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On maps preserving the spectrum of the skew Lie product of operators

A thesis submitted in fulfillment of the requirements for the degree of Master of Mathematical Sciences in Pure Mathematics

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Table of Contents

List of Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathcal{H}$	Hilbert space
$\langle , \rangle$	Inner product
$\sum_{n=1}^{\infty} x_n$	Summation of x_n for $n = 1, \cdots$
$\mathcal{H}'$	Dual of $\mathcal{H}$
$\lVert . \rVert_{\mathcal{H}}$	Norm in $\mathcal{H}$
$\mathcal{M}_n(\mathbb{C})$	The algebra of all $n \times n$ complex matrices
$\mathcal{B}(\mathcal{H})$	The set of all bounded linear operators on $\mathcal{H}$
T^*	The adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$
$\lVert T \rVert$	Norm of an operator T
$\mathcal{B}_s(\mathcal{H})$	The set of self-adjoint operators
$\mathcal{B}_a(\mathcal{H})$	The set anti-self-adjoint operators
$\mathcal{F}(\mathcal{H})$	The set of finite rank operators
tr	The trace of a finite rank operator
$\sigma(T)$	The spectrum of T
$\sigma_\pi(T)$	The peripheral spectrum of T
$\sigma_p(T)$	The point spectrum of T
$r(T)$	the spectral radius of T

Chapter 1

Introduction

The famous conjecture of Kaplansky (see [3, Pages 55-76]) says that:

Given two unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ semi-simple. Does any surjective linear application $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ which preserves invertibility (i.e $\phi(x)$ is invertible in $\mathcal{B}$ for any invertible element $x \in \mathcal{A}$) a Jordan morphism?

Note that this formulation of the problem is due to Aupetit [5]. This conjuncture has been solved in many cases:

(i) If $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional, [28].

(ii) If $\mathcal{B}$ is a commutative Banach algebra, [5].

(iii) If $\mathcal{A} = \mathcal{B}(X)$, $\mathcal{B} = \mathcal{B}(Y)$ where X, Y are two Banach spaces, [31, 26].

(iv) If $\mathcal{A}$ and $\mathcal{B}$ are two von Neumann algebras, [6].

(v) If $\mathcal{B}$ has a separating family of finite dimensional irreducible representations, [15].

But the problem remains unsolved even for C*-algebras. The interest reader may consult [5, 11, 26] for more details.

On the other hand, inspired by this conjecture, an important fields of research is the problem of describing maps on operators and matrices that preserve certain functions, subsets and relations has been widely studied in the literature, see [7], [9], [10], [11], [14], [15], [16], [17], [23], [29], [30], [32], [33] and their references therein. One of the classical problems in this area of research is to characterize maps preserving the spectra of the product of operators. Molnár in [29] studied maps preserving the spectrum of operator and matrix products. His results have been extended in several directions [8], [1], [2], [12], [13], [19], [21], [22], [24] and [25]. In [2], the problem of characterizing maps between matrix algebras preserving the spectrum of polynomial products of matrices is considered. In particular, the results obtained therein extend and unify several results obtained in [11] and [13].

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex infinite dimensional Hilbert spaces. Let $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}(\mathcal{K})$) denote the algebra of all bounded linear operators on $\mathcal{H}$ (resp. on $\mathcal{K}$). We say that a map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ preserves the skew Lie product of operators if

$$[\varphi(T), \varphi(S)]_* = [T, S]_*$$

where $[T, S]_* = TS - ST^*$ for any operators $S, T \in \mathcal{B}(\mathcal{H})$. Latter in [1], the form of all maps preserving the spectrum and the local spectrum of skew Lie product of matrices are determined.

In this thesis we will examine the form of surjective maps preserving the spectrum of skew Lie product of operators on an infinite dimensional complex Hilbert space. The plan is as follows

(i) Some of the basic definitions and terminology used in this thesis are introduced in Chapter 2.

(ii) In Chapter 3, maps preserving peripheral spectrum of Jordan products of self-adjoint operators are discussed

(iii) In the last chapter, we deal with the problem of characterizing surjective maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ preserves the skew Lie product of operators. Precisely, we shall prove the following.

Theorem 1.0.1. *A surjective map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$\sigma\left(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*\right) \;=\; \sigma\left(TS - ST^*\right), \;(T,\; S \in \mathcal{B}(\mathcal{H})), \tag{1.1}$$

if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(T) = \pm UTU^* \tag{1.2}$$

for all $T \in \mathcal{B}(\mathcal{H})$.

Chapter 2

Preliminaries

Through first chapter, we recall some usual notation and collect some elementary results that will be used. All of the vector spaces to be over the complex field $\mathbb{C}$.

2.1 Basic definitions and examples

Definition 2.1.1 (Inner product)**.** Let $\mathcal{H}$ be a vector space. An **inner product** is a function

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

such that, for all $x, y, z \in \mathcal{H}$ and scalar $\alpha \in \mathbb{C}$.

(1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(2) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(4) $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$

We then have

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

and

$$\langle x, \alpha z \rangle = \overline{\alpha} \langle x, z \rangle$$

We say that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ (or simply $\mathcal{H}$ is an **inner product space**).

Theorem 2.1.2 (Cauchy-Schwarz)**.** *In an inner product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$*, we have*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|, \quad \textit{for all } x, y \in \mathcal{H}$$

where $\|x\| = \sqrt{\langle x, x \rangle}$ *for every* $x \in \mathcal{H}$*.*

Corollary 2.1.3. *In an inner product space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$*, we have*

$$\|x + y\| \leq \|x\| + \|y\|,$$

for every $x, y \in \mathcal{H}$*.*

Remark 2.1.4. $(\mathcal{H}, \|.\|)$ is a normed vector space.

Definition 2.1.5 (Hilbert space)**.** If $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is complete with respect to $\|\cdot\|$ then it is called a **Hilbert space.**

Example 2.1.6.

(a) $\ell^2(\mathbb{N})$, where $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}$.

Cauchy-Schwarz then says

$$|\sum_{i=1}^{\infty} x_i\overline{y_i}| \leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2}\sqrt{\sum_{i=1}^{\infty} |y_i|^2}$$

(b) $L^2([a,b])$, where $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$.

Definition 2.1.7 (Orthogonality)**.** Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space. We say that two vectors $x, y \in \mathcal{H}$ are orthogonal if $\langle x, y\rangle = 0$.

Theorem 2.1.8. *Let $x_i, \ldots, x_n$ be pairwise orthogonal vectors in $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Then*

$$\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Theorem 2.1.9 (Parallelogram identity)**.** *In $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{$\star$}$$

for all $x, y \in \mathcal{H}$.

2.2 Projections

Definition 2.2.1. Let $\mathcal{H}$ be a vector space and M a non empty subset of $\mathcal{H}$.

(i) The subset M is said to be convex if for any $x, y \in M$ we have

$$tx + (1-t)y \in M$$

for every $t \in [0,1]$.

(ii) For any $x \in \mathcal{H}$, the distance between x and M is defined by

$$d(x, M) = \inf_{m\in M} \|x - m\|.$$

Theorem 2.2.2. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $M \subseteq \mathcal{H}$ be closed and convex. Let $x \in \mathcal{H}$. Then there exists a unique point $m_x \in M$ such that*

$$\|x - m_x\| = \inf_{m\in M} \|x - m\| = d(x, M).$$

Definition 2.2.3 (Projection operator on a closed convex subset)**.** Let $\mathcal{H}$ be a Hilbert space. Let $M \subseteq \mathcal{H}$ be closed and convex. Define

$$P_M : \mathcal{H} \to \mathcal{H}$$

by $P_M(x) = m_x$ from above. This is called the projection of $\mathcal{H}$ onto M.

Definition 2.2.4 (Orthogonal complement)**.** If $S \subseteq \mathcal{H}$, let

$$S^\perp = \{x \in \mathcal{H} \,|\, \langle x, y\rangle = 0 \quad \forall y \in S\}.$$

We call $S^\perp$ the orthogonal of S.

We can state the following.

Theorem 2.2.5. *Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$ be a closed subspace. We have*

(i) $x - P_M x \in M^\perp$ *for all* $x \in \mathcal{H}$.

(ii) $\mathcal{H} = M \oplus M^\perp$. *That is, each* $x \in \mathcal{H}$ *can be written in exactly one way as* $x = m + m^\perp$ *with* $m \in M$, $m^\perp \in M^\perp$.

Corollary 2.2.6. *Let $M \subseteq \mathcal{H}$ be a closed subspace. Then we have*

(a) $P_m(\mathcal{H}) = M, \ker P_M = M^\perp$.

(b) $P_M^2 = P_M$.

(c) $P_{M^\perp} = I - P_M$.

Definition 2.2.7. Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$ be a closed subspace. The orthogonal projection of $\mathcal{H}$ onto M is the function $P_M : \mathcal{H} \longrightarrow \mathcal{H}$ such that for $x \in \mathcal{H}$, $P_M(x)$ is the unique element in M such that $(x - P_M(x)) \perp M$.

Definition 2.2.8 (Orthonormal system)**.** As subset $S \subseteq \mathcal{H}$ is an **orthonormal system** (orthonormal) if

$$\langle e, e'\rangle = \delta_{e,e'} \quad \forall e, e' \in S.$$

Definition 2.2.9 (Complete orthonormal system or Hilbert basis)**.** An orthonormal system S is **complete** or a **Hilbert basis** if

$$\overline{\text{span } S} = \mathcal{H}$$

Remark 2.2.10. Every Hilbert space has a complete orthonormal system.

Lemma 2.2.11. *If $\{e_k \,|\, k \in \mathbb{N}\}$ is orthonormal, then $\sum_{k\geq 1} a_m e_k$ converges in $\mathcal{H}$ if and only if $\sum_{k\geq 1} |a_k|^2$ converges in $\mathbb{C}$.*

If either series converges, then

$$\left\| \sum_{k\geq 1} a_k e_k \right\|^2 = \sum_{k\geq 1} |a_k|^2$$

Lemma 2.2.12. *Let $\{e_1, \ldots, e_n\}$ be orthonormal. Then*

$$\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

for each $x \in \mathcal{H}$.

2.3 Linear operators on Hilbert spaces

Definition 2.3.1 (Linear operators on Hilbert spaces)**.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear operator is a function $T : \mathcal{H} \to \mathcal{K}$ such that

$$T(x+y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

for all $x, y \in X, \alpha \in \mathbb{C}$.

