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PROBABILISTIC STRATEGIES IN GROUP THEORY

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استراتيجيات احتمالية في نظرية الزمر

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الملخص

نظرية الزمر ونظرية الاحتمال هما علمان مستقلان، ومع تطور العلوم أصبح من الممكن ربط العلمين في إتجاه جديد يتيح لنا حساب احتمال وجود ميزة معينة في الزمرة. الهدف من هذا البحث هو مسح و دراسة وفهم والمساهمة في مفهوم حساب احتمالية وجود ميزة معينة في أي زمرة منتهية G ، بإستخدام التعريف الرياضي للاحتمال للمساعدة في حل بعض المشاكل في نظرية الزمر. يناقش هذا البحث مفهوم حساب الاحتمالية في أي زمرة منتهية G للميزات التالية: تبادل عنصرين في الزمرة G ، منتج الموتر غير الأبيلي لعنصرين في G عندما يكون متساوٍ مع العنصر المحايد لـ G ، منتج خارجي غير أبيلي لعنصرين في G عندما يكون متساوٍ مع العنصر المحايد لـ G ، و إختيار كتلة معينه مع الأخذ بعين الإعتبار للعدد الأولي المختار p . أخيرًا، سيتم إنشاء نظرية جديدة مكافئة تحسب احتمال تبادل عنصرين في الزمرة G بإستخدام مفهوم ثوابت البنية الجبرية مع تقديم أمثلة على النظرية الجديدة.

ننظم هذا البحث على النحو التالي: الباب ١ مفاهيم و معلومات أساسية ضرورية لتقديم موضوع البحث. الباب ٢ يتم فيه دراسة مفهوم درجة التبادلية النسبية. الباب ٣ يتم فيه دراسة مفهوم درجة التنسور النسبية. الباب ٤ يتم فيه دراسة مفهوم الدرجة الخارجية النسبية. الباب ٥ يتم فيه دراسة مفهوم طرق الاحتمالات في نظرية الكتلة ومناقشة النتائج الرئيسية في هذا الموضوع. بينما يحتوي الباب ٦ على مساهمتنا و بصمتنا في تطوير هذا العلم بإنشاء نظريه جديده هي الأولى من نوعها والتي تستخدم مفهوم ثوابت البنية الجبرية في هذا الإتجاه مع ذكر أمثله عده عليها، وبذلك يتكون لدينا الأساس اللذي يسمح بإستخدام هذا المفهوم و البناء عليه مستقبلاً في هذا الإتجاه.

Abstract

The aim of this research is to survey, study, understand and contribute to the concept of calculating the probability of a specific feature being in any finite group G, using the regular mathematical definition of probability. This concept is a new trend that connects group theory and probability theory to help solve some problems in group theory. We will study the concept of calculating the probability in any finite group G for the following features: commute of two elements in group G, the non-abelian tensor product for two elements in G when it is equal to the identity element of G, the non-abelian exterior product for two elements in G when it is equal to the identity element of G, and choosing a particular p-block B with respect to the chosen prime number p. Lastly, a new equivalent theorem will be established that calculates the probability of commute of two elements in group G by using the concept of structure constants. Examples of the new theorem will be provided.

Notations

$\mathbb{Z}$, $\mathbb{Z}^+$	Integers number, Positive integer
$\mathbb{C}$, $\mathbb{N}$	Complex number, Natural number
$\mathbb{R}$	Real number
X	Set
$\emptyset$	The empty set
G	Finite group
F	Field
S_n	Symmetric group of degree n
A_n	Alternating group of degree n
D_{2n}	Dihedral group of degree 2n
C_n	Cyclic group of degree n
V_4	Klein four-group $(\langle a, b : a^2 = b^2 = (ab)^2 = 1\rangle)$
$A \triangleleft B$	A is a normal subgroup of B
$\|G\|$	Order of G
1_G	The identity element of G
$ker(\alpha)$	Kernel of the function α
$Im(\alpha)$	Image of the function α
a^b	$b^{-1}ab$
$Stab_G(x)$	The stabilizer of x in G
$O(x)$	The orbit of x
$\sigma_g(x)$	Action of S_n on the set X
$C_G(x)$	Centralizer of x in G
$C(x)$	Conjugace class of x
$a \sim b$	a is a conjugate to b
p	Prime number
p-group	Group of order p^a, $a \in \mathbb{N}$
$Fix_X(G)$	All elements in a set X which fixed by the group G
$\alpha R \beta$	$\exists g \in G : \beta = g \cdot \alpha,\ \alpha, \beta \in X$
$\dot{\cup}$	Disjoint union
a/b	b divides a
$GL(n, F)$	General linear group of degree n over a field F
ρ	Group homomorphism function from G to GL(n,F)of degree n
χ	Function from G to F, determined by $\chi(g) = trace\,(\rho(g)\,)$
$k(G)$	Number of conjugace class of G
$\chi_1, ..., \chi_{k(G)}$	Irreducible characters of G
$Irr(G)$	Set of all irreducible characters of G $\left(\{\chi_1, ..., \chi_{k(G)}\}\right)$
$\equiv_p$	Congruent modulo p

