there exists a unique indecomposable p-permutation $\mathcal{O}G$-module $\mathrm{Sc}(G,P)$ such that $s_P(\mathrm{Sc}(G,P)) \neq 0$. We have $s_P(\mathrm{Sc}(G,P)) = 1$ and $\mathrm{Sc}(G,P)$ is isomorphic to its dual.

Chapter 3

Fusion Systems and Saturated Fusion Systems

In Section 1.3, we introduce Burnside theorem and we know it is starting point of the concepts of fusion and control fusion. In this chapter, we introduce this concepts.

In first section, we shall introduce concept of fused of elements in G and fused of subgroups of G. Also, We introduce concept of control fused. Then we give us some examples about them. After that, we introduce concept of fusion system and types of subgroups in fusion system. We give us relationship between some of these types of subgroups. We give us some examples of types of subgroups in fusion system.

In second section, we introduce when fusion system become to saturated. We introduce theorem gives us equivalent condition to definition of saturated fusion system. We give us relationship between types of subgroups in saturated fusion system. We give us a relationship between saturated fusion system and Brauer indecomposable module. We followed references [3], [9], [11], [12], [13], [18], [19], [22] and [24].

Throughout this chapter, G denotes a finite group and p a prime number dividing order G. Let $(K, \mathcal{O}, F)$ be a p-modular system. Let R be $\mathcal{O}$ or F.

3.1 Fusion systems

Definition 3.1.1. Let G be a finite group. Let H be a subgroup of G. Let g_1 and g_2 be two elements in H. If $g_1 \sim_G g_2$ but $g_1 \nsim_H g_2$, then g_1 and g_2 are fused in H by G. Similarly, for subgroups of H. Let H_1 and H_2 be two subgroups of H. If $H_1 \sim_G H_2$ but $H_1 \nsim_H H_2$, then H_1 and H_2 are fused in H by G. If $g_1 \sim_G g_2$ and $g_1 \sim_H g_2$, then we called H a control fused in G.

The following examples are about fused of elements and fused of subgroups and control fused of subgroups.

Example 3.1.1.

Consider $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is the symmetric group of three letters which has order 6. In case $p = 2$ the set of all elements in S_3 has order power of 2 is $X = \{(12), (13), (23)\}$. S_3 acts on X by conjugation. In this case S_3 has not fused elements and has not control fused of subgroups. In case $p = 3$ the set of all elements in S_3 has order power of 3 is $Y = \{(123), (132)\}$. S_3 acts on Y by conjugation. If we take the alternating subgroup $H = A_3 = \{(1), (123), (132)\}$ which has order 3. Then we have a fused elements in A_3 by S_3 are (123) and (132). Also we have fused subgroups in A_3 by S_3 are $\langle(123)\rangle$ and $\langle(132)\rangle$. But in this case S_3 has not control fused of subgroup.

Example 3.1.2.

Consider $G = S_4 = \{(1), (12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432)\}$ is the symmetric group of four letters which has order 24. In case $p = 2$ the set of all elements in S_4 which has order power of 2 is $X = \{(12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)\}$. S_4 acts on X by conjugation. If we take the subgroup $H_1 = V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$ is the Klein 4-group which has order 4. Then we have fused elements in H_1 by S_4 are $(12)(34), (13)(24)$ and $(14)(23)$. Also we have fused subgroups in in H_1 by S_4 are $\langle(12)(34)\rangle, \langle(13)(24)\rangle$ and $\langle(14)(23)\rangle$. H_1 is not control fused in this case. If we take the subgroup $H_2 = \{(1), (12)(34), (1324), (1423)\}$ which has order 4. Then we have fused elements in in H_2 by S_4 are (1324) and (1423). Also we have fused subgroups in H_2 by S_4 are $\langle(1324)\rangle$ and $\langle(1423)\rangle$. H_2 is not control fused in this case. If we take the subgroup $H_3 = \{(1), (13)(24), (1234), (1432)\}$ which has order 4. Then we have fused elements in H_3 by S_4 are (1234) and (1432). Also we have fused subgroups in H_3 by S_4 are $\langle(1234)\rangle$ and $\langle(1432)\rangle$. H_3 is not control fused in this case. If we take the subgroup $H_4 = \{(1), (14)(23), (1234), (1342)\}$ which has order 4. Then we have fused elements in H_4 by S_4 are (1234) and (1342). Also we have fused subgroups in H_4 by S_4 are $\langle(1234)\rangle$ and $\langle(1342)\rangle$. H_4 is not control fused in this case. If we take the subgroup $H_5 = \{(1), (12), (12)(34), (34)\}$ which has order 4. Then we have fused elements in H_5 by S_4 are $(12), (12)(34)$ and (34). Also we have fused subgroups in H_5 by S_4 are $\langle(12)\rangle, \langle(12)(34)\rangle$ and $\langle(34)\rangle$. H_5 is not control fused in this case. If we take the subgroup $H_6 = \{(1), (13), (13)(24), (24)\}$ which has order 4. Then we have fused elements in H_6 by S_4 are $(13), (13)(24)$ and (24). Also we have fused subgroups in H_6 by S_4 are $\langle(13)\rangle, \langle(13)(24)\rangle$ and $\langle(24)\rangle$. H_6 is not control fused in this case. If we take the subgroup $H_7 = \{(1), (14), (14)(23), (23)\}$ which has order 4. Then we have fused elements in H_7 by S_4 are $(14), (14)(23)$ and (23). Also we have fused subgroups in H_7 by S_4 are $\langle(14)\rangle, \langle(14)(23)\rangle$ and $\langle(23)\rangle$. H_7 is not control fused in this case. If we take the subgroup $H_8 = \{(1), (12), (13), (23), (123), (132)\}$ which has order 6. It is a control fused in S_4 in this case $p = 2$. If we take the subgroup $H_9 = \{(1), (14), (24), (12), (124), (142)\}$ which has order 6. It is a control fused in S_4 in this case $p = 2$. If we take a subgroup $H_{10} = \{(1), (13), (14), (34), (134), (143)\}$ which has order 6. It is a control fused in S_4 in this case $p = 2$. If we take the subgroup $H_{11} = \{(1), (23), (24), (34), (234), (243)\}$ which has order 6. It is a control fused in S_4 in this case $p = 2$. If we take the subgroup $H_{12} = \{(1), (12), (12)(34), (13)(24), (14)$

$(23), (34), (1324), (1423)\}$ which has order 8. Then we have fused elements in H_{12} by S_4 are $(12)(34)$ fused with $\{(12), (34), (13)(24), (14)(23)\}$, $(12), (34)$ fused with $\{(13)(24), (14)(23)\}$. Also we have fused subgroups in in H_{12} by S_4 are $\langle(12)(34)\rangle$ fused with $\{\langle(12)\rangle, \langle(34)\rangle, \langle(13)(24)\rangle, \langle(14)(23)\rangle\}$. $\langle(12)\rangle, \langle(34)\rangle$ fused with $\{\langle(13)(24)\rangle,$ $\langle(14)(23)\rangle\}$. H_{12} is not control fused in this case. If we take the subgroup

$$H_{13} = \{(1), (13), (12)(34), (13)(24), (14)(23), (24), (1234), (1432)\}$$

which has order 8. Then we have fused elements in in H_{13} by S_4 are $(13)(24)$ fused with $\{(13), (24), (12)(34), (14)(23)\}$, $(13), (24)$ fused with $\{(12)(34), (14)(23)\}$. Also we have fused subgroups in H_{13} by S_4 are $\langle(13)(24)\rangle$ fused with $\{\langle(13)\rangle, \langle(24)\rangle, \langle(12)(34)\rangle$ $, \langle(14)(23)\rangle\}$. $\langle(13)\rangle, \langle(24)\rangle$ fused with $\{\langle(12)(34)\rangle, \langle(14)(23)\rangle\}$. H_{13} is not control fused in this case. If we take the subgroup