We write $\mathcal{L}(\mathcal{H}, \mathcal{K}) = \{T : \mathcal{H} \to \mathcal{K} \,|\, T \text{ is linear}\}$

Definition 2.3.2. $T : \mathcal{H} \to \mathcal{K}$ is continuous at $x \in \mathcal{H}$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $y \in \mathcal{H}$ we have

$$\|x - y\|_{\mathcal{H}} < \delta \Rightarrow \|Tx - Ty\|_{\mathcal{K}} < \epsilon$$

Notations

$$\mathcal{B}(\mathcal{H}, \mathcal{K}) = \{T : \mathcal{H} \to \mathcal{K} \,|\, T \text{ is linear and continuous}\}$$

and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Definition 2.3.3 (Bounded linear operator)**.** Let $T : \mathcal{H} \to \mathcal{K}$ be linear, then T is **bounded** if T maps bounded sets in $\mathcal{H}$ to bounded sets in $\mathcal{K}$. That is: for each $M > 0$ there exists $M' > 0$ such that for any $x \in \mathcal{H}$ we have

$$\|x\|_{\mathcal{H}} \leq M \Rightarrow \|Tx\|_{\mathcal{K}} \leq M'$$

Theorem 2.3.4. *Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then the following are all equivalent.*

1) T is continuous

2) T is continuous at 0

3) T is bounded

4) There exists a constant $c > 0$ such that

$$\|Tx\|_{\mathcal{K}} \leq c\|x\|_{\mathcal{H}}, \quad \textit{for every } \; x \in \mathcal{H}.$$

Remark 2.3.5. If $\dim(\mathcal{H}) < \infty$ then $\mathcal{L}(\mathcal{H}, \mathcal{K}) = \mathcal{B}(\mathcal{H}, \mathcal{K})$. This is **not** true if $\mathcal{H}$ has infinite dimension.

Definition 2.3.6 (Operator norm)**.** The **operator norm** of $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\|T\|$ is defined by any one of the following equivalent expressions.

(a) $\|T\| = \inf\{c > 0 \,|\, \|Tx\| < c\|x\|\}$.

(b) $\|T\| = \sup\limits_{x \neq 0} \dfrac{\|Tx\|}{\|x\|}$.

(c) $\|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\|$.

(d) $\|T\| = \sup\limits_{\|x\| = 1} \|Tx\|$.

Corollary 2.3.7. *If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then T continuous $\iff$ T bounded $\iff$ there is $c \geq 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in \mathcal{H}$.*

Proposition 2.3.8. *The operator norm is a norm on $\mathcal{B}(\mathcal{H}, \mathcal{K})$.*

Definition 2.3.9. The **topological dual** (just dual) of a Hilbert space $\mathcal{H}$ is

$$\mathcal{H}' = \mathcal{B}(\mathcal{H}, \mathbb{C}) = \{\phi : \mathcal{H} \to \mathbb{C} \,|\, \phi \text{ is linear and continuous}\}.$$

Observe that if $y \in \mathcal{H}$ is fixed, then the map

$$\begin{aligned} \phi_y :\ & \mathcal{H} \to \mathbb{C} \\ & x \mapsto \langle x, y \rangle \end{aligned}$$

is in $\mathcal{H}'$.

Theorem 2.3.10 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space. The map*

$$\begin{aligned} \phi :\ & \mathcal{H} \to \mathcal{H}' \\ & y \mapsto \phi_y \end{aligned}$$

is a conjugate linear bijection, and $\|\phi_y\| = \|y\|$.

Proposition 2.3.11. *Let $M \subseteq \mathcal{H}$ be a closed subspace. Then we have*

(a) $P_M \in \mathcal{B}(\mathcal{H})$.

(b) $\|P_M\| \leq 1$. In fact $\|P_M\| = 1$ if $P_M \neq 0$.

Combining Definition 2.2.3 and Theorem 2.2.5 we can see that any projection is orthogonal. We have now the following.

Proposition 2.3.12. *Any projection $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection of $\mathcal{H}$ onto $P(\mathcal{H})$.*

2.4 Adjoint operators

Theorem 2.4.1. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, there exists a unique operator $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \,; \forall x \in \mathcal{H}, y \in \mathcal{K}.$$

Definition 2.4.2. The operator T^* defined above is called the adjoint of T.

Theorem 2.4.3. *For any operators $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, the following properties holds*

(i) $(T^*)^* = T$.

(ii) $(T + S)^* = T^* + S^*$.

(iii) $(\alpha T)^* = \overline{\alpha}\, T^*$.

(iv) $(TS)^* = S^* T^*$.

(v) $\|T\| = \|T^*\|$.

Proposition 2.4.4. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

(i) $\ker T = T^*(\mathcal{H})^\perp$.

(ii) $\overline{T(\mathcal{H})} = (\ker(T^*))^\perp$.

Proposition 2.4.5. *($C*$-property) If $T \in \mathcal{B}(\mathcal{H})$, then*

$$\|T^* T\| = \|T\|^2 .$$

Corollary 2.4.6. *If $T \in \mathcal{B}(\mathcal{H})$, then $T^* T = 0 \iff TT^* = 0 \iff T = 0$.*

2.5 Classes of operators

Definition 2.5.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is of finite rank if its range $T(\mathcal{H})$ has finite dimension (and that dimension is called the rank of T); the set of finite rank operators will be denoted $\mathcal{F}(\mathcal{H})$.

Clearly the sum of two operators of finite rank has finite rank, since the range is contained in the sum of the ranges. Also, since the range of a constant multiple of T is contained in the range of T it follows that $\mathcal{F}(\mathcal{H})$ a linear subspace of $\mathcal{B}(\mathcal{H})$.

Remark 2.5.2. If $T \in \mathcal{F}(\mathcal{H})$. Let $x_1, \ldots, x_n$ be an orthonormal basis for $T(\mathcal{H})$. Then

$$Tx = \sum_{i=1}^{n} \langle Tx, x_i \rangle x_i, \text{ for every } x \in \mathcal{H}.$$

Theorem 2.5.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded operator of rank n. Then there exists vectors $x_1, \ldots, x_n \in \mathcal{H}$ and vectors $y_1, \ldots, y_n \in \mathcal{H}$ such that for every $x \in \mathcal{H}$, we have*

$$Tx = \sum_{i=1}^{n} \langle x, x_i \rangle y_i.$$

The vectors $y_1, \ldots, y_n$ may be chosen to be any orthonormal basis for $T(\mathcal{H})$

Remark 2.5.4. For nonzero elements $x, y \in \mathcal{H}$. Consider the operator $x \otimes y$ defined by

$$(x \otimes y)z = \langle z, y \rangle x, \quad z \in H.$$

Clearly $x \otimes y$ is a rank one operator. By Theorem 2.5.3, it is clear that for any rank one operator T there is $x, y \in \mathcal{H}$ such that $T = x \otimes y$.

Moreover if T is of rank n, there exists vectors $x_1, \ldots, x_n \in \mathcal{H}$ and vectors $y_1, \ldots, y_n$ such that for every $x \in \mathcal{H}$, we have

$$T = \sum_{i=1}^{n} x_i \otimes y_i.$$

Proposition 2.5.5. *$\mathcal{F}(\mathcal{H})$ is a $*$-two-sided ideal of $\mathcal{B}(\mathcal{H})$, which is to say a linear subspace such that*

$$B_1, B_2 \in \mathcal{B}(\mathcal{H}),\ T \in \mathcal{F}(\mathcal{H}) \Longrightarrow B_1 T B_2, T^* \in \mathcal{F}(\mathcal{H}).$$

Remark 2.5.6. For finite rank operators in $\mathcal{B}(\mathcal{H})$ one can define a trace functional tr by

$$\operatorname{tr}(A) = \sum_{k=1}^{n} \langle x_k, f_k \rangle ,$$

when

$$A = \sum_{k=1}^{n} x_k \otimes f_k.$$

Lemma 2.5.7. *For any $x, y \in \mathcal{H}$, we have*

$$\|x \otimes y\| = \|x\| \; \|y\|.$$

Definition 2.5.8. An operator $T \in \mathcal{B}(\mathcal{H})$ is called

(i) *normal* if $TT^* = T^*T$.

(ii) *self-adjoint* if $T = T^*$.

(iii) *anti self-adjoint* if $T = -T^*$.

(iv) *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

(v) *unitary* if $T^*T = TT^* = I$

(vi) *projection* if $T = T^2 = T^*$.

Remark 2.5.9.

- Note that the definition of projection given in the above theorem is equivalent to the one (orthogonal projection) introduced in Definition 2.2.7.

- T positive $\Longrightarrow$ T self-adjoint $\Longrightarrow$ T normal.

- Any projection is a positive operator.

- T is unitary $\Longrightarrow$ T is normal.

Proposition 2.5.10. *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) *T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.*

(ii) *T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in \mathcal{H}$.*

(iii) *T is unitary if and only if T is an inner product preserving surjection.*

2.5.1. Example. (Discrete diagonal) Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $g : \mathbb{N} \to \mathbb{C}$ be a bounded function. Define

$$Tx = \sum_{n=1}^{\infty} g(n)\ \langle x, e_n\rangle\, e_n\,.$$

(This is a diagonal operator uniquely given by $Te_n = g(n)\, e_n$.)
Let us observe that

$$\|T\| = \sup_{n\in\mathbb{N}} |g(n)|\,.$$

As $\langle Te_n, e_n\rangle = \left\langle e_n, \overline{g(n)}e_n\right\rangle$ we see that

$$T^*x = \sum_{n=1}^{\infty} \overline{g(n)}\ \langle x, e_n\rangle\, e_n$$

for all $x \in H$.

- T is self-adjoint $\Longleftrightarrow$ $g(n) = \overline{g(n)}$ for all $n \in \mathbb{N}$ (g is real).

- T is positive $\Longleftrightarrow$ $\langle Tx, x\rangle \geq 0$ for all x $\Longleftrightarrow$$\sum_n g(n)|\,\langle x, e_n\rangle\,|^2 \geq 0$ for all $x \in H$. But this is equivalent to $g(n) \geq 0$ for all $n \in \mathbb{N}$.

- T is unitary $\iff T^*T = TT^* = I$.
  But $T^*Tx = \sum_{n=1}^{\infty} \overline{g(n)}g(n) \langle x, e_n\rangle\ e_n = \sum_{n=1}^{\infty} |g(n)|^2\ \langle x, e_n\rangle^2 e_n$.
  In other words, T is unitary $\iff |g(n)| = 1$ for all n.