B	p-block
$d(\chi) = d$	The defect number of an irreducible character χ
$\chi(1)_p$	The p-part of $\chi(1)$
$\|G\|_p$	The p-part of $\|G\|$
$d(B)$	The defect number of a p-block B
$h(\chi)$	The height number of an irreducible character χ
D	The defect group
$Irr(B)$	The set of all irreducible characters which belong to a p-block B
$k(B)$	The order of $Irr(B)$
B_0	The principal p-block
$a \equiv b(\mod n)$	$a - b$ is divisible by n
$P(G)$	Probability of G
G'	Commutator subgroup of G
$[H, K]$	$\langle\{[h, k] : h \in H, k \in K\}\rangle$
$\langle x_1, ..., x_n\rangle$	The subgroup generated by $\{x_1, ..., x_n\}$
$G_1 \times ... \times G_n$	The direct product of $G_1, ..., G_n$
I_n	$n \times n$ identity matrix
K	Class sum
$\mathbb{C}[G]$	Group algebra
ω_χ	Function from $Z(\mathbb{C}[G])$ to $\mathbb{C}$, which dependin on χ
X^g	Set of points of X which fixed by the element g of G
X/G	Number of orbits of G acting on X
$[x, y]$	$x^{-1}y^{-1}xy$
$\otimes$	Non-abelian tensor product
$\wedge$	Exterior product
κ	The map from a group of non-abelian tensor product to the commutator group
κ'	The map from a group of non-abelian exterior product to the commutator group
J(G,H,K)	Kernel of κ
M(G,H,K)	Kernel of κ'
$C_G^{\otimes}(x)$	Tensor centralizer of x with respect to G
$C_G^{\wedge}(x)$	Exterior centralizer of x with respect to G
Z(G)	Center of G
$Z^{\otimes}(G)$	Tensor center of G
$Z^{\wedge}(G)$	Exterior center of G
d(H,K)	Relative commutativity degree of H and K
$d^{\otimes}(H, K)$	Relative tensor degree of H and K
$d^{\wedge}(H, K)$	Relative exterior degree of H and K
$P(\chi)$	The probability of ordinary irreducible character χ
$P(B)$	The probability of a p-block of G
$C_G(H)$	Centralizer of a subgroup H in G
ξ	The probability of a p-block of defect zero
$[G : H]$	The index of H in G
S^*	The set of all p-blocks of the group G relative to a fixed prime number p
$dim_F(A)$	Dimension over F of A
iff	if and only if

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Introduction

Group theory and Probability theory are independent sciences. With the development of science, it became possible to link the two sciences together in a new direction. The new direction allows us to calculate the probability of a specific feature being in a group. The beginning of this trend is a question that was raised by David J. Rusin [24], who posed the question: what is the probability that two elements of a finite group commute?.

The relationship between the representation of finite groups and probability theory has grown up rapidly, to investigate and solve some problems in group theory. In 1968, Erdős and Turán [8] studied some problems in statistical group theory. In 1970, Gallagher [10] used character theory to investigate the probability of commuting elements. In 1973, Gustafson [12] initiated the probability that two group elements commute for infinite groups, in which he used differential geometry as well as the abstract harmonic analysis to get parallel results for finite groups. In 1988, Persi Diaconis [7] has made fundamental contributions in the relationship between representation of finite groups and probability theory. In 2006, Guralnick and Robinson [11] investigated several objectives such as giving general properties and giving elementary proofs of numerical properties of the commutativity degree. In 2010, Alghamdi and F. G. Russo attempted to discuss Dade's ordinary conjecture in this framework. Then in 2012, they investigated a generalization of the probability that the commutator of two group elements is equal to a given element. They studied the relative tensor degree of finite groups in articles published in 2014, see [2], [3] and [4] respectively.

On the other hand, block theory is a fascinating subject and a very rich topic in finite group theory. There are many approaches to tackle block theory. Character theory is one approach. Likewise, the theory of probability can be applied to tackle block theory. This is shown by Alghamdi's paper 2016, in which he attempted to get a link between block theory and probability theory.