$$H_{14} = \{(1), (23), (12)(34), (13)(24), (14)(23), (14), (1234), (1342)\}$$

which has order 8. Then we have fused elements in H_{14} by S_4 are $(14)(23)$ fused with $\{(14), (23), (13)(24), (12)(34)\}$. $(14), (23)$ fused with $\{(13)(24), (12)(34)\}$. Also we have fused subgroups in H_{14} by S_4 are $\langle(14)(23)\rangle$ fused with $\{\langle(14)\rangle, \langle(23)\rangle, \langle(13)(24)\rangle$ $, \langle(12)(34)\rangle\}$. $\langle(14)\rangle, \langle(23)\rangle$ fused with $\{\langle(13)(24)\rangle, \langle(12)(34)\rangle\}$. H_{14} is not control fused in this case. If we take the subgroup $H_{15} = A_4 = \{(1), (12)(34), (13)(24), (14)(23)$ $, (123), (132), (124), (142), (134), (143), (234), (243)\}$ is the alternating subgroup of degree 4 which has order 12. It is a control fused in S_4 in this case. In case $p = 3$ the set of all elements in S_4 which has order power of 3 is $Y = \{(123), (132), (124), (142), (134)$ $, (143), (234), (243)\}$. S_4 acts on Y by conjugation. If we take the subgroup $N_1 = \{(1), (123), (132)\}$ which has order 3. Then we have fused elements in N_1 by S_4 are (123) and (132). Also we have fused subgroups N_1 by S_4 are $\langle(123)\rangle$ and $\langle(132)\rangle$. N_1 is not control fused in this case. If we take the subgroup $N_2 = \{(1), (124), (142)\}$ which has order 3. Then we have fused elements N_2 by S_4 are (124) and (142). Also we have fused subgroups N_2 by S_4 are $\langle(124)\rangle$ and $\langle(142)\rangle$. N_2 is not control fused in this case. If we take the subgroup $N_3 = \{(1), (134), (143)\}$ which has order 3. Then we have fused elements N_3 by S_4 are (143) and (134). Also we have fused subgroups N_3 by S_4 are $\langle(143)\rangle$ and $\langle(134)\rangle$. N_3 is not control fused in this case. If we take the subgroup $N_4 = \{(1), (234), (243)\}$ which has order 3. Then we have fused elements N_4 by S_4 are (234) and (243). Also we have fused subgroups N_4 by S_4 are $\langle(234)\rangle$ and $\langle(243)\rangle$. N_4 is not control fused in this case. If we take the subgroup

$$N_5 = \{(1), (12), (13), (23), (123), (132)\}$$

which has order 6. N_5 is control fused in this case. If we take the subgroup

$$N_6 = \{(1), (14), (24), (12), (124), (142)\}$$

which has order 6. N_6 is control fused in this case. If we take the subgroup

$$N_7 = \{(1), (13), (14), (34), (134), (143)\}$$

which has order 6. N_7 is control fused in this case. If we take a subgroup

$$N_8 = \{(1), (23), (24), (34), (234), (243)\}$$

which has order 6. N_8 is control fused in this case. If we take a subgroup $N_9 = H_{15} = A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$ is the alternating subgroup of degree 4 which has order 12. Then we have fused elements N_9 by S_4 are $(123), (142), (134), (243)$ fused with $(132), (124), (143), (234)$. Also we have fused subgroups N_9 by S_4 are $\langle(123)\rangle, \langle(142)\rangle, \langle(134)\rangle, \langle(243)\rangle$ fused with $\langle(132)\rangle, \langle(124)\rangle, \langle(143)\rangle, \langle(234)\rangle$. N_9 is not control fused in this case.

The following lemma gives us equivalent conditions for the control fusion subgroup.

Lemma 3.1.1. Let p be a fixed prime number. Let H be a subgroup of a finite group G. Then H is a control fusion in G if and only if the following two conditions are satisfied:

(i) H contains a Sylow p-subgroup of G.

(ii) If $g \in G$ and if Q is a p-subgroup of H such that $Q^g \leqslant H$, then $g = ch$ where $c \in C_G(Q)$ and $h \in H$.

Proof. ($\Rightarrow$) Suppose that H is a control fusion in G. Thus p-subgroups of H are conjugate in H. Since that p-subgroup contained in a Sylow p-subgroup P of G, then some conjugate of every p-subgroup is contained in H if and only if some conjugate of P is contained in H. This means H contains a Sylow p-subgroup of G. Also since that if $Q \leqslant H$ where Q is a p-subgroup of H, then $Q^g \leqslant H$ where $g \in G$. Then there exists $q \in Q$ and $h \in H$ such that $g^{-1}qg = h$. Thus $g = q^{-1}gh = ch$ where $c = q^{-1}g \in C_G(Q)$. Hence two conditions are satisfied.
($\Leftarrow$) Suppose that two conditions are satisfied. From (i) since that Sylow p-subgroups are conjugate and H contains it. Also from (ii) H contains a p-subgroup and conjugate of it. Hence H is a control fusion in G. □

Definition 3.1.2. Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. A fusion system $\mathcal{F} = \mathcal{F}_P(G)$ of G over P is the category whose object $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of P and whose morphism $\mathrm{Mor}_{\mathcal{F}}(Q, S)$ is the set of all group monomorphisms from Q to S induced by conjugation with elements in G :

$$\mathrm{Mor}_{\mathcal{F}}(Q, S) = \mathrm{Hom}_{\mathcal{F}}(Q, S) = \mathrm{Hom}_P(Q, S), \qquad \forall\, Q, S \leq P.$$

The following definition deals with conjugation on fusion system.

Definition 3.1.3. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. The subgroups Q and S of P are $\mathcal{F}$-conjugate if:

(i) $|Q| = |S|$.

(ii) $\mathrm{Mor}_{\mathcal{F}}(Q,S) \neq \varnothing$.

i.e., if there exists $g \in G$ such that $Q = g^{-1}Sg$. We denoted $Q^{\mathcal{F}}$ for the set of all subgroup of P which are $\mathcal{F}$-conjugate to Q.

Remark 3.1.1.

- From above definition we have all morphisms between $\mathcal{F}$-conjugate subgroup of P are isomorphisms induced by conjugation

$$\mathrm{Mor}_{\mathcal{F}}(Q,S) = \mathrm{Iso}_{\mathcal{F}}(Q,S) = \mathrm{Iso}_P(Q,S).$$

- For all $Q \leq P$ we have $\varphi \in \mathrm{Mor}_{\mathcal{F}}(Q,Q) = \mathrm{Aut}_{\mathcal{F}}(Q)$ is an automorphism.

The following definition of control fusion in fusion system $\mathcal{F}_P(G)$.

Definition 3.1.4. Let p be a fixed prime number. Let G be a finite group. Let H be a subgroup of G and P be a p-subgroup of G where $P \leq H$. Let $\mathcal{F}_P(G)$ be a fusion system of G over P. Then H is a control fusion in P if $\mathcal{F}_P(H) = \mathcal{F}_P(G)$.

We introduced the theory of Burnside in the Section 1.3. Now we will introduce and study it on fusion system.