2.5.2. Example. Let $x, y \in H$ be distinct. Take the rank one operator $x \otimes y$ defined by

$$(x \otimes y)z = \langle z, y\rangle\, x, \quad z \in H.$$

For $u, v \in H$ we obtain

$$\langle (x \otimes y)u, v\rangle = \langle u, y\rangle \langle x, v\rangle = \left\langle u, \overline{\langle x, v\rangle} y\right\rangle = \langle u, \langle v, x\rangle\, y\rangle = \langle u, (y \otimes x)v\rangle\ ,$$

implying

$$(x \otimes y)^* = y \otimes x\,.$$

In particular

$$(x \otimes x)^* = x \otimes x\,.$$

Now

$$(x \otimes y)(x \otimes y)^* = \|y\|^2\,(x \otimes x)\,.$$

By exchanging the roles of x and y, we obtain

$$(x \otimes y)^*(x \otimes y) = \|x\|^2\,(y \otimes y)\,.$$

Remark 2.5.11. Assume that x and y are unit vectors. Since

$$(x \otimes y)(x \otimes y)^* = (x \otimes x)$$

and

$$(x \otimes y)^*(x \otimes y) = (y \otimes y)$$

then $(x \otimes x)^2 = x \otimes x$ and $(x \otimes x)^* = x \otimes x$. Thus $x \otimes x$ is an orthogonal projection.

2.6 Spectrum of an operator

Definition 2.6.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be invertible if there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$. Here I denotes the identity operator in $\mathcal{H}$.

The operator S when it exists is unique and is called the inverse of T. We denote $S = T^{-1}$.

Theorem 2.6.2. *An operator $T \in \mathcal{B}(\mathcal{H})$ is invertible if and only if $T^* \in \mathcal{B}(\mathcal{H})$ is invertibe and $(T^*)^{-1} = (T^{-1})^*$.*

Theorem 2.6.3. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| < 1$. Then $I - T$ is invertible and*

$$(I - T)^{-1} = \sum_{n \geq 0} T^n.$$

Definition 2.6.4. Let $T \in \mathcal{B}(\mathcal{H})$.

(i) The spectrum $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $T - \lambda I$ does not have an inverse in $\mathcal{B}(\mathcal{H})$. More precisely

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ has not an inverse in } \mathcal{B}(\mathcal{H})\}\,.$$

(ii) The point spectrum of $T \in \mathcal{B}(\mathcal{H})$ is defined as

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \,|\, (T - \lambda I) \text{ is not one-to one}\}.$$

In other words, for each $\lambda \in \sigma_p(T)$ there is a nonzero y in $\mathcal{H}$ such that

$$Ty = \lambda\, y\,.$$

Basic properties of $\sigma(T)$

(i) For any $\lambda \in \sigma(T)$ we have $|\lambda| \leq \|T\|$.

(ii) The spectrum of a bounded operator T is always a closed, bounded and non-empty subset of the complex plane.

(iii) The spectral radius, $r(T)$ of T is the radius of the smallest circle in the complex plane which is centered at the origin and contains the spectrum $\sigma(T)$ inside of it. That is

$$r(T) = \sup\{|\lambda| \,|\, \lambda \in \sigma(T)\}.$$

Note that $r(T) < \infty$ since the spectrum $\sigma(T)$ is always nonempty and compact subset of $\mathbb{C}$.

(iv) The spectral radius formula says that for any element $T \in \mathcal{B}(\mathcal{H})$, we have

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Theorem 2.6.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$, Then*

(i) $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$.

(ii) *If T is invertible then* $\sigma(T^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$.

(iii) $\sigma(\alpha I + T) = \alpha + \sigma(T)$.

Proposition 2.6.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then the following statements hold:*

(i) *If $Tx = \lambda\, x$ for some $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}$, then $T^*x = \overline{\lambda}x$.*

(ii) *If $\lambda_1 \neq \lambda_2$ are complex numbers, then*

$$\mathrm{Ker}(T - \lambda_1 I) \perp \mathrm{Ker}(T - \lambda_2 I)\,.$$

Definition 2.6.7. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be bounded bellow if there exists some constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for any $x \in \mathcal{H}$.

Proposition 2.6.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then*

$$T \text{ is invertible in } \mathcal{B}(\mathcal{H}) \iff T \text{ is bounded bellow.} \tag{2.1}$$

Corollary 2.6.9. *If $T \in \mathcal{B}(\mathcal{H})$ is normal and $\lambda \in \sigma(T) \setminus \sigma_p(T)$, then $(T - \lambda I)(\mathcal{H})$ is not closed.*

Proof. If $T-\lambda I$ is one-to-one and $(T-\lambda I)(\mathcal{H})$ is closed, then, by the Inverse Mapping Theorem, there is a continuous linear map
$S : (T - \lambda I)(\mathcal{H}) \to \mathcal{H}$ such that $S(T - \lambda I)x = x$ for all $x \in \mathcal{H}$. It means that $\|x\| \leq \|S\|\|(T - \lambda I)x\|$. As $\|S\| \neq 0$, we see that

$$\|(T - \lambda I)x\| \geq \frac{1}{\|S\|}\|x\| .$$

In view of Proposition 2.6.8, $\lambda \notin \sigma(T)$. □

Corollary 2.6.10. *(Approximate Spectrum)*
If $T \in \mathcal{B}(\mathcal{H})$ is normal, then $\lambda \in \sigma(T)$ if and only if there is a sequence (x_n) of unit vectors such that $\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$.

Proof. By Proposition 2.6.8 $\lambda \in \sigma(T) \iff \inf\limits_{\|x\|=1} \|(T - \lambda I)x\| = 0.$ □

Remark 2.6.11. Spectrum of a normal operator is equal to approximate point spectrum. Note that the approximate spectrum of T is the set of $\lambda \in \mathbb{C}$ such that $T-\lambda I$ is not bounded bellow.

Theorem 2.6.12. *If $T \in \mathcal{B}(\mathcal{H})$ is a normal operator, then the following statements hold:*

(i) *T is self-adjoint if and only $\sigma(T) \subset \mathbb{R}$.*

(ii) *T is positive if and only $\sigma(T) \subset \mathbb{R}^+$.*

(iii) *T is unitary if and only $\sigma(T) \subset \{z \in \mathbb{C} \mid |z| = 1\}$.*

(iv) *T is a projection if and only $\sigma(T) \subset \{0, 1\}$.*

Proposition 2.6.13. *If $T \in \mathcal{B}(\mathcal{H})$ is normal, then*

$$r(T) = \|T\| .$$

Proposition 2.6.14. *For $T \in \mathcal{B}(\mathcal{H})$ the following conditions are equivalent:*

(i) *T is positive*

(ii) *$T = A^*A$ for some $A \in \mathcal{B}(\mathcal{H})$.*

(iii) *$T = S^2$ for some self-adjoint $S \in \mathcal{B}(\mathcal{H})$. ($S$ is denoted by $T^{1/2}$ and called the square root of T).*

Remark 2.6.15. If T is self-adjoint, then e^{iT} is unitary. The converse also holds:

Proposition 2.6.16. *For any unitary operator $U \in \mathcal{B}(\mathcal{H})$ there is a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 2\pi$ such that $U = e^{iT}$.*

Lemma 2.6.17. *For any self-adjoint operator A we have $\|A\|$ or $-\|A\|$ belongs to $\sigma(A)$.*

Chapter 3

Maps Preserving Spectrum of Jordan Products of Self-Adjoint Operators

In this chapter, we give a sketch of the proof of the main result of [17] concerning maps preserving spectrum of Jordan products of self-adjoint operators.

3.1 Statement of the main result

Throughout this chapter, $\mathcal{H}$ denotes a complex Hilbert spaces, and $\mathcal{B}(\mathcal{H})$ denotes the C*-algebra of all bounded linear operators on $\mathcal{H}$. We denotes the set of self-adjoint (resp. anti-self-adjoint) operators by $\mathcal{B}_s(\mathcal{H})$ (resp. $\mathcal{B}_a(\mathcal{H})$). For $T \in \mathcal{B}(\mathcal{H})$, $\sigma(T)$ and $r(T)$ stand for the spectrum and the spectral radius of T, respectively. Recall that if $T \in \mathcal{B}(\mathcal{H})$ is normal, then

$$\|A\| = r(A).$$

Definition 3.1.1. The peripheral spectrum of an element $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_\pi(T) = \{z \in \sigma(T) : |z| = r(T)\}.$$

Note that $\sigma_\pi(T) \subset \sigma(T)$.

Definition 3.1.2. Fix a positive integer $k \geq 2$ and a finite sequence $(i_1, i_2, \cdots, i_m)$ such that $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, k\}$ and there is an i_p not equal to i_q for all other q; that is, i_p appears just one time in the sequence. For operators $A_1, \cdots, A_k$, the operator,

$$A_1 \circ A_2 \circ \ldots \circ A_k = A_{i_1} A_{i_2} \cdots A_{i_m} + A_{i_m} \cdots A_{i_2} A_{i_1}$$

is called generalized Jordan product of $A_1, A_2, \cdots, A_k$ while m is called the width of the product.

In particular when $k = 2$, we get the well known Jordan product defined by

$$A \circ B = AB + BA$$

for any A and B in $\mathcal{B}(\mathcal{H})$.

Set $\mathcal{A} = \mathcal{B}(\mathcal{H})$ or $\mathcal{B}_s(\mathcal{H})$ and let $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$.

Definition 3.1.3.

(i) We say that φ preserves the peripheral spectrum of the generalized Jordan product of operators if

$$\sigma_\pi\left(\varphi(A_1)\circ\varphi(A_2)\circ\cdots\circ\varphi(A_k)\right) \;=\; \sigma_\pi\left(A_1\circ A_2\circ\cdots\circ A_k\right). \tag{3.1}$$

for any operators $A_1, A_2, \cdots, A_k \in \mathcal{A}$.

(ii) In particular, we say that φ preserves the peripheral spectrum of the Jordan product of operators if

$$\sigma_\pi\left(\varphi(A)\circ\varphi(B)\right) \;=\; \sigma_\pi\left(A\circ B\right) \tag{3.2}$$

for any operators A and $B \in \mathcal{A}$.

(iii) We say that φ preserves the spectrum of the Jordan product of operators if

$$\sigma\left(\varphi(A)\circ\varphi(B)\right) \;=\; \sigma\left(A\circ B\right) \tag{3.3}$$

for any operators A and $B \in \mathcal{A}$.