Let p be a prime number and B a p-block of a finite group G. The thesis is divided to six chapters as follows:

Chapter 1, contains the basic concepts, on which the thesis depends. The first section includes concepts that are presented in a topic of group action on a set, and under these concepts we will study Orbit-Stabilizer Theorem, when the group G acts on itself as a set, and we will get the changes that will be used in the coming chapters along with the Burnside's Lemma in this concept. In the second section, definitions of representation and character theory will be presented with some theories related to them. In the third section, we will review some concepts in block theory on the approach of character theory, and study some examples that will be deduced and built upon in the next sections, especially in Chapter 5.

Chapter 2, in the first section, we will give a solution to Rusin's question [24] by applying the classical definition of probability to measure the property of the commute of two elements in group G. Furthermore, theories will be mentioned which give us an equivalent definition for calculating this possibility and upper bounded.

For instance, for any group G the probability of commute two elements in it must be equal to or less than 5/8, and some examples will be given in the second section. In the third and last section, we will study the definition of a probability that has a randomly chosen commutator which is equal to a given element of group G. Moreover, we will present states theorem, which gives us an equivalent definition via character theory with some examples applied, and provide some recent investigations in this concept.

Chapter 3, in the first section, we will introduce definitions of a compatiplity action and non-abelian tensor product. Furthermore, we will establish some basic properties that describe the main calculus rules in the non-abelian tensor product, with some theories and relationships under this concept. In the second section, we will study the definitions of tensor centralizer and tensor center which have been based on the definitions which introduced in previous section, and we will study the algebraic structures of these concepts, the definition of the relative tensor degree, and explanation of the relation between the tensor centralizer and the tensor degree. In the third and last section, we will study the relation between the relative commutative degree and relative tensor degree.

Chapter 4, in the first section, we will introduce the definition of a non-abelian exterior product, with some relations under this concept. In the second section, we will study the definitions of exterior centralizer and exterior center which have been based on the definitions which introduced in previous section. Moreover, the algebraic structures of these concepts along with the previous concepts, and the definition of the relative exterior degree are introduced. In the third and last section, we will study the general relation among the relative commutative degree, relative tensor degree, and relative exterior degree.

Chapter 5, in the first and second sections, we will study the notion of the probability of a p-block B with applied examples. In the third and last section, we will study some facts about the probability of irreducible ordinary character and principal p-block B_0 in group G, by Brauer-Feit Theorem as in [25, Theorem 2.4]. Furthermore, we will show the relation between the probability of the principal p-block B_0 in group G, and the order of irreducible ordinary character of G and some current conjectures in this concept.

Chapter 6, in this chapter, we will establish a new equivalent theorem that calculates the probability of commute of two elements in group G, by using the concept of structure constants. Therefore, we will begin with the concept of algebra over field F, from which we will deduce the definition of structure constants that will be discussed in the first section. In the second section, we will study the definition of the group algebra G over F, which is denoted by F[G] and mention the basis for it. Moreover, in the same section, we will study the definition of the center of F[G] which is denoted by Z(F[G]), and the impotent theorem in this section, which explains the basis for Z(F[G]). In the third and last section, we will establish a new theorem that calculates the probability of commute of two elements in group G by using the concept of structure constants, and examples of the new theorem will be provided. All groups mentioned in this thesis are supposed to be finite.

Chapter 1

Basic Concepts

In this introductory chapter, we shall review some basic notions, definitions, theories and examples in group theory. Especially those terms related to the terminology employed in this thesis. In the first section, we will mention the concept of group acts on a set and the most important theorem in this concept the Orbit-Stabilizer Theorem, which will be used and rely on it as a basis of understanding and proofs in the next subjects of this work, and then record what changes to the theorem in special cases, like when the group G acts on at self as a set or when the set $X = \{(x, y) \in G \times G\}$, as regarding and serving my purpose. In the second section, we will mention all concepts of representation, ordinary representation, character, and some theories and properties that will be useful for this work. In the third section, we will define an equivalence relation on the characters by fixing a prime number p, which divides them for equivalence classes which are called by p-blocks, and study some examples that will be deduced and built upon in the next sections, especially in Chapter 5. The basic references of this chapter are [5],[9],[14],[15],[16],[20] and [21]. All the groups presented in this work are supposed to be finite.

1.1 Group acts on a set

In this section, we shall review some concepts related to the group acts on a set.

Definition 1.1. *Let G be a group. Let X be a finite set. A (left) action of G on X is a map $\alpha : G \times X \to X$ given by $\alpha((g, x)) = g \cdot x$, which satisfies the following conditions:*
1)$1_G \cdot x = x$.
2)$(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.
For all $x \in X$ and $g_1, g_2 \in G$.
In the similar way we can define a right action by the map $\alpha : X \times G \to X$ given by $\alpha((x, g)) = x \cdot g$, which satisfies the following conditions:
1) $x \cdot 1_G = x$.
2) $x \cdot (g_1g_2) = (x \cdot g_1) \cdot g_2$.
For all $x \in X$ and $g_1, g_2 \in G$.