Theorem 3.1.1. *(Burnside). Let p be a fixed prime number. Let G be a finite group. Let P be a Sylow p-subgroup of G. If P is abelian, then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$.*

Proof. Suppose that $\phi \in \mathrm{Mor}_{\mathcal{F}_P(G)}(Q,S)$ where Q,$S \leq P$. For some $g \in G$ we have $\phi(q) = g^{-1}qg$. Since P is abelian then $Q^g \leq P$. Also, if $g^{-1}pg \in P^g$ and $g^{-1}qg \in Q^g$ then

$$g^{-1}pgg^{-1}qg = g^{-1}pqg = g^{-1}qpg = g^{-1}qgg^{-1}pg.$$

Hence P^g centralizes Q^g. Then P and P^g are Sylow p-subgroup of $C_G(Q^g)$. Thus there exists $x \in C_G(Q^g)$ such that $P = P^{gx}$ this mean $gx \in N_G(P)$. For $y \in Q$ satisfy $y^{gx} = x^{-1}g^{-1}ygx = g^{-1}yg = \phi(y)$. Hence $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$. □

The following example shows the importance of the abelian condition for the Sylow p-subgroup to become control fusion.

Example 3.1.3. Consider $G = S_4$ is the symmetric group of four letters which has order 24. The non abelian Sylow 2-subgroup of S_4 is $P = \langle (1234), (12)(34) \rangle = \{(1), (1234), (12)(34), (13), (14)(23), (24), (1432), (13)(24)\}$. It is has order 8 and $N_{S_4}(P) = P$. Then we have $(13)(24) \sim_{S_4} (12)(34)$ but $(13)(24) \nsim_P (12)(34)$. Hence $N(P)$ is not control fusion in P.

Definition 3.1.5. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. A fully normalized subgroup Q in $\mathcal{F}_P(G)$ satisfy $|N_P(Q)| \geq |N_P(S)|$ for all $S \in Q^{\mathcal{F}}$.

Definition 3.1.6. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. A subgroup Q in $\mathcal{F}_P(G)$ is called fully centralized if $|C_P(Q)| \geq |C_P(S)|$ for all $S \in Q^{\mathcal{F}}$.

Definition 3.1.7. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. A fully automized subgroup Q in $\mathcal{F}_P(G)$ satisfy $\mathrm{Aut}_P(Q) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.

Definition 3.1.8. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. A receptive subgroup Q in $\mathcal{F}_P(G)$ satisfy for every morphisms ϕ from S to Q where $S \leq Q^{\mathcal{F}}$ such that: $\phi(S)$ is fully normalized in $\mathcal{F}_P(G)$, extends to a morphism ψ from N_ϕ to P where $N_\phi = \{y \in N_P(S) : \exists z \in N_P(\phi(S)) \text{ such that } \phi(y^{-1}uy) = z^{-1}\phi(u)z, \forall u \in S\}$.

Remark 3.1.2.

- For N_ϕ in above definition we have $C_P(S) \leq N_\phi$ and $SC_P(S) \trianglelefteq N_\phi \trianglelefteq N_P(S)$.
- For ψ and ϕ in above definition we have $\psi|_S = \phi$.
- If subgroup in $\mathcal{F}_P(G)$ is fully normalized and fully centralized, then it is must be fully automized subgroup.

The following lemmas give us equivalent condition to fully normalized subgroup and fully centralized subgroup when we take fusion system over a Sylow p-subgroup of G.

Lemma 3.1.2. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a Sylow p-subgroup of G. Let Q be a subgroup of P. Then Q is fully centralized subgroup in $\mathcal{F}_P(G)$ if and only if $C_P(Q) \in Syl_p(C_G(Q))$.

Proof. Suppose that $S \in Syl_p(C_G(Q))$ such that $C_P(Q) \leq S$. From Sylow's theorem we have there exists $g \in G$ such that $(QS)^g \leq P$ and we have $Q \cong Q^g$. Suppose that $x \in S^g$ and $g^{-1}xg \in C_G(Q)$ which implies that $g^{-1}xgyg^{-1}x^{-1}g = y \Leftrightarrow x(gyg^{-1})x^{-1} = gyg^{-1}$ for all $y \in Q$. Hence $S \leq C_G(Q^g) \cap P = C_P(Q^g)$. Thus $|C_P(Q)| \leq |S| \leq |C_P(Q^g)|$. Hence Q is fully centralized subgroup if and only if $|S| = |C_P(Q)|$ means Q is fully centralized subgroup if and only if $C_P(Q) \in Syl_p(C_G(Q))$. □

Lemma 3.1.3. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a Sylow p-subgroup of G. Let Q be a subgroup of P. Then Q is fully normalized subgroup in $\mathcal{F}_P(G)$ if and only if $N_P(Q) \in Syl_p(N_G(Q))$.

Proof. Suppose that $S \in Syl_p(N_G(Q))$ such that $N_P(Q) \leq S$. From Sylow's theorem we have there exists $g \in G$ such that $(QS)^g \leq P$ and we have $Q \cong Q^g$. Suppose that $x \in S^g$ and $g^{-1}xg \in N_G(Q)$ which implies that $g^{-1}xgyg^{-1}x^{-1}g = y \Leftrightarrow x(gyg^{-1})x^{-1} = gyg^{-1}$ for all $y \in Q$. Hence $S \leq N_G(Q^g) \cap P = N_P(Q^g)$. Thus $|N_P(Q)| \leq |S| \leq |N_P(Q^g)|$. Hence Q is fully normalized subgroup if and only if $|S| = |N_P(Q)|$ means Q is fully normalized subgroup if and only if $N_P(Q) \in Syl_p(N_G(Q))$. □

The following examples are about of fully normalized subgroups and fully centralized subgroups.

Example 3.1.4.

Consider $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is the symmetric group of three letters which has order 6. In case $p = 2$ we take $P = \langle(12)\rangle$ is the Sylow 2-subgroup of S_3 which has order 2. The fusion system of S_3 over $\langle(12)\rangle$ has one subgroup of P is $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$. The centralizer of $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ in G is $C_G(Q) = G = S_3$. The centralizer of $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ in P is $C_P(Q) = P = \langle(12)\rangle$. Since $\langle(12)\rangle \in Syl_2(S_3)$, then from Lemma 3.1.2 we have $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ is fully centralized subgroup in $\mathcal{F}_{\langle(12)\rangle}(S_3)$. The normalizer of $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ in G is $N_G(Q) = G = S_3$. The normalizer of $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ in P is $N_P(Q) = P = \langle(12)\rangle$. Since $\langle(12)\rangle \in Syl_2(S_3)$, then from Lemma 3.1.3 we have $Q = \left\langle 1_{\langle(12)\rangle}\right\rangle$ is fully normalized subgroup in $\mathcal{F}_{\langle(12)\rangle}(S_3)$. In case $p = 3$ we take $P = A_3$ is the alternating Sylow 3-subgroup of S_3 which has order 3. The fusion system of S_3 over A_3 has one subgroup is $S = \langle 1_{A_3}\rangle$. S is fully centralized and fully normalized subgroup in $\mathcal{F}_{A_3}(S_3)$ because the centralizer of $S = \langle 1_{A_3}\rangle$ in G is $C_G(S) = G = S_3$. The centralizer of $S = \langle 1_{A_3}\rangle$ in P is $C_P(S) = P = A_3$. Since $A_3 \in Syl_3(S_3)$, then from Lemma 3.1.2 we have $S = \langle 1_{A_3}\rangle$ is fully centralized subgroup in $\mathcal{F}_{A_3}(S_3)$. The normalizer of $S = \langle 1_{A_3}\rangle$ in G is $N_G(S) = G = S_3$. The normalizer of $S = \langle 1_{A_3}\rangle$ in P is $N_P(S) = P = A_3$. Since $A_3 \in Syl_3(S_3)$, then from Lemma 3.1.3 we have $S = \langle 1_{A_3}\rangle$ is fully normalized subgroup in $\mathcal{F}_{A_3}(S_3)$.