Lemma 3.1.4. *If $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is a bijective map preserving the spectrum of the Jordan product of operators then it preserves also the peripheral spectrum of the Jordan product of operators.*

Proof. Assume that relation (3.3) holds for any $A, B \in \mathcal{A}$. Let $\alpha \in \sigma_\pi(A\circ B)$ and then $r(A\circ B) = |\alpha|$ and $\alpha \in \sigma(A\circ B)$. Since

$$\sigma\left(\varphi(A)\circ\varphi(B)\right) \;=\; \sigma\left(A\circ B\right),$$

then $\alpha \in \sigma(\varphi(A)\circ\varphi(B))$. But from (3.3), we have $r(\varphi(A)\circ\varphi(B)) \;=\; r(A\circ B)$. Hence $|\alpha| = r(\varphi(A)\circ\varphi(B))$ and therefore $\alpha \in \sigma_\pi(\varphi(A)\circ\varphi(B))$. Thus $\sigma_\pi(A\circ B) \subset \sigma_\pi(\varphi(A)\circ\varphi(B))$.

Since φ is a bijection, easy computation shows that

$$\sigma\left(\varphi^{-1}(A)\circ\varphi^{-1}(B)\right) \;=\; \sigma\left(A\circ B\right) \tag{3.4}$$

for any operators A and $B \in \mathcal{A}$. By a similar reasoning one can easily shows that $\sigma_\pi(A\circ B) \subset \sigma_\pi\left(\varphi^{-1}(A)\circ\varphi^{-1}(B)\right), \forall A, B \in \mathcal{A}$. Accordingly $\sigma_\pi(A\circ B) \supset \sigma_\pi(\varphi(A)\circ\varphi(B))$. $\square$

In [17, 18, 23, 33, 34, 35], several authors study mappings through their action on Jordan product. When $\mathcal{A} = \mathcal{B}_s(\mathcal{H})$, the authors in [17] show that φ satisfies condition (3.2) for all $A, B \in \mathcal{B}_s(\mathcal{H})$ if and only if there exist a scalar $\lambda \in \{-1, 1\}$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(A) = \lambda U A U^*$$

for all $A \in \mathcal{A}$, or

$$\varphi(A) = \lambda U A^\top U^*$$

for all $A \in \mathcal{A}$, where $A^\top$ is the transpose of A with respect to an arbitrarily but fixed orthonormal basis of $\mathcal{H}$.
In [34], a similar result have been obtained for the generalized Jordan product of operators. Note that maps preserving the spectrum of Jordan product of self-adjoint operators has been established in [23].
The purpose of this chapter is to revisit the proof given in [17], since it will be needed for the proof of our main result in Chapter 4. We use some arguments from [23] and [34].

We shall prove the following

Theorem 3.1.5. *A surjective map φ (no linearity of φ is assumed) from $\mathcal{B}_s(\mathcal{H})$ onto itself satisfies*

$$\sigma\left(\varphi(A) \circ \varphi(B)\right) \;=\; \sigma\left(A \circ B\right) \tag{3.5}$$

for any operators A and $B \in \mathcal{B}_s(\mathcal{H})$ if and only if there exist a scalar $\lambda \in \{-1, 1\}$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(A) = \lambda U A U^*$$

for all $A \in \mathcal{B}_s(\mathcal{H})$, or

$$\varphi(A) = \lambda U A^\top U^*$$

for all $A \in \mathcal{B}_s(\mathcal{H})$. Here $A^\top$ is the transpose of A with respect to an arbitrarily but fixed orthonormal basis of $\mathcal{H}$.

In the sequel, we focus our attention to prove Theorem 3.1.5. We assume always that $\varphi : \mathcal{B}_s(\mathcal{H}) \longrightarrow \mathcal{B}_s(\mathcal{H})$ satisfies the conditions in Theorem 3.1.5.

3.2 Auxiliary results

Let us firstly establish several lemmas. Recall that for two nonzero vectors x and y in $\mathcal{H}$, we denote by $x \otimes y$ the operator of rank one defined by

$$(x \otimes y)(z) := \langle z, y \rangle x.$$

Note that $(x \otimes y)^* = y \otimes x$ and that every rank one operator in $\mathcal{B}(\mathcal{H})$ can be written as $x \otimes y$.

Lemma 3.2.1. *For any $x \in \mathcal{H}$ and $A \in \mathcal{B}_s(\mathcal{H})$, we have*

$$\sigma(A(x \otimes x) + (x \otimes x)A) \;=\; \left\{0, \langle Ax, x\rangle \pm \|Ax\|\,\|x\|\right\}, \tag{3.6}$$

$$\sigma_\pi(A(x \otimes x) + (x \otimes x)A) \;=\; \begin{cases} \{\pm\|Ax\|\,\|x\|\} & \text{if } \langle Ax, x\rangle = 0 \\ \{\langle Ax, x\rangle + \|Ax\|\,\|x\|\} & \text{if } \langle Ax, x\rangle > 0 \\ \{\langle Ax, x\rangle - \|Ax\|\,\|x\|\} & \text{if } \langle Ax, x\rangle < 0. \end{cases} \tag{3.7}$$

and

$$\|A(x \otimes x) + (x \otimes x)A\| \;=\; |\langle Ax, x\rangle| + \|Ax\|\,\|x\|. \tag{3.8}$$

Proof. For the proof of equalities (3.6) and (3.7), see [17, Lemma 2.2] or [34, Lemma 4]. See also Corollary 4.2.7 for another proof.
For the last equality, firstly note that

$$\|A(x \otimes x) + (x \otimes x)A\| = r\left(A(x \otimes x) + (x \otimes x)A\right)$$

as $A \in \mathcal{B}_s(\mathcal{H})$. The rest follows immediately from (3.7). □

Lemma 3.2.2. *For $A \in \mathcal{B}_s(\mathcal{H})$, then*

$$\|A(x \otimes x) + (x \otimes x)A\| = 2 \tag{3.9}$$

for all unit vectors $x \in \mathcal{H}$ implies that $A = \lambda I$ with $\lambda \in \{-1, 1\}$.

Proof. Recall firstly that for any $x, y \in \mathcal{H}$ we have (see Lemma 2.5.7)

$$\|x \otimes y\| = \|x\| \, \|y\|.$$

Assume now that $\|A(x \otimes x) + (x \otimes x)A\| = 2$ for all unit vectors $x \in \mathcal{H}$. By the triangle inequality, we have

$$\begin{aligned} 2 &= \|A(x \otimes x) + (x \otimes x)A\| \\ &\leq \|Ax\| \, \|x\| + \|x\| \, \|Ax\| \\ &\leq 2\|Ax\| \\ &\leq 2\|A\| \, \|x\| = 2\|A\|. \end{aligned}$$

Thus $\|A\| \geq 1$ and $\|Ax\| \geq 1$ for any unit vector $x \in \mathcal{H}$. Now, since $A \in \mathcal{B}_s(\mathcal{H})$, we have always $\|A\|$ or $-\|A\|$ belongs to $\sigma(A)$; (see Lemma 2.6.17). Therefore

$$2 = \|A(x \otimes x) + (x \otimes x)A\| = r(A(x \otimes x) + (x \otimes x)A).$$

From Lemma 3.2.1, we have

$$\begin{aligned} r(A(x \otimes x) + (x \otimes x)A) &= \max(\langle Ax, x\rangle + \|Ax\| \, \|x\|, \langle Ax, x\rangle - \|Ax\| \, \|x\|) \\ &= |\langle Ax, x\rangle| + \|Ax\| \end{aligned}$$

Thus

$$|\langle Ax, x\rangle| + \|Ax\| \, \|x\| = 2 \tag{3.10}$$

On the other hand, for all unit vectors $x \in \mathcal{H}$,

$$2\,|\langle Ax, x\rangle| = |\langle Ax, x\rangle| + |\langle Ax, x\rangle| \leq |\langle Ax, x\rangle| + \|Ax\| = \|A(x \otimes x) + (x \otimes x)A\| = 2.$$

Therefore $|\langle Ax, x\rangle| \leq 1$ for all unit vectors $x \in \mathcal{H}$. Since A is self adjoint we infer that $\|A\| \leq 1$ and therefore $\|A\| = 1$ and $\|Ax\| = 1$ for all unit vectors $x \in \mathcal{H}$. Equation (3.10) writes

$$|\langle Ax, x\rangle| + \|Ax\| \, \|x\| = 2 = 2\|Ax\| \, \|x\|.$$

Accordingly,

$$|\langle Ax, x\rangle| = \|Ax\| \, \|x\| = 1$$

for any unit vector $x \in \mathcal{H}$. Thus Ax and x are linearly dependent for any unit vector $x \in \mathcal{H}$. Therefore $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Clearly $\lambda \in \{-1, 1\}$ by Equation (3.9) and since $A \in \mathcal{B}_s(\mathcal{H})$. This complete the proof. □

Lemma 3.2.3. *$\varphi(I) = \lambda I$ with $\lambda \in \{-1, 1\}$.*

Proof. Since φ satisfies (3.2), then if $A = B$ we get

$$2r(A)^2 = r(2A^2) = r(2\varphi(A)^2) = 2r(\varphi(A))^2.$$

Accordingly

$$r(A) = r(\varphi(A)).$$

For any $A \in \mathcal{B}_s(\mathcal{H})$, we have

$$\sigma_\pi(AI + IA) = \sigma_\pi\left(\varphi(A)\varphi(I) + \varphi(I)\varphi(A)\right).$$

It yields that

$$2\sigma_\pi(A) = \sigma_\pi\left(\varphi(A)\varphi(I) + \varphi(I)\varphi(A)\right),$$

for every $A \in \mathcal{B}_s(\mathcal{H})$. Since the spectral radius and the norm coincide on $\mathcal{B}_s(\mathcal{H})$, it yields

$$2\|A\| = \|\varphi(A)\varphi(I) + \varphi(I)\varphi(A)\|$$

for every $A \in \mathcal{B}_s(\mathcal{H})$. . Pick a unit vector $y \in \mathcal{H}$, there exists $A \in \mathcal{B}_s(\mathcal{H})$ such that $\varphi(A) = y \otimes y$. We have $\|A\| = r(A) = r(\varphi(A)) = r(y \otimes y) = 1$. From which we infer that

$$\|\varphi(A)\varphi(I) + \varphi(I)\varphi(A)\| = 2,$$

for any $A \in \mathcal{B}_s(\mathcal{H})$. By Lemma 3.2.2, it yields that $\varphi(I) = \lambda I$ with $\lambda \in \{-1, 1\}$. □

Remark 3.2.4. If $\varphi(I) = -I$, considering $-\varphi$, then $-\varphi$ satisfies the conditions in Theorem 3.1.5. So we may as well assume $\varphi(I) = I$ in the following, and hence

$$\sigma\left(\varphi(A)\right) = \sigma\left(A\right) \tag{3.11}$$

for every $A \in \mathcal{B}_s(\mathcal{H})$.