Remark. *If we have an action of G on X, then we say that G acts on X or that X is a G-set.*

In this thesis virtually all actions considered will be left actions.

Theorem 1.1. *Let G be a group acts on a set X. Then the relation on X defined by relation α R β if $\exists g \in G : \beta = g \cdot \alpha$ is an equivalence relation, where $\alpha, \beta \in X$.*

Proof. Since $1_G \cdot \alpha = \alpha$ for all $\alpha \in X \Rightarrow \alpha R\alpha$ (Reflexive). Since α R $\beta \Rightarrow \exists g \in G : \beta = g \cdot \alpha \Rightarrow g^{-1} \cdot \beta = (g^{-1}g) \cdot \alpha \Rightarrow g^{-1} \cdot \beta = \alpha \Rightarrow \beta R\alpha$ (Symmetric). $\alpha R\beta$ and $\beta R\delta$, since $\alpha R\beta \Rightarrow \exists g \in G : \beta = g \cdot \alpha$ and since $\beta R\delta \Rightarrow \exists h \in G : \delta = h \cdot \beta$. Then $\delta = h \cdot \beta = h \cdot (g \cdot \alpha) = (hg) \cdot \alpha \Rightarrow \alpha R\delta$ (Transitive). Thus R is equivalent relation on X.

□

Remark. *The equivalence class of the equivalent relation are called the* ***Otbits*** *of G on X.*

Definition 1.2. *Let G be a group acts on the set X. Let $x \in X$. The* ***orbit*** *of x under G is the set of all elements in X of the form $g \cdot x$ for all $g \in G$, which denoted by $O(x)$. In symbols*

$$O(x) = \{g \cdot x | g \in G\} \subseteq X.$$

Remark. *If $O(x) = X$ for all $x \in X$. Then the action is called* ***transitive****.*

Definition 1.3. *Let G be a group acts on the set X. Let $x \in X$. The* ***stabilizer*** *of x in G is the set of elements $g \in G$ such that $g \cdot x = x$, which is denoted by $Stab_G(x)$. In symbols*

$$Stab_G(x) = \{g \in G | g \cdot x = x\}.$$

Lemma 1.1. *Let G be a group acts on the set X. Let $x \in X$. Then $Stab_G(x)$ is a subgroup of G for all $x \in X$.*

Proof. The subgroup conditions are verified as follows: $Stab_G(x)$ is not an empty set since by Definition 1.1 we have $1_G \cdot x = x$, hence $1_G \in Stab_G(x)$ for all $x \in X$, consider any two elements g_1 and $g_2 \in Stab_G(x)$ i.e. $g_1 \cdot x = x$ and $g_2 \cdot x = x$. Then

$$g_1 \cdot x = x \Rightarrow g_1 \cdot (g_2^{-1} \cdot x) = x \Rightarrow (g_1 g_2^{-1}) \cdot x = x$$

hence $g_1 g_2^{-1} \in Stab_G(x)$. Thus $Stab_G(x)$ is a subgroup of G.

□

The next theorem is the most important theorem in the theory of group actions, as in [5, Corollary 5].

Theorem 1.2 (Orbit-Stabilizer Theorem)**.** *Let G be a group acts on a finite set X. For any $x \in X$, we have*

$$|O(x)| = \frac{|G|}{|Stab_G(x)|}.$$

Proof. Let $x' \in O(x)$, then there is g $\in G$ such that $x' = g \cdot x$, also there is $gStab_G(x)$ where $\forall y \in gStab_G(x)$ there is $g' \in Stab_G(x)$ such that $y = g \cdot g'$. Then $y \cdot x = (gg') \cdot x = g \cdot (g' \cdot x) = g \cdot x = x'$. This means that there are at most element in $O(x)$, similar to the set of cossets of $Stab_G(x)$ in G. Let g and $y \in G$ such that $g \cdot x = y \cdot x \Rightarrow g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (y \cdot x) \Rightarrow (g^{-1}g) \cdot x = (g^{-1}y) \cdot x \Rightarrow 1_G \cdot x = (g^{-1}y) \cdot x \Rightarrow x = (g^{-1}y) \cdot x$. This means that $g^{-1}y \in Stab_G(x) \Rightarrow y \in gStab_G(x)$. Thus g and y belong to the same set of cosset $Stab_G(x)$. It follows that the α: $\frac{G}{Stab_G(x)} \mapsto O(x)$ which is given by $g'Stab_G(x) \mapsto g' \cdot x$. The map α is a well-defined bijective of the set cosset of $Stab_G(x)$ in G onto the orbit $O(x)$ =$\{g' \cdot x : g' \in G\}$, hence |O(x)| = [$G$:Stab$_G(x)$]= $|G|/|Stab_G(x)|$.