Example 3.1.5. Consider $G = A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132)$ $, (124), (142), (134), (143), (234), (243)\}$ is the alternating group of degree 4 which has order 12. In case $p = 2$ we take $P = V_4$ is the Klein 4-group and Sylow 2-subgroup of A_4 which has order 4. The fusion system of A_4 over V_4 has four subgroups in $\mathrm{Ob}(\mathcal{F}_{V_4}(A_4))$ are $Q_1 = \langle 1_{V_4}\rangle$, $Q_2 = \langle(12)(34)\rangle$, $Q_3 = \langle(13)(24)\rangle$ and $Q_4 = \langle(14)(23)\rangle$. This subgroups are fully normalized and fully centralized and fully subgroups in $\mathcal{F}_{V_4}(A_4)$ because the centralizer of the subgroup $Q_1 = \langle 1_{V_4}\rangle$ in G is $C_G(Q_1) = G = A_4$. The centralizer of Q_1 in P is $C_P(Q_1) = P = V_4$. Since $V_4 \in Syl_2(A_4)$, then from Lemma 3.1.2 we have $Q_1 = \langle 1_{V_4}\rangle$ is fully centralized subgroup in $\mathcal{F}_{V_4}(A_4)$. The normalizer of $Q_1 = \langle 1_{V_4}\rangle$ in G is $N_G(Q_1) = G = A_4$. The normalizer of Q_1 in P is $N_P(Q_1) = P = V_4$. Since $V_4 \in Syl_2(A_4)$, then from Lemma 3.1.3 we have $Q_1 = \langle 1_{V_4}\rangle$ is fully normalized subgroup in $\mathcal{F}_{V_4}(A_4)$. The centralizer of the subgroup $Q_2 = \langle(12)(34)\rangle$ in G is $C_G(Q_2) = P = V_4$. The centralizer of Q_2 in P is $C_P(Q_2) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.2 we have $Q_2 = \langle(12)(34)\rangle$ is fully centralized subgroup in $\mathcal{F}_{V_4}(A_4)$. The normalizer of $Q_2 = \langle(12)(34)\rangle$ in G is $N_G(Q_2) = P = V_4$. The normalizer of Q_2 in P is $N_P(Q_2) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.3 we have $Q_2 = \langle(12)(34)\rangle$ is fully normalized subgroup in $\mathcal{F}_{V_4}(A_4)$. The centralizer of the subgroup $Q_3 = \langle(13)(24)\rangle$ in G is $C_G(Q_3) = P = V_4$. The centralizer of Q_3 in P is $C_P(Q_3) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.2 we have $Q_3 = \langle(13)(24)\rangle$ is the fully centralized subgroup in $\mathcal{F}_{V_4}(A_4)$. The normalizer of $Q_3 = \langle(13)(24)\rangle$ in G is $N_G(Q_3) = P = V_4$. The normalizer of Q_3 in P is $N_P(Q_3) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.3 we have $Q_3 = \langle(13)(24)\rangle$ is fully normalized subgroup in $\mathcal{F}_{V_4}(A_4)$. The centralizer of the subgroup

$Q_4 = \langle(14)(23)\rangle$ in G is $C_G(Q_4) = P = V_4$. The centralizer of Q_4 in P is $C_P(Q_4) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.2 we have $Q_4 = \langle(14)(23)\rangle$ is fully centralized subgroup in $\mathcal{F}_{V_4}(A_4)$. The normalizer of $Q_4 = \langle(14)(23)\rangle$ in G is $N_G(Q_4) = P = V_4$. The normalizer of Q_4 in P is $N_P(Q_4) = P = V_4$. Since $V_4 \in Syl_2(V_4)$, then from Lemma 3.1.3 we have $Q_4 = \langle(14)(23)\rangle$ is fully normalized subgroup in $\mathcal{F}_{V_4}(A_4)$. In case $p = 3$ we take $P = \langle(123)\rangle$ is a Sylow 3-subgroup of A_4 which has order 3. The fusion system of A_4 over $P = \langle(123)\rangle$ has one subgroup in $\mathrm{Ob}(\mathcal{F}_{\langle(123)\rangle}(A_4))$ is $S = \left\langle 1_{\langle(123)\rangle}\right\rangle$. The centralizer of the subgroup $S = \left\langle 1_{\langle(123)\rangle}\right\rangle$ in G is $C_G(S) = G = A_4$. The centralizer of S in P is $C_P(S) = P = \langle(123)\rangle$. Since $\langle(123)\rangle \in Syl_3(A_4)$, then from Lemma 3.1.2 we have $S = \left\langle 1_{\langle(123)\rangle}\right\rangle$ is fully centralized subgroup in $\mathcal{F}_{\langle(123)\rangle}(A_4)$. The normalizer of $S = \left\langle 1_{\langle(123)\rangle}\right\rangle$ in G is $N_G(S) = G = A_4$. The normalizer of S in P is $N_P(S) = P = \langle(123)\rangle$. Since $\langle(123)\rangle \in Syl_3(A_4)$, then from Lemma 3.1.3 we have $S = \left\langle 1_{\langle(123)\rangle}\right\rangle$ is fully normalized subgroup in $\mathcal{F}_{\langle(123)\rangle}(A_4)$.

The following lemma gives us relationship between receptive subgroup and fully centralized subgroup in $\mathcal{F}_P(G)$ when P is a p-subgroup of G.

Lemma 3.1.4. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. Then all $Q \leq P$ which is receptive subgroup in $\mathcal{F}_P(G)$ is fully centralized subgroup in $\mathcal{F}_P(G)$.

Proof. Suppose that $Q \leq P$ receptive subgroup in $\mathcal{F}_P(G)$ and $S \in Q^{\mathcal{F}}$. Also, suppose that $\phi \in \mathrm{Iso}_{\mathcal{F}}(S, Q)$. Since Q is receptive subgroup in $\mathcal{F}_P(G)$. Then ϕ extension to ψ from N_ϕ to P. Since that $C_P(S) \leq N_\phi$ and since that ψ is the homomorphism, then $\psi(C_P(S)) \leq C_P(\psi(S)) = C_P(Q)$. Thus $|C_P(S)| \leq |C_P(Q)|$. Since this holds for all $S \in Q^{\mathcal{F}}$. Hence Q is a fully centralized subgroup in $\mathcal{F}_P(G)$. $\square$

The following lemma gives us a relationship between receptive subgroup, fully automized subgroup and fully normalized subgroup in $\mathcal{F}_P(G)$ when P is a p-subgroup of G.

Lemma 3.1.5. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. Then all $Q \leq P$ which is fully automized and receptive subgroups in $\mathcal{F}_P(G)$ is fully normalized subgroup in $\mathcal{F}_P(G)$.

Proof. Suppose that Q is a fully automized and receptive subgroup in $\mathcal{F}_P(G)$ and $S \in Q^{\mathcal{F}}$. Since that Q is receptive subgroup in $\mathcal{F}_P(G)$, then from Lemma 3.1.4 it is fully centralized subgroup $|C_P(S)| \leq |C_P(Q)|$. Since Q is a fully automized subgroup, then $\mathrm{Aut}_P(Q) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. So $\mathrm{Aut}_P(S) \leq \mathrm{Aut}_P(Q)$. Thus $|\mathrm{Aut}_P(S)| \leq |\mathrm{Aut}_P(Q)|$. Since that $\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$ and $\mathrm{Aut}_P(S) \cong N_P(S)/C_P(S)$, thus

$$|\mathrm{Aut}_P(S)| = |N_P(S)/C_P(S)| \leq |\mathrm{Aut}_P(Q)| = |N_P(Q)/C_P(Q)|.$$

Then

$$|N_P(S)| = |\mathrm{Aut}_P(S)|.|C_P(S)| \leq |\mathrm{Aut}_P(Q)|.|C_P(Q)| = |N_P(Q)|.$$

Since this holds for all $S \in Q^{\mathcal{F}}$. Hence Q is a fully normalized subgroup in $\mathcal{F}_P(G)$. $\square$

The following lemma gives us a relationship between fully normalized subgroup and receptive subgroup in $\mathcal{F}_P(G)$ when P is a Sylow p-subgroup of G.