Lemma 3.2.5. *Let $\mathcal{H}$ be a Hilbert space of dimension at least three, and let $0 \neq A \in \mathcal{B}_s(\mathcal{H})$. Then the following statements are equivalent.*

(1) A has rank one.

(2) For any $X \in \mathcal{B}_s(\mathcal{H})$, $\sigma\left(AX + XA\right)$ contains 0 and at most two nonzero elements.

Proof. See the proof of Lemma 4.2 in [23]. □

The next lemma was proved in [20, Lemma 3.3].

Lemma 3.2.6. *Let $A, B \in \mathcal{B}_s(\mathcal{H})$. If*

$$|\langle Ax, x\rangle| + \|Ax\|\ \|x\| = |\langle Bx, x\rangle| + \|Bx\|\ \|x\|$$

holds for all $x \in \mathcal{H}$, then $A = \pm B$.

Lemma 3.2.7. *For any $A, B \in \mathcal{B}_s(\mathcal{H})$, the following statements are equivalent.*

(1) $A = B$.

(2) $\sigma(AX + XA) = \sigma(BX + XB)$ for all $X \in \mathcal{B}_s(\mathcal{H})$.

Proof. This implication (1) $\Longrightarrow$ (2) is trivial. Conversely, assume that

$$\sigma\left(AX + XA\right) = \sigma\left(BX + XB\right) \tag{3.12}$$

for all $X \in \mathcal{B}_s(\mathcal{H})$. By Lemma 3.2.1, we get

$$\begin{aligned} \|A(x \otimes x) + (x \otimes x)A\| &= r\left(|\langle Ax, x\rangle| + \|Ax\|\ \|x\|\right) \\ &= \|B(x \otimes x) + (x \otimes x)B\| \\ &= |\langle Bx, x\rangle| + \|Bx\|\ \|x\|. \end{aligned}$$

for any $x \in \mathcal{H}$. Then by Lemma 3.2.6, we infer that $A = \pm B$. Evidently, when $X = I$, then from (3.12) we get $\sigma(A) = \sigma(B)$. Thus $A = B$. □

Corollary 3.2.8. *φ is bijective.*

Proof. φ is surjective by assumption. Injectivity follows from previous lemma. $\square$

Lemma 3.2.9. *φ preserves rank one projections in both directions.*

Proof. Let A be a rank one projection. Then there exists $x \in \mathcal{H}$ such that $A = x \otimes x$. Set $B = \varphi(A)$. We will prove that B is a rank one projection. Since $B \in \mathcal{B}_s(\mathcal{H})$, then from Eq. (3.11), we have

$$\|B\| = r(B) = r(\varphi(A)) = r(A) = 1.$$

Thus $B \neq 0$. Now, by relation (3.2), we have

$$\begin{aligned} \sigma_\pi\left(B \circ \varphi(X)\right) &= \sigma_\pi\left(\varphi(A) \circ \varphi(X)\right) \\ &= \sigma_\pi\left(A \circ X\right) \end{aligned}$$

for any operators $X \in \mathcal{B}_s(\mathcal{H})$. Since A is of rank one, then by Lemma 3.2.5 we have $\sigma_\pi\left(A \circ X\right)$ contains 0 and at most two nonzero elements. Thus $\sigma_\pi\left(B \circ \varphi(X)\right) = \sigma_\pi\left(A \circ X\right)$ contains 0 and at most two nonzero elements, for every $X \in \mathcal{B}_s(\mathcal{H})$. Since φ is onto, again Lemma 3.2.5 implies that $B = \varphi(A)$ is also of rank one. Since $B \in \mathcal{B}_s(\mathcal{H})$ and every rank one self-adjoint operator has the form $\alpha_x x \otimes x$. So, $B = \varphi(x \otimes x) = \alpha_y y \otimes y$ for some $\alpha_y \in \{-1, 1\}$ and $y \in \mathcal{H}$. Also, by (3.3) we have

$$\{0, 1\} = \sigma(A) = \sigma(\varphi(A)) = \sigma(\alpha_y y \otimes y) = \{0, \|y\|^2\}.$$

Thus $\|x\| = \|y\| = 1$ and B is a rank one projection. $\square$

Lemma 3.2.10. *Let $A, C \in \mathcal{B}_s(\mathcal{H})$. If $\operatorname{tr}(A \circ B) = \operatorname{tr}(C \circ B)$ for every rank-one projection $B \in \mathcal{B}_s(\mathcal{H})$, then $A = C$.*

Proof. Let $B = x \otimes x$, where x is a unit vector. Then B is a rank-one projection and every rank-one projection takes this form. Also, we have

$$A \circ B = A(x \otimes x) + (x \otimes x)A = Ax \otimes x + x \otimes A^* x = Ax \otimes x + x \otimes Ax.$$

Therefore

$$\begin{aligned} \operatorname{tr}\left(A \circ B\right) &= \operatorname{tr}\left(Ax \otimes x + x \otimes Ax\right) \\ &= \langle Ax, x\rangle + \langle x, Ax\rangle \\ &= \langle Ax, x\rangle + \langle Ax, x\rangle = 2\left\langle Ax, x\right\rangle. \end{aligned}$$

Similarly, $\operatorname{tr}\left(C \circ B\right) = 2\left\langle Cx, x\right\rangle$.

Now by assumption, we have $\operatorname{tr}(A \circ B) = \operatorname{tr}(C \circ B)$. Thus $\langle Ax, x\rangle = \langle Cx, x\rangle$ holds for every unit vector $x \in \mathcal{H}$, which entails $A = C$ since $\mathcal{H}$ is complex. $\square$

3.3 Proof of Theorem 3.1.5

Let $A \in \mathcal{B}_s(\mathcal{H})$ and $y \otimes y$ be a rank one projection. Note that there exists $x \in \mathcal{H}$ such that $\varphi(x \otimes x) = y \otimes y$, by Lemma 3.2.9. Then Eq. (3.5) implies that

$$\sigma\left(A \circ x \otimes x\right) = \sigma\left(\varphi(A) \circ \varphi(x \otimes x)\right) = \sigma\left(\varphi(A) \circ y \otimes y\right)$$

Or by Lemma 3.2.1 we have

$$\sigma\left(A \circ x \otimes x\right) = \sigma\left(A(x \otimes x) + (x \otimes x)A\right) = \left\{0, \langle Ax, x\rangle \pm \|Ax\|\ \|x\|\right\},$$

and

$$\sigma\left(\varphi(A) \circ y \otimes y\right) = \sigma\left(\varphi(A)(y \otimes y) + (y \otimes y)\varphi(A)\right) = \left\{0, \langle \varphi(A)y, y\rangle \pm \|\varphi(A)y\|\ \|y\|\right\}.$$

Which entails that

$$\begin{aligned} \operatorname{tr}\left(\varphi(A) \circ \varphi(B)\right) &= \operatorname{tr}\left(\varphi(A) \circ y \otimes y\right) \\ &= \operatorname{tr}\left(A \circ x \otimes x\right). \end{aligned}$$

Now, let us check that φ is linear. Let $A, A' \in \mathcal{B}_s(\mathcal{H})$ be arbitrarily given and let $B = x \otimes x \in \mathcal{B}(\mathcal{H})$ be a rank-one projection. By Lemma 3.2.9, we know that $\varphi(B)$ is a also rank one projection. Set $\varphi(B) = y \otimes y$. Then we get

$$\begin{aligned} \operatorname{tr}\left(\varphi(A + A') \circ \varphi(B)\right) &= \operatorname{tr}\left((A + A') \circ x \otimes x\right) \\ &= \operatorname{tr}\left(A \circ x \otimes x\right) + \operatorname{tr}\left(A' \circ x \otimes x\right) \\ &= \operatorname{tr}\left(\varphi(A) \circ \varphi(B)\right) + \operatorname{tr}\left(\varphi(A') \circ \varphi(B)\right) \\ &= \operatorname{tr}\left((\varphi(A) + \varphi(A')) \circ \varphi(B)\right). \end{aligned}$$

By Lemma 3.2.9, $\varphi(B)$ runs over all rank-one projections when B runs over all rank-one projection. Hence Lemma 3.2.10 ensures that

$$\varphi(A + A') = \varphi(A) + \varphi(A'),$$

i.e., φ is additive. Similarly, we can check that $\varphi(\alpha x) = \alpha\varphi(x)$ for any $x \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

Thus φ is linear. Moreover it is a bijection by Corollary 3.2.8. Thus φ is a linear and bijective map such that

$$\|A\| = r(A) = r(\varphi(A)) = \|\varphi(A)\|$$

for all $A \in \mathcal{B}_s(\mathcal{H})$, by 3.11. The conclusion follows now from [27, Theorem 2] and from the classical result that every automorphism (respectively, anti-automorphism) of $\mathcal{B}(\mathcal{H})$ is inner.

Chapter 4

Maps preserving spectrum of skew Lie products of operators

This chapter deals with the problem of characterizing surjective maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ preserving the spectrum of skew Lie products of operators. The content of this chapter is a continuation of [1] and an extended version of our paper [4] accepted for publication in **Kragujevac Journal of Mathematics**.

4.1 Introduction and statement of the main result.

Throughout this chapter, $\mathcal{H}$ denotes an infinite-dimensional complex Hilbert spaces.

Definition 4.1.1. The peripheral spectrum of an element $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$\sigma_\pi(T) = \{z \in \sigma(T) : |z| = r(T)\}.$$

Given two operators A and B in $\mathcal{B}(\mathcal{H})$. The product $AB + BA^*$ is called the skew Jordan product of A and B. Also, the skew Lie product of A and B is defined by

$$[A, B]_* := AB - BA^*.$$

Definition 4.1.2. Let φ be a map from $\mathcal{B}(\mathcal{H})$ into itself. We say that φ preserves the skew Lie product of operators if

$$\sigma([\varphi(A), \varphi(B)]_*) = \sigma([A, B]_*) \tag{4.1}$$

for any operators A and $B \in \mathcal{B}(\mathcal{H})$.

The objective of chapter is to describe surjective non linear maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserving the skew Lie product of operators. Namely we shall prove the following.