□

Example 1.1. *An action of the symmatric group S_3 on the set $X = \{1, 2, 3\}$ is given by $g \cdot x = \sigma_g(x)$. First we want to prove that it is a group action*

$$\begin{aligned} 1)i)(g_1 g_2) \cdot x &= \sigma_{g_1 g_2}(x) \\ &= (\sigma_{g_1}\sigma_{g_2})(x) \\ &= \sigma_{g_1}(\sigma_{g_2}(x)) && \textit{(from Definition 1.1)} \\ &= g_1 \cdot (g_2 \cdot x).\ \forall g_1, g_2 \in G,\ \forall x \in X. \\ ii)1_G \cdot x &= \sigma_{1_G}(x) = x.\ \forall x \in X. \end{aligned}$$

So, this is indeed a group action.

$$\begin{aligned} 2)Stab_{S_3}(1) &= \{g \in S_3 : g \cdot 1 = \sigma_g(1) = 1\} = \{(1),(23)\}.\\ Stab_{S_3}(2) &= \{g \in S_3 : g \cdot 2 = \sigma_g(2) = 2\} = \{(1),(13)\}.\\ Stab_{S_3}(3) &= \{g \in S_3 : g \cdot 3 = \sigma_g(3) = 3\} = \{(1),(12)\}. \end{aligned}$$

$$\begin{aligned} 3)O(1) &= \{g \cdot 1 = \sigma_g(1) : g \in G\} = \{1,2,3\} = X.\\ O(2) &= \{g \cdot 2 = \sigma_g(2) : g \in G\} = \{1,2,3\} = X.\\ O(3) &= \{g \cdot 3 = \sigma_g(3) : g \in G\} = \{1,2,3\} = X. \end{aligned}$$

We can see by Theorem 1.2

$$|O(1)| = \frac{|G|}{|Stab_G(1)|} = \frac{6}{2} = 3,$$

and the same way for all $x \in X$.

Remark. *The exponential notation for the conjugation of two elements x and y in a group G, that is the notation $x^y = y^{-1}xy$.*

Example 1.2. *Let G be a group acts on itself as a set(i.e. $X = G$) by conjugation, that is for all $x \in X = G$ and $g \in G$, we define $g \cdot x = g^{-1}xg$. First we want to prove that it is a group action*

$$\begin{aligned} 1)i)g_1 \cdot (g_2 \cdot x) &= g_1 \cdot (g_2^{-1}xg_2)\\ &= g_1^{-1}(g_2^{-1}xg_2)g_1\\ &= (g_2g_1)^{-1}x(g_2g_1)\\ &= (g_2g_1) \cdot x.\\ ii)1_G \cdot x &= 1_G^{-1}x1_G = x. \end{aligned}$$

So, this is indeed a group action.

$$\begin{aligned} 2)Stab_G(x) &= \{g \in G : g \cdot x = x\}\\ &= \{g \in G : g^{-1}xg = x\}\\ &= \{g \in G : xg = gx\}\\ &= C_G(x) \text{ (centralizer of } x \text{ in } G \text{).} \end{aligned}$$

$$\begin{aligned} 3)O(x) &= \{g \cdot x : g \in G\}\\ &= \{g^{-1}xg : g \in G\}\\ &= C(x) \text{ (conjugace class of } x \text{).} \end{aligned}$$

Special case. *Let G be a group acts on a finite set X.*
1) If $X = G$, and if G acts on X by conjugation. Then from the previous example we found that

$$O(x) = C(x) \text{ (conjugace class of } x\text{) and } Stab_G(x) = C_G(x) \text{ (centralizer).}$$

Thus, we have from the Orbit-Stabilizer Theorem for all $x \in X$

$$|C(x)| = \frac{|G|}{|C_G(x)|}.$$

2) If $X = \{(x,y) \in G \times G\} \neq \emptyset$, and if G acts on X by $g \cdot a = (x^g, y^g) \in G \times G$, for all $g \in G$ and for all $a = (x,y) \in X$. Then

$$\begin{aligned} Stab_G(a) &= \{g \in G : g \cdot a = a\} \\ &= \{g \in G : g \cdot (x, y) = (x, y)\} \\ &= \{g \in G : (x^g, y^g) = (x, y)\} \\ &= \{g \in G : (x, y)^g = (x, y)\} \\ &= \{g \in G : g^{-1}(x, y)g = (x, y)\} \\ &= \{g \in G : (x, y)g = g(x, y)\} \\ &= C_G(a), \end{aligned}$$

and

$$\begin{aligned} O(a) &= \{g \cdot a : g \in G\} \\ &= \{g \cdot (x, y) : g \in G\} \\ &= \{(x^g, y^g) : g \in G\} \\ &= \{(x, y)^g : g \in G\} \\ &= \{g^{-1}(x, y)g : g \in G\} \\ &= C(a). \end{aligned}$$