Lemma 3.1.6. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a Sylow p-subgroup of G. If $Q \leq P$ and satisfy $N_P(Q) \in Syl_p(N_G(Q))$ then Q is receptive subgroup in $\mathcal{F}_P(G)$.

Proof. Suppose that $Q \leq P$ and satisfy $N_P(Q) \in Syl_p(N_G(Q))$ then from Lemma 3.1.3 we have Q is a fully normalized subgroup in $\mathcal{F}_P(G)$. Also, suppose that $\phi \in \mathrm{Iso}_{\mathcal{F}}(S, Q)$ where $S \in Q^{\mathcal{F}}$ and $N_\phi = \{y \in N_P(S) : \exists z \in N_P(\phi(S))$ such that $\phi(y^{-1}uy)$ $= z^{-1}\phi(u)z, \forall u \in S\}$. Thus for conjugation map we have $f_{y^{-1}} \circ \phi^{-1} \circ f_z \circ \phi$ centralizes S and $f_z \circ \phi \circ f_{y^{-1}} \circ \phi^{-1}$ centralizes $\phi(S) = Q$. Then $\phi \circ f_y = \overline{\phi} \circ \phi$ where $\overline{\phi}$ is induced by conjugation with some element $g \in C_G(Q)$. Hence $\phi(N_\phi) \leqslant N_P(Q)C_G(Q)$. Since $N_P(Q) \in Syl_p(N_G(Q))$ and N_ϕ is a p-group then we have morphism $\psi \in \mathrm{Mor}_{\mathcal{F}_P(G)}$ induced by $q \in C_G(Q)$ such $\psi(\phi(N_\phi)) \leqslant N_P(Q)$. Hence we can extension ψ to θ where $\theta = \psi \circ \phi$ such that $\theta : N_\phi \longrightarrow N_P(Q)$. Then Q is receptive subgroup in $\mathcal{F}_P(G)$. □

3.2 Saturated fusion systems

Definition 3.2.1. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. A subgroup Q in $\mathcal{F}_P(G)$ called saturated if it is fully automized and receptive subgroup in $\mathcal{F}_P(G)$.

Example 3.2.1.

Consider $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is the symmetric group of three letters which has order 6. In case $p = 3$ if we take $P = A_3$ is the alternating Sylow 3-subgroup of S_3 which has order 3. From Example 3.2.3. we have $\langle 1_{A_3} \rangle$ is a fully centralized, fully normalized and fully automized subgroup in $\mathcal{F}_{A_3}(S_3)$. Since $A_3 \in Syl_3(S_3)$, $\langle 1_{A_3} \rangle \leqslant A_3$ and $\langle 1_{A_3} \rangle$ is a fully normalized, then from Lemma 3.1.6 we have $\langle 1_{A_3} \rangle$ is the receptive subgroup in $\mathcal{F}_{A_3}(S_3)$. Hence $\langle 1_{A_3} \rangle$ is the saturated subgroup in $\mathcal{F}_{A_3}(S_3)$.

Definition 3.2.2. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. Then $\mathcal{F}_P(G)$ is a saturated fusion system if each subgroup in $\mathcal{F}_P(G)$ is $\mathcal{F}$-conjugate to a saturated subgroup in $\mathcal{F}_P(G)$.

The following theorem describes the fusion system over a Sylow p-subgroup.

Theorem 3.2.1. *(Puig). Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a Sylow p-subgroup of G. Then $\mathcal{F}_P(G)$ is a saturated fusion system.*

Proof. Since P is a Sylow p-subgroup of G, then from properties of subgroup of Sylow p-subgroup we have each subgroup Q of P is G-conjugate to a subgroup $S \leq P$ such that $N_P(S) \in Syl_p(N_G(S))$ and $\mathrm{Aut}_P(S) \in Syl_p(\mathrm{Aut}_G(S))$. Hence each subgroup S of P is fully automized subgroup in $\mathcal{F}_P(G)$. Since each subgroup S of P is satisfy $N_P(S) \in Syl_p(N_G(S))$ thus from Lemma 3.1.6 each subgroup is receptive subgroup in $\mathcal{F}_P(G)$. Hence each subgroup S of P is saturated subgroup. Then $\mathcal{F}_P(G)$ is a saturated fusion system. □

Example 3.2.2. The fusion system of the symmetric group S_3 over A_3 is a saturated fusion system.

Example 3.2.3. The fusion system of the alternating group A_4 over V_4 is a saturated fusion system.

The following lemmas give us the relationship between subgroups of the saturated fusion system.

Lemma 3.2.1. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a saturated fusion system of G over P where G a finite group and P a p-subgroup of G. Then $Q \leq P$ is receptive subgroup in $\mathcal{F}_P(G)$ if and only if it is fully centralized subgroup in $\mathcal{F}_P(G)$.

Lemma 3.2.2. Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a saturated fusion system of G over P where G a finite group and P a p-subgroup of G. Then $Q \leq P$ is fully automized and receptive subgroup in $\mathcal{F}_P(G)$ if and only if it is fully normalized subgroup in $\mathcal{F}_P(G)$.

Proof. ($\Rightarrow$) Suppose that $Q \leq P$ is fully automized and receptive subgroup in $\mathcal{F}_P(G)$ then Q is fully normalized in $\mathcal{F}_P(G)$ from Lemma 3.1.5.
($\Leftarrow$) Suppose that $Q \leq P$ is fully normalized in $\mathcal{F}_P(G)$. Since $\mathcal{F}_P(G)$ is a saturated fusion system then there exists $S \leq P$ is fully automized and receptive subgroup in $\mathcal{F}_P(G)$. Then from Lemma 3.1.5 we have S is fully normalized in $\mathcal{F}_P(G)$. Since Q is also fully normalized in $\mathcal{F}_P(G)$ then $|N_P(Q)| = |N_P(S)|$. .Thus

$$|N_P(Q)| = |\mathrm{Aut}_P(Q)||C_P(Q)| = |\mathrm{Aut}_P(S)||C_P(S)| = |N_P(S)|.$$

Since S is fully automized and receptive subgroup then $|\mathrm{Aut}_P(S)| \geq |\mathrm{Aut}_P(Q)|$ and $|C_P(S)| \geq |C_P(Q)|$. Thus we must have $|\mathrm{Aut}_P(S)| = |\mathrm{Aut}_P(Q)|$ and $|C_P(S)| = |C_P(Q)|$. Hence Q is fully automized and fully centralized. Since Q is fully centralized then from Lemma 3.2.1 we have Q is receptive. □

We get the following result from the previous lemmas.

Corollary 3.2.1. *Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. Then $\mathcal{F}_P(G)$ is a saturated fusion system if and only if the following two conditions hold:*

(i) *Every fully normalized subgroup in $\mathcal{F}_P(G)$ is fully automized subgroup in $\mathcal{F}_P(G)$ and fully centralized subgroup in $\mathcal{F}_P(G)$ and*

(ii) *Every fully centralized subgroup in $\mathcal{F}_P(G)$ is receptive subgroup in $\mathcal{F}_P(G)$.*

The following definition gives us equivalent conditions to definition of saturated fusion system.

Definition 3.2.3. Let p be a fixed prime number. Let G a finite group. Let P be a p-subgroup of G. The fusion system $\mathcal{F}_P(G)$ of G over P is saturated fusion system if the following two conditions are satisfied:

(i) P is fully normalized subgroup in $\mathcal{F}_P(G)$.

(ii) For every subgroup Q of P, if Q is fully normalized subgroup in $\mathcal{F}_P(G)$, then Q is receptive subgroup in $\mathcal{F}_P(G)$.