Theorem 4.1.3. *A surjective map φ (no linearity of φ is assumed) from $\mathcal{B}(\mathcal{H})$ onto itself satisfies* (4.1) *if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(A) = UAU^*, \text{ for all } A \in \mathcal{B}(\mathcal{H})$$

or

$$\varphi(A) = -UAU^*, \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

As an immediate corollary of the above theorem, we characterize non linear maps preserving the skew Jordan product.

Corollary 4.1.4. *A map φ (no linearity of φ is assumed) from $\mathcal{B}(\mathcal{H})$ onto itself satisfies*

$$\sigma\left(\varphi(A)\varphi(B)+\varphi(B)\varphi(A)^*\right) \;=\; \sigma\left(AB+BA^*\right), \tag{4.2}$$

for every $A,\ B \in \mathcal{B}(\mathcal{H})$ if and only if there exists a unitary matrix $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(A) = UAU^*, \quad \textit{for any } A \in \mathcal{B}(\mathcal{H})$$

or

$$\varphi(A) = -UAU^*, \ \textit{for any } A \in \mathcal{B}(\mathcal{H}).$$

Proof. Set $T \mapsto \phi(A) := i\varphi(iA)$. Easy computations entail that ϕ satisfies

$$\sigma\left([\varphi(A),\varphi(B)]_*\right) \;=\; \sigma\left([A,B]_*\right) \tag{4.3}$$

for any $A, B \in \mathcal{B}(\mathcal{H})$. By Theorem 4.1.3 φ has the desired form. $\square$

Remark 4.1.5. . Firstly, note that the only restriction on the map φ is surjectivity; no linearity or additivity or continuity is assumed. Also, we point out that the consideration of maps φ from $\mathcal{B}(\mathcal{H})$ onto itself is for the sake of simplicity. Our result and its proof remains valid in the case where φ is a surjective map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are two different Hilbert spaces.

The case of finite dimensional Hilbert spaces was considered in [1] where it is shown that the theorem 4.1.3 remains valid without the surjectivity assumption of the map φ. The proof given therein is based on a density argument and is completely different from the one presented in the current thesis.

4.2 Preliminary results

To prove Theorem 4.1.3 some further tools are needed which are developed in this section.

Given an operator $A \in \mathcal{B}(\mathcal{H})$ and $h \in \mathcal{H}$, note that

$$\begin{aligned}
[x\otimes y, A]_*(h) &= ((x\otimes y)A - A(x\otimes y)^*)\,h \\
&= ((x\otimes y)A - A(y\otimes x))\,h \\
&= (x\otimes y)Ah - A(y\otimes x)h \\
&= \langle Ah, y\rangle\, x - \langle h, x\rangle\, Ay \\
&= \langle h, A^*y\rangle\, x - \langle h, x\rangle\, Ay, \quad (\text{since} \quad \langle Ah, y\rangle = \langle h, A^*y\rangle) \\
&= (x\otimes A^*y)h - (Ay\otimes x)h,
\end{aligned}$$

Thus

$$[x\otimes y, A]_* = x\otimes(A^*y) \;-\; (Ay)\otimes x,$$

which is an operator of rank at most two. Similarly, we have

$$[A, x\otimes y]_* = Ax\otimes y - x\otimes Ay.$$

It is well known that every Hilbert space $\mathcal{H} \neq \{0\}$ has an orthonormal basis $(e_k)_{k\in I}$. For any $A \in \mathcal{B}(\mathcal{H})$, the transpose $A^\top$ of A with respect to the basis $(e_k)_{k\in I}$ is defined as the unique operator such that

$$\langle Ae_k, e_j\rangle = \left\langle A^\top e_j, e_k\right\rangle,$$

for any $j, k \in I$.

For any $x = \sum_{k\in I} x_k e_k = \sum_{k\in I} \langle x, e_k\rangle e_k$, we write

$$\bar{x} = \sum_{k\in I} \overline{x}_k \; e_k.$$

We begin with the following.

Lemma 4.2.1. *For any $x, y \in \mathcal{H}$, we have*

$$(x \otimes y)^\top = \bar{y} \otimes \bar{x}.$$

Proof. For any $h = \sum_{k\in I} h_k e_k$, we have

$$\begin{aligned}
(x\otimes y)^\top h &= \sum_{j\in I} \left\langle (x\otimes y)^\top h, e_j\right\rangle e_j \\
&= \sum_{j\in I}\sum_{k\in I} h_k \left\langle (x\otimes y)^\top e_k, e_j\right\rangle e_j \\
&= \sum_{j\in I}\sum_{k\in I} h_k \left\langle (x\otimes y) e_j, e_k\right\rangle e_j \\
&= \sum_{j\in I}\sum_{k\in I} h_k \overline{\langle (x\otimes y)^* e_k, e_j\rangle} e_j \\
&= \sum_{j\in I}\overline{\sum_{k\in I} \overline{h}_k \left\langle (x\otimes y)^* e_k, e_j\right\rangle} e_j \\
&= \sum_{j\in I}\overline{\left\langle (x\otimes y)^* \overline{h}, e_j\right\rangle} e_j \\
&= \sum_{j\in I}\overline{\left\langle (y\otimes x) \overline{h}, e_j\right\rangle} e_j \\
&= \sum_{j\in I}\overline{\left\langle \overline{h}, x\right\rangle \langle y, e_j\rangle} e_j = \left\langle x, \bar{h}\right\rangle \sum_{j\in I} \overline{\langle y, e_j\rangle} e_j = \left\langle x, \bar{h}\right\rangle \bar{y}.
\end{aligned}$$

Or it is easy to see that $\langle x, \bar{h}\rangle = \langle h, \bar{x}\rangle$. Thus $(x\otimes y)^\top h = \langle h, \bar{x}\rangle \bar{y}$. Accordingly $(x\otimes y)^\top = \bar{y}\otimes\bar{x}$. $\square$

The next Lemma is quoted from [1]. We include its proof for the sake of completeness.

Lemma 4.2.2.
For any nonzero vectors $x, y \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have

(1)

$$\sigma\left([x\otimes y, A]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x,y)}\right\} \tag{4.4a}$$

(2)

$$\sigma\left([A, x\otimes y]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle x, Ay\rangle \pm \sqrt{\Lambda_A(x,y)}\right\} \tag{4.5a}$$

where

$$\Delta_A(x,y) = \big(\langle Ax,y\rangle + \langle Ay,x\rangle\big)^2 - 4\|x\|^2 \left\langle A^2y,y\right\rangle,$$

and

$$\Lambda_A(x,y) = (\langle x,Ay\rangle + \langle Ax,y\rangle)^2 - 4\langle x,y\rangle \langle Ax,Ay\rangle .$$

Proof. Assume that there is a nonzero scalar α in $\sigma([x\otimes y,A]_*)$ and let h be a nonzero vector in $\mathcal{H}$ such that $[x\otimes y,A]_*h = \alpha h$. It follows that

$$\langle Ah,y\rangle x - \langle h,x\rangle Ay = \alpha h, \tag{4.6}$$

and consequently,

$$\langle Ah,y\rangle \|x\|^2 - \langle h,x\rangle\langle Ay,x\rangle = \alpha\langle h,x\rangle, \tag{4.7}$$

and

$$\langle Ah,y\rangle\langle Ax,y\rangle - \langle h,x\rangle\langle A^2y,y\rangle = \alpha\langle Ah,y\rangle. \tag{4.8}$$

Observe that $\langle h,x\rangle \neq 0$. Indeed, if $\langle h,x\rangle = 0$, then from (4.7) it follows that $\langle Ah,y\rangle = 0$, which is impossible since $\alpha \neq 0$ and $h \neq 0$. Now, let us distinguish two cases.
Case 1. If $\langle A^2y,y\rangle \neq 0$. Then, by Eq. (4.8) yields $\langle Ah,y\rangle \neq 0$. Hence, from (4.7) and (4.8), we see that

$$-\alpha^2 + (\langle Ax,y\rangle - \langle Ay,x\rangle)\,\alpha + \langle Ay,x\rangle\langle Ax,y\rangle - \|x\|^2\langle A^2y,y\rangle = 0. \tag{4.9}$$

Accordingly, $\alpha = \frac{1}{2}\Big(\langle Ax,y\rangle - \langle Ay,x\rangle \pm \sqrt{\Delta_A(x,y)}\Big)$.
Case 2. If $\langle A^2y,y\rangle = 0$. Then, $((x\otimes y)A - A(y\otimes x))\,Ay = -\langle Ay,x\rangle Ay$. Moreover, if $\langle Ax,y\rangle + \langle Ay,x\rangle \neq 0$ we have $((x\otimes y)A - A(y\otimes x))\,z = \langle Ax,y\rangle z$ where we put $z = x - \frac{\|x\|^2}{\langle Ax,y\rangle + \langle Ay,x\rangle}Ay$. Thus

$$\sigma\left((x\otimes y)A - A(y\otimes x)\right) = \{0, \langle Ax,y\rangle, -\langle Ay,x\rangle\}.$$

Finally, if $\langle Ax,y\rangle + \langle Ay,x\rangle = 0$ then $((x\otimes y)A - A(y\otimes x))$ is of rank at most one and $\sigma\left((x\otimes y)A - A(y\otimes x)\right) = \{0, \langle Ax,y\rangle\}$.

For the next statement, observe that

$$[A, x\otimes y]_* = Ax\otimes y - x\otimes Ay.$$

Let α be a nonzero scalar in $\sigma([x\otimes y,A]_*)$ and let h be a nonzero vector in $\mathcal{H}$ such that $[A,x\otimes y]_*h = \alpha h$. It follows that

$$\langle h,y\rangle Ax - \langle h,Ay\rangle x = \alpha h, \tag{4.10}$$

and consequently,

$$\langle h,y\rangle \langle Ax,y\rangle - \langle h,Ay\rangle \langle x,y\rangle = \alpha\langle h,y\rangle, \tag{4.11}$$

and

$$\langle h,y\rangle \langle Ax,Ay\rangle - \langle h,Ay\rangle \langle x,Ay\rangle = \alpha\langle h,Ay\rangle. \tag{4.12}$$

Accordingly

$$\langle h,y\rangle\langle h,Ay\rangle\left((\alpha - \langle Ax,y\rangle)(\alpha + \langle x,Ay\rangle) + \langle x,y\rangle \langle Ax,Ay\rangle\right) = 0. \tag{4.13}$$

A similar argument as above yields that

$$\alpha = \frac{1}{2}\left(\langle Ax,y\rangle - \langle x,Ay\rangle \pm \sqrt{(\langle x,Ay\rangle + \langle Ax,y\rangle)^2 - 4\langle x,y\rangle \langle Ax,Ay\rangle}\right).$$

□

Corollary 4.2.3. *For any two operators A and B in $\mathcal{B}(\mathcal{H})$ the following statements are equivalent.*

(1) $A = B$.