Thus, we have from the Orbit-Stabilizer Theorem for all $a \in X$

$$|C(a)| = \frac{|G|}{|C_G(a)|}.$$

Lemma 1.2. *Let p be a prime number. Let G be a p-group acts on a finite set X. Then*

$$|X| = |Fix_X(G)| mod\ p.$$

Proof. Since G is a p-group then $|G| = p^\alpha, \alpha \geq 0$, and since

$$X = O(x_1)\dot{\cup}O(x_2)\dot{\cup}...\dot{\cup}O(x_r)$$

where $x_i \in X, i = 1, ..., r, r \in \mathbb{N}$. Then

$$|X| = |O(x_1)| + |O(x_2)| + ... + |O(x_r)|,$$

then by the fact that $|O(x_i)| = 1$ iff $x_i \in Fix_X(G)$, for all $i = 1, ..., r$. Hence

$$|X| = |Fix_X(G)| + \sum_{x_i \notin Fix_X(G)} |O(x_i)|.$$

From Orbit-Stabilizer Theorem we have

$$|X| = |Fix_X(G)| + \sum_{x_i \notin Fix_X(G)} \frac{|G|}{|Stab_G(x_i)|}.$$

But $Stab_G(x_i) \leq G$ then $|Stab_G(x_i)| = p^\beta$, where $\beta \leq \alpha$ and $p^\alpha/p^\beta = \lambda\ p$, $\lambda \in \mathbb{N}$. Then

$$|X| = |Fix_X(G)| + \lambda p$$

thus

$$|X| = |Fix_X(G)|\ mod\ p.$$

□

Burnside's lemma was formulated and proven by Burnside in 1897, but historically it was already discovered in 1887 by Frobenius, and even earlier in 1845 by Cauchy. Because of that it is sometimes also named as Cauchy-Frobenius lemma. This allows us to count the number of orbits in sets.

Lemma 1.3 (Burnside's lemma)**.** *Let G be a group acts on a finite set X. Let X^g be the set of points of X which are fixed by g, where $g \in G$. Then*

$$|\textit{Orbits of } G \textit{ acting on } X\,| = \frac{\sum_{g\in G}|X^g|}{|G|}.$$

i.e. (The number of orbits is equal to the average number of points fixed by an element of G). The number of orbits will be denoted by X/G.

Proof. Let $X^g = \{x \in X : g \cdot x = x\}$. Then

$$\textstyle\sum_{g\in G}|X^g| = \{(g,x) \in G \times X : g \cdot x = x\} = \sum_{x\in X}|Stab_G(x)|.$$

From Orbit-Stabilizer Theorem we have

$$\begin{aligned}\sum_{x\in X}|Stab_G(x)| &= \sum_{x\in X}\frac{|G|}{|O(x)|}\\ &= |G|\sum_{x\in X}\frac{1}{|O(x)|}\\ &= |G|\left(\sum_{O(x)\in X/G}\sum_{x\in O(x)}\frac{1}{|O(x)|}\right)\\ &= |G|\sum_{O(x)\in X/G}1\\ &= |G|.|X/G|\end{aligned}$$

hence

$$\sum_{g\in G}|X^g| = |G|.|X/G|$$

thus

$$|X/G| = \frac{\sum_{g\in G}|X^g|}{|G|}.$$

□

Lemma 1.4. *Let G be a group acts on itself as a set by conjugation, $g \cdot x = g^{-1}xg \in X = G$. Let $k(G)$ be the number of conjugacy classes of G. Then*

$$k(G) = \frac{\sum_{x\in G}|C_G(x)|}{|G|},$$

where $C_G(x)$ is the centraliser of x in G.

Proof. Since G acts on itself as a set, then we have

$$\begin{aligned} O(x) &= \{g \cdot x : g \in G\} \\ &= \{g^{-1}xg : g \in G\} \\ &= C(x), \end{aligned}$$

for all x $\in X = G$, by Lemma 1.3 we have

$$|Orbits| = \frac{\sum_{g \in G} |X^g|}{|G|}$$
$$|Orbits| = \frac{\sum_{x \in G} |X^x|}{|G|}$$