The following theorem gives us equivalent conditions to definition of saturated fusion system.

Theorem 3.2.2. *Let p be a fixed prime number. Let $\mathcal{F}_P(G)$ be a fusion system of G over P where G a finite group and P a p-subgroup of G. Then $\mathcal{F}_P(G)$ is saturated fusion system if and only if*

(i) *P is fully automized in $\mathcal{F}_P(G)$ and*

(ii) *Every subgroup in $\mathcal{F}_P(G)$ is $\mathcal{F}$-conjugate to a fully normalized subgroup is receptive.*

We introduce Brauer indecomposable module in Section 2.3. Now we will introduce in this section the important relationship between Brauer indecomposable module and saturated fusion system.

Theorem 3.2.3. *Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let M be an indecomposable p-permutation FG-module with vertex P. If M is Brauer indecomposable FG-module then $\mathcal{F}_P(G)$ is a saturated fusion system of G over P.*

The converse of the above theorem hold in special cases of G, M and P. We will study it in the following theorems.

Theorem 3.2.4. *Let p be a fixed prime number. Let G be a finite group. Let P be an abelian p-subgroup of G. Let $\mathrm{Sc}(G,P)$ be a Scott FG-module with vertex P. If $\mathcal{F}_P(G)$ is saturated fusion system of G over P then $\mathrm{Sc}(G,P)$ is Brauer indecomposable FG-module.*

Theorem 3.2.5. *Let p be a fixed prime number. Let G be a finite group. Let P be a p-subgroup of G. Let $\mathrm{Sc}(G,P)$ be a Scott FG-module with vertex P. Let $\mathcal{F}_P(G)$ be a saturated fusion system of G over P. Then the following conditions are equivalent:*

(i) *$\mathrm{Sc}(G,P)$ is Brauer indecomposable FG-module.*

(ii) *For all fully normalized subgroup Q of P we have*

$$\mathrm{Res}^{N_G(Q)}_{QC_G(Q)}\mathrm{Sc}(N_G(Q), N_P(Q))$$

is indecomposable $FQC_G(Q)$-module.

Remark 3.2.1.

- If the conditions in above theorem are satisfies then

$$\mathrm{Sc}(G, P)(Q) \cong \mathrm{Sc}(N_G(Q), N_P(Q))$$

  for all fully normalized subgroup Q of P.

- $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)}\mathrm{Sc}(N_G(Q), N_P(Q))$ in above theorem is indecomposable if the following conditions are satisfying:

(i) $N_P(Q) \in Syl_p(H)$ where $H \leq N_G(Q)$.

(ii) $[N_G(Q) : H] = p^a$ where $a \geq 0$.

Theorem 3.2.6. *Let p be a fixed prime number. Let G be a finite p-group. Let P be a p-subgroup of G. Let* $\mathrm{Sc}(G, P)$ *be a Scott FG-module with vertex P. If $\mathcal{F}_P(G)$ is saturated fusion system of G over P then* $\mathrm{Sc}(G, P)$ *is Brauer indecomposable FG-module.*

The following corollary, we obtain from above theorem.

Corollary 3.2.2. *Let p be a fixed prime number. Let G be a finite group has cyclic Sylow p-subgroup. Let P be a p-subgroup of G. Then the Scott FG-module* $\mathrm{Sc}(G, P)$ *with vertex P is Brauer indecomposable FG-module.*

Chapter 4

Tensor Product of Related Objects

In this chapter, we will study the exterior tensor product of three algebra structures, Brauer indecomposable module where it is FG-module has Brauer quotient is indecomposable or zero, Scott module where it is a unique indecomposable summand of induced F which contains F and fusion system where it is a category has object is the set of all p-subgroups and morphisms is the set of all group homomorphism induced by conjugation.

4.1 Tensor product of Brauer indecomposable modules

Throughout this section, G denotes a finite group, p a prime number and F an algebraically closed field of characteristic p. We followed the references [1], [4] and [10].

We introduce in Section 1.1 equivalent condition to definition of local F-algebra. Now we introduce in the following theorem the tensor product of two local F-algebras is local F-algebra.

Theorem 4.1.1. *Let A_i for $i = 1, 2$ be local F-algebra then $A_1 \otimes_F A_2$ is also local F-algebra.*

Proof. Since A_1 and A_2 are local F-algebras then from Lemma 1.1.1 we have

$$\frac{A_1}{J(A_1)} \cong F \quad \text{and} \quad \frac{A_2}{J(A_2)} \cong F.$$

If we take a map

$$\psi : A_1 \otimes_F A_2 \longrightarrow \frac{A_1}{J(A_1)} \otimes_F \frac{A_2}{J(A_2)}$$

such that

$$\psi(a_1 \otimes_F a_2) = (a_1 + J(A_1)) \otimes_F (a_2 + J(A_2))$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. ψ is F-epimorphism and it has a kernel

$$\ker\psi = (J(A_1) \otimes_F A_2) + (A_1 \otimes_F J(A_2)).$$

Thus from the first isomorphism theorem we have

$$\frac{(A_1 \otimes_F A_2)}{(J(A_1) \otimes_F A_2) + (A_1 \otimes_F J(A_2))} \cong (\frac{A_1}{J(A_1)}) \otimes_F (\frac{A_2}{J(A_2)}).$$

Since

$$(J(A_1) \otimes_F A_2) + (A_1 \otimes_F J(A_2)) \cong J(A_1 \otimes_F A_2).$$

Then

$$\begin{aligned}
\frac{(A_1 \otimes_F A_2)}{J(A_1 \otimes_F A_2)} &\cong (\frac{A_1}{J(A_1)}) \otimes_F (\frac{A_2}{J(A_2)}) \\
&\cong F \otimes_F F \\
&\cong F.
\end{aligned}$$

Hence $A_1 \otimes_F A_2$ is local F-algebra. □

Now we will show in the following theorem that the endomorphism over an algebraically closed field of tensor product of two modules is the tensor product of their endomorphisms.

Theorem 4.1.2. *Let A_i for $i = 1, 2$ be F-algebra. Let M_i be indecomposable A_i-module, with $i = 1, 2$. Then*

$$\mathrm{End}_{A_1 \otimes_F A_2}(M_1 \otimes_F M_2) \cong \mathrm{End}_{A_1}(M_1) \otimes_F \mathrm{End}_{A_2}(M_2).$$

Proof. Define a map

$$\rho : \mathrm{End}_{A_1}(M_1) \otimes_F \mathrm{End}_{A_2}(M_2) \longrightarrow \mathrm{End}_{A_1 \otimes_F A_2}(M_1 \otimes_F M_2)$$

$$\rho(\Phi \otimes_F \psi)(m_1, m_2) = \Phi(m_1) \otimes_F \psi(m_2)$$

for all $m_1 \in M_1$, $m_2 \in M_2$, $\Phi \in \mathrm{End}_{A_1}(M_1)$ and all $\psi \in \mathrm{End}_{A_2}(M_2)$. ρ is isomorphism to show it, suppose that $\Phi_1, \Phi_2 \in \mathrm{End}_{A_1}(M_1)$, $\psi_1, \psi_2 \in \mathrm{End}_{A_2}(M_2)$, $m_1 \in M_1$ and $m_2 \in M_2$ thus

$$\begin{aligned}
\rho((\Phi_1 \otimes_F \psi_1)(\Phi_2 \otimes_F \psi_2))(m_1, m_2) &= \rho(\Phi_1\Phi_2 \otimes_F \psi_1\psi_2)(m_1, m_2) \\
&= (\Phi_1\Phi_2)(m_1) \otimes_F (\psi_1\psi_2)(m_2) \\
&= \Phi_1(m_1)\Phi_2(m_1) \otimes_F \psi_1(m_2)\psi_2(m_2) \\
&= (\Phi_1(m_1) \otimes_F \psi_1(m_2))(\Phi_2(m_1) \otimes_F \psi_2(m_2)) \\
&= \rho(\Phi_1 \otimes_F \psi_1)(m_1, m_2)\rho(\Phi_2 \otimes_F \psi_2)(m_1, m_2).
\end{aligned}$$