(2) $\sigma([X, A]_*) = \sigma([X, B]_*)$ for every operator $X \in \mathcal{B}(\mathcal{H})$.

(3) $\sigma([X, A]_*) = \sigma([X, B]_*)$ for every operator $X \in \mathcal{B}_a(\mathcal{H})$.

Proof. The implications (1)$\Longrightarrow$(2)$\Longrightarrow$ (3) are trivial. Now, assume that

$$\sigma([X, A]_*) = \sigma([X, B]_*) \tag{4.14}$$

for every operator $X \in \mathcal{B}_a(\mathcal{H})$. In particular, when $X = i(x \otimes x)$, we get from (4.14) and Lemma 4.2.2-(1)

$$\begin{aligned} \sigma\left([i(x \otimes x), A]_*\right) &= \{0, \langle Ax, x\rangle \pm \sqrt{\langle A^2x, x\rangle}\} \\ &= \sigma\left([i(x \otimes x), B]_*\right) = \{0, \langle Bx, x\rangle \pm \sqrt{\langle B^2x, x\rangle}\} \end{aligned}$$

for any unit vector $x \in \mathbb{C}^n$. We shall consider two cases.
Case 1. Suppose that $\langle Ax, x\rangle + \sqrt{\langle A^2x, x\rangle} \neq 0$ and $\langle Ax, x\rangle - \sqrt{\langle A^2x, x\rangle} \neq 0$. Then, the scalars $\langle Bx, x\rangle \pm \sqrt{\langle B^2x, x\rangle}$ are nonzero also. If $\langle Ax, x\rangle + \sqrt{\langle A^2x, x\rangle} = \langle Bx, x\rangle + \sqrt{\langle B^2x, x\rangle}$ and $\langle Ax, x\rangle - \sqrt{\langle A^2x, x\rangle} = \langle Bx, x\rangle - \sqrt{\langle B^2x, x\rangle}$. By adding the two previous equations yields that $\langle Ax, x\rangle = \langle Bx, x\rangle$. Similarly, if $\langle Ax, x\rangle + \sqrt{\langle A^2x, x\rangle} = \langle Bx, x\rangle - \sqrt{\langle B^2x, x\rangle}$ and $\langle Ax, x\rangle - \sqrt{\langle A^2x, x\rangle} = \langle Bx, x\rangle + \sqrt{\langle B^2x, x\rangle}$, one can get easily that $\langle Ax, x\rangle = \langle Bx, x\rangle$.
Case 2. Suppose that $\langle Ax, x\rangle + \sqrt{\langle A^2x, x\rangle} = 0$ or $\langle Ax, x\rangle - \sqrt{\langle A^2x, x\rangle} = 0$. It follows that $\langle Bx, x\rangle + \sqrt{\langle B^2x, x\rangle} = 0$ or $\langle Bx, x\rangle - \sqrt{\langle B^2x, x\rangle} = 0$. A similar argument as in the first case entails that $\langle Ax, x\rangle = \langle Bx, x\rangle$. Accordingly, $\langle Ax, x\rangle = \langle Bx, x\rangle$, for any unit vector $x \in \mathbb{C}^n$. Thus $A = B$ as desired. □

Lemma 4.2.4. *Let A be in $\mathcal{B}(\mathcal{H})$. Then $\sigma([A, X]_*) = \{0\}$ holds for any operator $X \in \mathcal{B}(\mathcal{H})$ if and only if $A = \alpha I$, for some $\alpha \in \mathbb{R}$.*

Proof. The "if" part is obvious. To check the "only if" part, assume that

$$\sigma([A, X]_*) = \{0\}$$

holds for any operator $X \in \mathcal{B}(\mathcal{H})$. As $A - A^*$ is anti-self-adjoint then

$$\|A - A^*\| = r(A - A^*) = r([A, I]_*) = 0,$$

it follows that $A = A^*$. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\{x, Ax\}$ is a linearly independent set. Take $X = x \otimes x$. Then by Lemma 4.2.2-(2) we have

$$\sigma\left([A, X]_*\right) = \frac{1}{2}\left\{0, \pm\sqrt{\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2}\right\}.$$

This is a contradiction since $\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2 \neq 0$. Therefore A is a scalar operator. □

Lemma 4.2.5. *If $A \in \mathcal{B}(\mathcal{H})$ is nonzero operator, then*

(1) A is a self-adjoint operator if and only if $\sigma([X, A]_*) \subset i\mathbb{R}$, for any operator $X \in \mathcal{B}(\mathcal{H})$.

(2) A is anti-self-adjoint if and only if $\sigma([X,A]_*) \subset \mathbb{R}$, for any operator $X \in \mathcal{B}(\mathcal{H})$.

Proof.

(1) If $A = A^*$ then $\sigma([X,A]_*) \subset i\mathbb{R}$, since $[X,A]_* = XA - AX^* = XA - (XA)^*$. To prove the converse, assume that $\sigma([X,A]_*) \subset i\mathbb{R}$ for any operator $X \in \mathcal{B}(\mathcal{H})$. In particular by Lemma 4.2.2-(1) we get

$$\sigma\left([x \otimes y, A]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x,y)}\right\} \subset i\mathbb{R}$$

for any $x, y \in \mathcal{H}$. Which yields that

$$0 \;=\; \Re\left(\langle Ax, y\rangle - \langle Ay, x\rangle\right) = \langle (A-A)^* x, y\rangle + \langle y, (A - A^*)x\rangle\,.$$

Replace x by ix in the above equality, we get

$$\langle (A-A)^* x, y\rangle - \langle y, (A - A^*)x\rangle = 0.$$

Accordingly $\langle (A-A)^* x, y\rangle = 0$, for any $x, y \in \mathcal{H}$. Thus $A = A^*$.

(2) We have

$$\begin{aligned} A \in \mathcal{B}_a(\mathcal{H}) &\iff iA \in \mathcal{B}_s(\mathcal{H}) \\ &\iff \sigma([X, iA]_*) \subset i\mathbb{R}, \forall X \in \mathcal{B}(\mathcal{H}) \quad \text{(By Lemma 4.2.5-(1))} \\ &\iff i\sigma([X, A]_*) \subset i\mathbb{R}, \forall X \in \mathcal{B}(\mathcal{H}) \quad \text{(Since } \sigma([X, iA]_*) = i\sigma([X, A]_*)\text{)} \\ &\iff \sigma([X, A]_*) \subset \mathbb{R}, \forall X \in \mathcal{B}(\mathcal{H}). \end{aligned}$$

□

Lemma 4.2.6. *For any $x \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have,*

$$\sigma(A(x \otimes x) + (x \otimes x)A) \;=\; \{0, \alpha_1, \alpha_2\},$$

where

$$\alpha_1 = \langle Ax, x\rangle + \sqrt{\langle A^2 x, x\rangle \|x\|^2}, \text{ and } \alpha_2 = \langle Ax, x\rangle - \sqrt{\langle A^2 x, x\rangle \|x\|^2}$$

.

Proof. Straightforward computation entails that

$$[(ix) \otimes x, A]_* = i\sigma(A(x \otimes x) + (x \otimes x)A) = \sigma(Ax \otimes x + x \otimes A^* x).$$

Hence, by Lemma 4.2.2 the result follows. □

Corollary 4.2.7. *For any $x \in \mathcal{H}$ and $A \in \mathcal{B}_s(\mathcal{H})$, we have*

(i)

$$\sigma(A(x \otimes x) + (x \otimes x)A) \;=\; \{0, \langle Ax, x\rangle \pm \|Ax\|\;\|x\|\}\,.$$

(ii)

$$\sigma_\pi(A(x\otimes x)+(x\otimes x)A) = \begin{cases} \{\pm\|Ax\|\ \|x\|\} & \textit{if } \langle Ax,x\rangle = 0 \\ \{\langle Ax,x\rangle + \|Ax\|\ \|x\|\} & \textit{if } \langle Ax,x\rangle > 0. \\ \{\langle Ax,x\rangle - \|Ax\|\ \|x\|\} & \textit{if } \langle Ax,x\rangle < 0. \end{cases}$$

Proof. By Lemma 4.2.6, we have

$$\begin{aligned} \sigma(A(x\otimes x)+(x\otimes x)A) &= \left\{0, \langle Ax,x\rangle \pm \sqrt{\langle A^2x,x\rangle\ \|x\|^2}\right\} \\ &= \left\{0, \langle Ax,x\rangle \pm \sqrt{\langle Ax,A^*x\rangle\ \|x\|^2}\right\} \\ &= \left\{0, \langle Ax,x\rangle \pm \sqrt{\langle Ax,Ax\rangle\ \|x\|^2}\right\} \\ &= \left\{0, \langle Ax,x\rangle \pm \sqrt{\|Ax\|^2\ \|x\|^2}\right\} \\ \{0, \langle Ax,x\rangle \pm \|Ax\|\ \|x\|\} \end{aligned}$$

Therefore

$$\sigma_\pi(A(x\otimes x)+(x\otimes x)A) = \begin{cases} \{\pm\|Ax\|\ \|x\|\} & \text{if } \langle Ax,x\rangle = 0 \\ \{\langle Ax,x\rangle + \|Ax\|\ \|x\|\} & \text{if } \langle Ax,x\rangle > 0. \\ \{\langle Ax,x\rangle - \|Ax\|\ \|x\|\} & \text{if } \langle Ax,x\rangle < 0. \end{cases}$$

□

4.3 Proof of Theorem 4.1.3

The "if" part is obvious. We will complete the proof of the "only if" part after proving several claims.

Claim 1. φ is injective.

Proof. For $A, B \in \mathcal{B}(\mathcal{H})$, assume that $\varphi(A) = \varphi(B)$. Then, for every $X \in \mathcal{B}(\mathcal{H})$, we have

$$\sigma([X,A]_*) = \sigma([\varphi(X),\varphi(A)]_*) = \sigma([\varphi(X),\varphi(B)]_* = \sigma([X,B]_*).$$

It then follows from Corollary 4.2.3 that $A = B$, and φ is injective. □

Claim 2. φ preserves self-adjoint and anti-self adjoint operators in both directions. In particular, we have $\varphi(0) = 0$.