$$\begin{aligned} |conjugacy\ classes| &= \frac{\sum_{x \in G} |X^x|}{|G|} \\ k(G) &= \frac{|X^{x_1}| + ... + |X^{x_{|G|}}|}{|G|} \\ &= \frac{|\{y \in X : x_1 \cdot y = y\}| + ... + |\{y \in X : x_{|G|} \cdot y = y\}|}{|G|} \\ &= \frac{|\{y \in X : y^{x_1} = y\}| + ... + |\{y \in X : y^{x_{|G|}} = y\}|}{|G|} \\ &= \frac{|\{y \in X : x_1^{-1} y x_1 = y\}| + ... + |\{y \in X : x_{|G|}^{-1} y x_{|G|} = y\}|}{|G|} \\ &= \frac{|\{y \in X : yx_1 = x_1 y\}| + ... + |\{y \in X : yx_{|G|} = x_{|G|} y\}|}{|G|} \\ &= \frac{|\{y \in G : yx_1 = x_1 y\}| + ... + |\{y \in G : yx_{|G|} = x_{|G|} y\}|}{|G|} \\ &= \frac{|C_G(x_1)| + ... + |C_G(x_{|G|})|}{|G|} \\ &= \frac{\sum_{x \in G} |C_G(x)|}{|G|} \end{aligned}$$

thus

$$k(G) = \frac{\sum_{x \in G} |C_G(x)|}{|G|}.$$

□

1.2 Characters

In this section, we shall review some concepts related to the character theory.

Definition 1.4. *Let G be a group. The* ***representation*** *of G is a group homomorphism function $\rho : G \to GL(n, F)$. Where $GL(n, F)$ is the general linear group of degree n over the field F. Which is denoted by F-representation, and the integer number n is called the degree(or dimension) of the representation.*

Remark. *If the field F has a characteristic zero then we call the representation by* ***ordinary representation****.*

Definition 1.5. *Let G be a group. Let ρ be the representation of G. The **character** of ρ is a function $\chi : G \to F$ such that $\chi(g) = trace\,(\rho(g)\,)$, for all $g \in G$ and $\rho(g) \in GL(n, F)$. It is a class function (i.e. constant on each conjugacy class of G).*

Remark.
1)The irreducible representation gives an irreducible character.
2)$Irr(G)$ is the set of all irreducible character of G.

Proposition 1.1. *The character has the following properties:*
1)Equivalent representations have the same characters.
2)The conjugate elements have the same character.
3)The number of irreducible characters is equal to the number of conjugacy classes i.e. $|Irr(G)| = k(G)$.

Theorem 1.3 (Generalized Orthogonality Relation)**.** *Let G be a group and $g \in G$. Then*

$$\frac{1}{|G|}\sum_{h\in G}\chi_i(hg)\chi_j(h^{-1}) = \begin{cases} \dfrac{\chi_i(g)}{\chi_i(1)} & i = j, \\ 0 & i \neq j. \end{cases}$$

Proof. The proof can be found in [15, Theorem 2.13]. □

If $g = 1_G$, then we will get the next theorem, as in [13, Corollary (2.14)].

Theorem 1.4 (First Orthogonality Relation)**.** *Let G be a group. Then*

$$\frac{1}{|G|}\sum_{h\in G}\chi_i(h)\chi_j(h^{-1}) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Theorem 1.5 (Second Orthogonality Relation)**.** *Let G be a group. Let $g, h \in G$. Then*

$$\sum_{\chi\in Irr(G)}\chi(g)\chi(h^{-1}) = \begin{cases} 0 & \text{if } g \nsim h, \\ |C_G(g)| & \text{if } g \sim h. \end{cases}$$

Proof. The proof can be found in [15, Theorem 2.18]. □

Definition 1.6. *[15, Definition 2.20] Let χ be a character of a group G. The **kernal of** χ is all elements g of G which achieve the relation $\chi(g) = \chi(1)$. In symbol*

$$ker\,(\chi) = \{g \in G : \chi(g) = \chi(1)\}.$$

Definition 1.7. *[15, Definition 2.26] Let χ be a character of a group G. Then*

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}.$$

Lemma 1.5. *[15, Corollary 2.23] Let G be a group with commutator subgroup G'. Then*

$$G' = \cap\{\ ker\ \chi : \chi \in Irr(G), \chi(1) = 1\}.$$

Example 1.3. *Let G be a group.*
1) When $G = S_3$, S_3 has two linear characters χ_1 and χ_2. Then

$$G' = ker\chi_1 \cap ker\chi_2$$

$$= \{g \in G : \chi_1(g) = \chi_1(1)\} \cap \{g \in G : \chi_2(g) = \chi_2(1)\}$$

$$= G \cap \{(1), (123), (132)\} = A_3.$$

2) When $G = D_8$, D_8 has four linear characters χ_1, χ_2, χ_3 and χ_4. Then

$$G^{'} = ker\chi_1 \cap ker\chi_2 \cap ker\chi_3 \cap ker\chi_4$$

$$= \{1, a^2\} = \langle a^2 \rangle.$$

Definition 1.8. *Let ρ be a $\mathbb{C}$-representation of a group G which affords an irreducible character χ. If $z \in Z(\mathbb{C}[G])$, then $\rho(z) = \varepsilon \cdot I$ for some root of unity $\varepsilon \in \mathbb{C}$. Then we call the function which depends on χ by ω_χ, such that:*

$$\omega_\chi : Z(\mathbb{C}[G]) \longrightarrow \mathbb{C}$$
$$\omega_\chi(z) = \varepsilon.$$

The root ε does not depend on the choice of particular $\mathbb{C}$-representation affording χ and we observe that $\rho(z) = \omega(z) \cdot I$.