Hence ρ is a homomorphism. Now we will show ρ is one to one. Since

$$\begin{aligned}
\ker(\rho) &= \{\Phi \otimes_F \psi \in \mathrm{End}A_1(M_1) \otimes_F \mathrm{End}_{A_2}(M_2) : \rho((\Phi \otimes_F \psi)(m_1, m_2)) = 0\} \\
&= \{\Phi \otimes_F \psi \in \mathrm{End}_{A_1}(M_1) \otimes_F \mathrm{End}_{A_2}(M_2) : \Phi(m_1) \otimes_F \psi(m_2) = 0\} \\
&= \{\Phi \otimes_F \psi = 0\} = \{0\}.
\end{aligned}$$

Hence ρ is one to one. Also ρ is onto we will show it, suppose that $\Theta \in \mathrm{End}_{A_1\otimes_F A_2}(M_1 \otimes_F M_2)$ where $\Theta(m_1 \otimes_F m_2) = n_1 \otimes_F n_2$ for all $m_1, n_1 \in M_1$ and $m_2, n_2 \in M_2$. So we define $\Phi \in \mathrm{End}_{A_1}(M_1)$ where $\Phi(m_1) = n_1$. Also we define $\psi \in \mathrm{End}_{A_2}(M_2)$ where $\psi(m_2) = n_2$. Thus $\rho(\Phi \otimes_F \psi)(m_1, m_2) = \Phi(m_1) \otimes_F \psi(m_2) = n_1 \otimes_F n_2$. Then ρ is onto. Hence ρ is isomorphism. Then $\mathrm{End}_{A_1\otimes_F A_2}(M_1 \otimes_F M_2) \cong \mathrm{End}_{A_1}(M_1) \otimes_F \mathrm{End}_{A_2}(M_2)$. □

Now we will show that the tensor product of two indecomposable modules is indecomposable module.

Corollary 4.1.1. *Let A_i be F-algebra, with $i = 1, 2$. Let M_i for $i = 1, 2$ be A_i-module. If M_1 and M_2 are indecomposable modules then $M_1 \otimes_F M_2$ is indecomposable $A_1 \otimes_F A_2$-module.*

Proof. Suppose that M_1 and M_2 are indecomposable modules then from Corollary 1.1.1 we have $\mathrm{End}_{A_1}(M_1)$ and $\mathrm{End}_{A_2}(M_2)$ are local F-algebras. But from Theorem 4.1.2 we have

$$\mathrm{End}_{A_1}(M_1) \otimes_F \mathrm{End}_{A_2}(M_2) \cong \mathrm{End}_{A_1\otimes_F A_2}(M_1 \otimes_F M_2).$$

Then from Theorem 4.1.1 we have $\mathrm{End}_{A_1\otimes_F A_2}(M_1 \otimes_F M_2)$ is local F-algebra. So from Corollary 1.1.1 we have $M_1 \otimes_F M_2$ is indecomposable $A_1 \otimes_F A_2$-module. □

The Brauer map for the exterior tensor product of two FG-modules can be expressed as a tensor product of their Brauer map.

Theorem 4.1.3. *Let p be a fixed prime number. Let G_1 and G_2 be finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let M_i be FG_i-module for $i = 1, 2$. Then*

$$\mathrm{Br}^{M_1\otimes_F M_2}_{P_1\times P_2} \cong \mathrm{Br}^{M_1}_{P_1} \otimes_F \mathrm{Br}^{M_2}_{P_2}.$$

Corollary 4.1.2. *Let p be a fixed prime number. Let G_1 and G_2 be two finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let M_i be FG_i-module for $i = 1, 2$. Then*

$$(M_1 \otimes_F M_2)(P_1 \times P_2) \cong M_1(P_1) \otimes_F M_2(P_2.)$$

Now we will show the main theory in our research.

Theorem 4.1.4. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let M_i be Brauer indecomposable FG_i-module, with $i = 1, 2$. Then $M_1 \otimes_F M_2$ is a Brauer indecomposable $FG_1 \otimes_F FG_2$-module.*

Proof. Case 1, If either $M_1 = 0$ or $M_2 = 0$ then clearly $M_1 \otimes_F M_2 = 0$. Thus $(M_1\otimes_F M_2)(P_1\times P_2) = 0$ where P_1 and P_2 are p-subgroups of G_1 and G_2 respectively. Hence $M_1 \otimes_F M_2$ is a Brauer indecomposable $FG_1 \otimes_F FG_2$-module.
Case 2, If $M_1 \neq 0$ and $M_2 \neq 0$ then from Corollary 4.1.1 we have $M_1 \otimes_F M_2$ is indecomposable $FG_1 \otimes_F FG_2$-module. From Corollary 4.1.2 we have $(M_1 \otimes_F M_2)(P_1 \times P_2) \cong M_1(P_1) \otimes_F M_2(P_2)$. Since M_1 and M_2 are Brauer indecomposable modules then $M_1(P_1)$ and $M_2(P_2)$ are zero or indecomposables. If either $M_1(P_1) = 0$

or $M_2(P_2) = 0$ then $(M_1 \otimes_F M_2)(P_1 \times P_2) \cong M_1(P_1) \otimes_F M_2(P_2) = 0$. Hence $M_1 \otimes_F M_2$ is a Brauer indecomposable $FG_1 \otimes_F FG_2$-module. If $M_1(P_1) \neq 0$ and $M_2(P_2) \neq 0$ then $M_1(P_1)$ is an indecomposable $FP_1C_G(P_1)$-module and $M_2(P_2)$ is an indecomposable $FP_2C_G(P_2)$-module. Thus from Corollary 4.1.1 we have $M_1(P_1) \otimes_F M_2(P_2)$ is indecomposable $F(P_1 \times P_2)C_G(P_1 \times P_2)$-module. Then $(M_1 \otimes_F M_2)(P_1 \times P_2)$ is indecomposable $F(P_1 \times P_2)C_G(P_1 \times P_2)$-module. Hence $M_1 \otimes_F M_2$ is a Brauer indecomposable $FG_1 \otimes_F FG_2$-module. □

4.2 Tensor product of Scott modules

Throughout this section, G denotes a finite group, p a prime number and F an algebraically closed field of characteristic p.

Lemma 4.2.1. Let G_1 and G_2 be two finite groups. Then

$$FG_1 \otimes_F FG_2 \cong F(G_1 \times G_2)$$

as F-algebras.

In the following lemma we will show important property for exterior tensor product of induced modules.

Theorem 4.2.1. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let H_i be subgroup of G_i, with $i = 1, 2$. Let W_i for $i = 1, 2$ be FH_i-module. Then*

$$\mathrm{Ind}_{H_1}^{G_1}(W_1) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(W_2) \cong \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(W_1 \otimes_F W_2).$$

Proof. From Definition 2.1.1 and Lemma 4.2.1 we have

$$\begin{aligned} \mathrm{Ind}_{H_1}^{G_1}(W_1) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(W_2) &\cong (W_1 \otimes_{FH_1} FG_1) \otimes_F (W_2 \otimes_{FH_2} FG_2) \\ &\cong (W_1 \otimes_F W_2) \otimes_{F(H_1 \times H_2)} (FG_1 \otimes_F FG_2) \\ &\cong (W_1 \otimes_F W_2) \otimes_{F(H_1 \times H_2)} F(G_1 \times G_2) \\ &\cong \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(W_1 \otimes_F W_2). \end{aligned}$$

□

Now we will show the second main theory in our research.