Proof. Pick up an operator $A \in \mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}_s(\mathcal{H})$, then

$$\sigma([\varphi(X),\varphi(A)]_*) = \sigma([X,A]_*) \subset i\mathbb{R}.$$

As φ is surjective then Lemma 4.2.5-(1) entails that $\phi(A) \in \mathcal{B}_s(\mathcal{H})$. Similarly if $A \in \mathcal{B}_a(\mathcal{H})$, we have $\sigma([\varphi(X),\varphi(A)]_*) \subset \mathbb{R}$. By Lemma 4.2.5-(2), we get $\phi(A) \in \mathcal{B}_a(\mathcal{H})$.

For the converse, note that φ is bijective and φ^{-1} satisfies Eq. (4.1) A similar discussion entails that if $\varphi^{-1}(A) \in \mathcal{B}_s(\mathcal{H})$ (resp. $\varphi^{-1}(A) \in \mathcal{B}_a(\mathcal{H})$ then so is A. □

Claim 3. φ is homogenous, i.e $\varphi(\alpha A) = \alpha A$, for any $\alpha \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$.

Proof. For any $\alpha \in \mathbb{C}$ and $A, X \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} \sigma([\varphi(X), \varphi(\alpha A)]_*) &= \sigma([X, \alpha A]_*) \\ &= \alpha\ \sigma([X, A]_*) \\ &= \alpha\ \sigma([\varphi(X), \varphi(A)]_*) \\ &= \sigma([\varphi(X), \alpha\varphi(A)]_*). \end{aligned}$$

Hence

$$\sigma([\varphi(X), \varphi(\alpha A)]_*) = \sigma([\varphi(X), \alpha\varphi(A)]_*)$$

for any $X \in \mathcal{B}(\mathcal{H})$. Since φ is bijective, we infer from Lemma 4.2.4 that $\varphi(\alpha A) = \alpha\varphi(A)$. □

Claim 4. There exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

- $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$, or
- $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$. From Claim 3 and (4.1), we have

$$\begin{aligned} \sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) &= -\sigma\left(\varphi(iA)\varphi(iB) - \varphi(iB)\varphi(iA)^*\right) \\ &= -\sigma\left(-AB - BA^*\right) \\ &= \sigma\left(AB + BA^*\right). \end{aligned}$$

Thus

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{4.15}$$

for any $A, B \in \mathcal{B}(\mathcal{H})$. Now Claim 2 implies that $\varphi(A) \in \mathcal{B}_s(\mathcal{H})$ whenever $A \in \mathcal{B}_s(\mathcal{H})$. This together with Claim 1. entail that the restriction $\varphi_{|\mathcal{B}_s(\mathcal{H})} : \mathcal{B}_s(\mathcal{H}) \longrightarrow \mathcal{B}_s(\mathcal{H})$ is well defined and bijective. Moreover Eq. (4.15) implies that

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)\right) = \sigma\left(AB + BA\right),$$

for any $A, B \in \mathcal{B}_s(\mathcal{H})$. Therefore by [17, Theorem 3.1] (see also [34, Theorem 2] there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

- $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$, or
- $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

□

In particular Claim 4 implies that $\varphi(I) = c = \pm I$. In the sequel we may and shall assume that $\phi(I) = I$. Define a map $\psi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ by

$$\psi(A) = U^*\varphi(A)U$$

for every $A \in \mathcal{B}(\mathcal{H})$. Then ψ is a bijective maps a satisfying

$$\sigma\left(\psi(A)\psi(B) + \psi(B)\psi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{4.16}$$

for every $A, B \in \mathcal{B}(\mathcal{H})$. Moreover, we have either

$$\psi(A) = U^*\varphi(A)U = U^*(UAU^*)U = A, \tag{4.17}$$

for any $A \in \mathcal{B}_s(\mathcal{H})$, or

$$\psi(A) = U^*\varphi(A)U = U^*(UA^\top U^*)U = A^\top, \tag{4.18}$$

for any $A \in \mathcal{B}_s(\mathcal{H})$.

Claim 5. The form (4.18) cannot occur.

Proof. Assume for the sake of contradiction that $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. Let $\{e_j, j \in I\}$ be the orthonormal basis with respect to which $A^\top$ is computed, for every $A \in \mathcal{B}_s(\mathcal{H})$. To get a contradiction we shall prove that $\langle Ax, x\rangle = \langle \psi(A)x, x\rangle$ for any $x \in \mathcal{H}$ and $A \in \mathcal{B}_s(\mathcal{H})$. To do so it suffices to prove that

$$\langle Ae_k, e_l\rangle = \langle \psi(A)e_k, e_l\rangle \tag{4.19}$$

for any k and l in I and $A \in \mathcal{B}(\mathcal{H})$.

Let $A \in \mathcal{B}(\mathcal{H})$ and pick up two vectors e_k and e_l in $\{e_j, j \in I\}$. For any $\alpha, \beta \in \mathbb{R}$, set $a = \alpha e_k + \beta e_l$. Note that

$$\psi(a \otimes a) = (a \otimes a)^\top = a \otimes a.$$

Now, by (4.16) we have

$$\begin{aligned}
\sigma\left((a \otimes a)A + A(a \otimes a)\right) &= \sigma\left((a \otimes a)A + A(a \otimes a)^*\right)\\
&= \sigma\left(\psi(a \otimes a)\psi(A) + \psi(A)\psi(a \otimes a)^*\right)\\
&= \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)^*\right)\\
&= \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right).
\end{aligned}$$

Accordingly

$$\sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right) = \sigma\left((a \otimes a)A + A(a \otimes a)\right). \tag{4.20}$$

Corollary 4.2.6 together with (4.20) entail that

$$\left\{0, \langle \psi(A)a, a\rangle \pm \|a\|\sqrt{\langle \psi(A)^2 a, a\rangle}\right\} = \left\{0, \langle Aa, a\rangle \pm \sqrt{\langle A^2 a, a\rangle\, \|a\|^2}\right\}.$$

Accordingly $\langle \psi(A)a, a\rangle = \langle Aa, a\rangle$. Since α and β are arbitrary, we infer that

$$\langle Ae_k, e_k\rangle = \langle \psi(A)e_k, e_k\rangle$$

and

$$\langle A(e_k + e_l), (e_k + e_l)\rangle = \langle \psi(A)(e_k + e_l), (e_k + e_l)\rangle$$

for every $k, l \in I$. Since A and $\psi(A)$ are in $\mathcal{B}_s(\mathcal{H})$, we infer that

$$\langle Ae_k, e_l\rangle = \langle \psi(A)e_k, e_l\rangle \tag{4.21}$$

This in particular implies that $\psi(A) = A$ for every for any $A \in \mathcal{B}_s(\mathcal{H})$. This is impossible since $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. □

Claim 6. $\psi(A) = A$ for any $A \in \mathcal{B}(\mathcal{H})$.

Proof. We have $\psi(A) = A$ for any $A \in \mathcal{B}_s(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, using a similar reasoning as above, one can show that $\langle Ax, x\rangle = \langle \psi(A)x, x\rangle$ for any $x \in \mathcal{H}$. Hence, $\psi(A) = A$ as desired. □

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ملخص الرسالة

ليكن H, K فضائا هيلبرت المركب ذو البعد الغير منتهي , حيث أن$B(H)$, ($B(K)$على التوالي) ترمز إلى الجبر لكل المؤثرات الخطية المحدودة على H أو(K على التوالي) .

في بحوث ودراسات سابقة نجد ان تم دراسة وصف التطبيقات على المؤثرات الخطية والمصفوفات التي تحافظ على دوال معينة ,ومجموعات جزئية وعلاقات تم دراستها على نطاق واسع ,أنظر *[7], [9], [10],* *[11], [14], [15], [16], [17], [22], [28], [29], [31], [32]*. أحد هذه المسائل التقليدية في هذا المجال من البحث هو تمييز و وصف الدوال التي تحافظ على أطياف ضرب المؤثرات الخطية .

(Molnar) في *[29]* , قام بدراسة الدوال التي تحافظ على ضرب المؤثرات الخطية والمصفوفات ,كما أن نتائجه التي حصل عليها تم تمديدها وتوسيعها في العديد من الدراسات , أنظر *[8], [1], [2], [12],* *[13], [19], [20], [21], [23] [24],*.

نلاحظ في *[2]*, أن مسألة توصيف التطبيقات بين جبور المصفوفات التي تحافظ على أطياف ضرب كثيرات الحدود للمصفوفات تم أخذها في عين الإعتبار . وعلى وجه الخصوص النتائج التي تم الحصول عليها تم تمديدها وتوحيدها في العديد من الدراسات نجدها في *[11]* , *[13]*.

في هذه الأطروحة سوف نقوم بدراسة صياغة الدوال الشاملة التي تحافظ على أطياف ضرب (*skew Lie*) للمؤثرات الخطية على فضاء هيلبرت المركب ذو البعد الغير المنتهي .

نقول أن الدالة $\varphi: B(H) \to B(K)$ تحافظ على ضرب (*skew Lie*) للمؤثرات الخطية إذا حققت أن $[\varphi(T), \varphi(S)]_*$ =$[T, S]_*$, حيث يُعرف ضرب (*skew Lie*) بأنه $[T, S]_* = TS - ST^*$ وذلك لأي T, S مؤثران خطيان ينتميان إلى $B(H)$.

ستكون الخطة في هذه الرسالة كالتالي:

- الفصل الأول عبارة عن مقدمة تصف المسألة التي نريد دراستها وكيف أنه تم دراستها في العديد من الحالات ومجالات البحث .
- الفصل الثاني يحتوي على تعاريف ونتائج أساسية تم إستخدامها في الرسالة .
- الفصل الثالث عبارة عن دراسة الدوال التي تحافظ على الطيف المحيطي لضرب جوردن للمؤثرات ذاتية الترافق بالإضافة إحتوائه على العديد من المفاهيم والنظريات المهمة حول هذا الموضوع والبراهين لذلك.
- الفصل الأخير إستعراض النتيجة الأساسية لهذه الأطروحة وبرهانها ,حيث سنقوم بدراسة توصيف الدوال الشاملة $\varphi: B(H) \to B(K)$ التي تحافظ على طيف ضرب (*skew Lie*) للمؤثرات الخطية , ويحتوي الفصل أيضاً على العديد من المفاهيم المهمة والنتائج الأساسية والبراهين على ذلك .

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كلية العلوم التطبيقية
قسم العلوم الرياضية

الدوال التي تحافظ على ضرب لي للمؤثرات الخطية على فضاء هيلبرت المركب

هذه الرسالة مقدمة لنيل درجة الماجستير في الرياضيات (2020م -1441هـ)

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