Theorem 4.2.2. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let H_i be subgroup of G_i, with $i = 1, 2$. Let $\mathrm{Sc}(G_i, H_i)$ for $i = 1, 2$ be Scott FG_i-module. Then*

$$\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2)$$

is a Scott $FG_1 \otimes_F FG_2$- module.

Proof. Since $\mathrm{Sc}(G_1, H_1)$ is a Scott FG_1-module then from definition of Scott module we have $\mathrm{Sc}(G_1, H_1)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_1}^{G_1}(F)$ and $F \subseteq \mathrm{Sc}(G_1, H_1)$. Since $\mathrm{Sc}(G_2, H_2)$ is a Scott FG_2-module then from definition of Scott module we have $\mathrm{Sc}(G_2, H_2)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_2}^{G_2}(F)$ and $F \subseteq \mathrm{Sc}(G_2, H_2)$. Then

$$\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2) \cong \mathrm{Ind}_{H_1}^{G_1}(F) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(F).$$

From Theorem 4.2.1 we have

$$\mathrm{Ind}_{H_1}^{G_1}(F) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(F) \cong \mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F \otimes_F F) \cong \mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F).$$

Hence

$$\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2) \cong \mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F).$$

Also from Corollary 4.1.1 we have $\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2)$ is indecomposable $FG_1 \otimes_F FG_2$- module. Now we want to prove $\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F)$. Suppose that N is an another indecomposable summand $FG_1 \otimes_F FG_2$-module of $\mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F)$. Then N is an indecomposable summand of $\mathrm{Ind}_{H_1}^{G_1}(F) \otimes_F \mathrm{Ind}_{H_2}^{G_2}(F)$. So N is an indecomposable summand of $\mathrm{Ind}_{H_1}^{G_1}(F)$ or N is an indecomposable summand of $\mathrm{Ind}_{H_2}^{G_2}(F)$ or N is an indecomposable summand of $\mathrm{Ind}_{H_1}^{G_1}(F)$ and $\mathrm{Ind}_{H_2}^{G_2}(F)$. But $\mathrm{Sc}(G_1, H_1)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_1}^{G_1}(F)$ and $\mathrm{Sc}(G_2, H_2)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_2}^{G_2}(F)$. Hence $N = \mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2)$. So $\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2)$ is the unique indecomposable summand of $\mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(F)$. Also since F is an algebraically closed field of characteristic p, $F \subseteq \mathrm{Sc}(G_1, H_1)$ and $F \subseteq \mathrm{Sc}(G_2, H_2)$ then

$$F \cong F \otimes_F F \subseteq \mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2).$$

Hence $\mathrm{Sc}(G_1, H_1) \otimes_F \mathrm{Sc}(G_2, H_2) = \mathrm{Sc}(G_1 \times G_2, H_1 \times H_2)$ is a Scott $FG_1 \otimes_F FG_2$-module. $\square$

4.3 Tensor product of fusion systems

In this section, we shall construct the cartesian product of two fusion systems. As a consequence, we shall define the notion of tensor product of fusion systems.

Now let p be a fixed prime number. Let G_1 and G_2 be two finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let $\mathcal{F}_i$ be the fusion of G_i over P_i. Then we have the following definition:

Definition 4.3.1. With the notation above, we define the cartesian fusion system to of $G_1 \times G_2$ over the p-group $P_1 \times P_2$ to be the $\mathcal{F}_1 \times \mathcal{F}_2$.

Now let F be an algebraically closed field of characteristic the prime number p. We construct the vector space $F\mathcal{F}$ which consists of all linear combinations of the objects of the fusion system $\mathcal{F}$. Let us record this observation as in the following lemma:

Lemma 4.3.1. With the notation above, $F\mathcal{F}$ is a finite dimensional vector space over F.

Proof. Since $F\mathcal{F} \neq \varnothing$ and $(F\mathcal{F}, +)$ is an abelian group. Also for all $\alpha, \beta \in F$ and $M, N \in F\mathcal{F}$ satisfies:

(i) $\alpha(M + N) = \alpha M + \alpha N$.

(ii) $(\alpha + \beta)M = \alpha M + \beta M$.

(iii) $(\alpha\beta)M = \alpha(\beta M) = \beta(\alpha M)$.

(iv) $1_F M = M$.

Hence $F\mathcal{F}$ is a finite dimensional vector space over F which has a basis is object of $\mathcal{F}$. □

We remark that, the motivation to do such construction of this linear vector space is to carry some results in fusion systems to the linear vector space structure. The other thing is for build a tensor product of fusion system as in the following proposition:

proposition 4.3.1. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let $\mathcal{F}_i$ be the fusion of G_i over P_i. Let $F\mathcal{F}_i$ for $i = 1, 2$ be the vector spaces which are associated with the fusion systems $\mathcal{F}_{P_i}(G_i)$. Then*

$$F[\mathcal{F}_1 \times \mathcal{F}_2] \cong F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$$

as a vector space isomorphism.

Proof. Since cartesian product of two categories is a category. Then the cartesian product of two fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ is again fusion system $\mathcal{F}_1 \times \mathcal{F}_2$. Now suppose that $(P_1, P_2) \in F(\mathcal{F}_1 \times \mathcal{F}_2)$ thus $\exists (Q_1, Q_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ such that $(P_1, P_2)(Q_1, Q_2) = (1_{G_1}, 1_{G_2})$. Then $P_1 Q_1 = 1_{G_1}$ and $P_2 Q_2 = 1_{G_2}$. Hence $P_1 \in F\mathcal{F}_1$ and $P_2 \in F\mathcal{F}_2$. So $(P_1, P_2) \in F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$. Hence the required isomorphism is identity map as a vector space. □

Note that the proposition above says that the tensor product we define of two fusion systems is again fusion system over the cartesian product. However, we just construct the vector space. Then we shall use the direct product of p-subgroups of G_1 and G_2 to generate an algebra product. This yields the following theorem.

Theorem 4.3.1. *The vector space $F\mathcal{F}$ has the structure of finite dimensional algebra over F.*

Proof. Since $F\mathcal{F}$ is a finite dimensional vector space over F which has a basis is object of $\mathcal{F}$ and it is a ring satisfy $\alpha(MN) = (\alpha M)N = M(\alpha N)$ for all $\alpha \in F$ and $M, N \in F\mathcal{F}$. Hence $F\mathcal{F}$ is a finite dimensional algebra over F. □

Now we arrive to the main construction that we aim to do.

proposition 4.3.2. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. Let G_1 and G_2 be two finite groups. Let P_i be p-subgroup of G_i, with $i = 1, 2$. Let $\mathcal{F}_i$ be the fusion system of G_i over P_i. Let $F\mathcal{F}_i$ for $i = 1, 2$ be the finite dimensional algebra over F which is associated with the fusion system $\mathcal{F}_{P_i}(G_i)$. Then*

$$F[\mathcal{F}_1 \times \mathcal{F}_2] \cong F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$$

as an algebra isomorphism.

The following theorem gives us the first result we are seeking to in this construction:

Theorem 4.3.2. *Let p be a fixed prime number. Let F be an algebraically closed field of characteristic p. If $\mathcal{F}_i$ is saturated fusion system for $i = 1, 2$ then the fusion system $F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$ is a saturated fusion system.*

Proof. Since cartesian product of two categories is a category. Then the cartesian product of two saturated fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ is again saturated fusion system $\mathcal{F}_1 \times \mathcal{F}_2$. From Proposition 4.3.2 we have $F[\mathcal{F}_1 \times \mathcal{F}_2] \cong F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$. Hence $F\mathcal{F}_1 \otimes_F F\mathcal{F}_2$ is a saturated fusion system. □

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