Lemma 1.6. *The function ω_χ is an algebra homomorphism.*

1.3 Blocks

In this section, we shall review some concepts related to the block theory by using character theory.

$Irr(G)$ is the set of all irreducible characters. If X =$Irr(G)$ = $\{\chi_1, \chi_2, ..., \chi_{k(G)}\}$ where $k(G)$ is the number of conjugacy classes, and if $g \in G$. Then we can define a relation on X. By fixing a prime number p and for $\chi_i, \chi_j \in Irr(G)$, where $i, j = 1, ..., k(G)$ such that

$$\chi_i \sim \chi_j \leftrightarrow \frac{|C(g)|\chi_i(g)}{\chi_i(1)} \equiv_p \frac{|C(g)|\chi_j(g)}{\chi_j(1)}$$

where $\equiv_p$ means congruent modulo p, for all $g \in G$, where $|C(g)|$ is the order of the conjugacy class of g. This relation is an equivalence relation on $Irr(G)$, since if we take $\chi : G \to F$,

$\varphi : G \to F$ *and* $\psi : G \to F$. *Where* χ, φ *and* $\psi \in Irr(G)$. *Then* $\frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\chi(g)}{\chi(1)}$ $\leftrightarrow \chi \sim \chi$ *is a reflexive. If* $\frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\psi(g)}{\psi(1)}$, *then* $\frac{|C(g)|\psi(g)}{\psi(1)} \equiv_p \frac{|C(g)|\chi(g)}{\chi(1)}$ *means that if* $\chi \sim \psi$ *then* $\psi \sim \chi$ *is symmetric. If* $\chi \sim \psi$ *and* $\psi \sim \varphi$, *then* $\chi \sim \varphi$. *We have* $\frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\psi(g)}{\psi(1)} \equiv_p \frac{|C(g)|\varphi(g)}{\varphi(1)}$. *Then* $\frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\varphi(g)}{\varphi(1)}$. *Thus* $\psi \sim \varphi$ *is transitive.*
The equivalence relation gives equivalence classes (partitions). The corresponding equivalence classes are called the ***p-blocks*** *of* G. $X = Irr(G) = B_1 \dot{\cup} B_2 \dot{\cup} \dots \dot{\cup} B_t$. *Where* t *is natural number.* B_i *is a p-block for all* $i = 1, .., t$. *Now we can define the p-blocks of the group G by the following definition, and we will mention some definitions related to this topic.*

Definition 1.9. *Let* G *be a group. Let* χ *and* ψ *be an irreducible character. Let* p *be a prime number. The* ***p-blocks*** $B_1, ..., B_t$ *where* $t \in \mathbb{N}$ *of* G *are the equivalence classes which are given by the equivalence relation on Irr(G) such that*

$$\chi \sim \psi \leftrightarrow \frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\psi(g)}{\psi(1)},$$

for all $g \in G$.

Definition 1.10. *Let* G *be a group. Let* p *be a prime number. The* ***principal p-block*** *is the p-block which contains the trivial character, and will be denoted it by* B_0.

Definition 1.11. *Let* G *be a group. Let* $\chi \in Irr(G)$. *Let* p *be a prime number. The* ***defect number of*** χ *which is denoted by* $d(\chi) = d$ *is a positive integer that achieves the relation*

$$p^d \chi(1)_p = |G|_p,$$

where $\chi(1)_p$ *is the p-part of* $\chi(1)$ *and* $|G|_p$ *is the p-part of the order of G.*

Definition 1.12. *Let* G *be a group. Let* p *be a prime number. Let* B *be a p-block of* G. *The* ***defect number of*** B *which is denoted by* $d(B)$ *is a maximal number of* $d(\chi)$, *for all* χ *belongs to* B. *In symbols*

$$d(B) = max\{d(\chi) : \chi \in B\}.$$

Definition 1.13. *Let* G *be a group. Let* p *be a prime number. Let* B *be a p-block of G. If* χ *belongs to* B, *then the* ***height number of*** χ *which is denoted by* $h(\chi)$ *is given by subtraction* $d(\chi)$ *of* $d(B)$. *In symbols*

$$h(\chi) = d(B) - d(\chi).$$

Remark. *If we say an irreducible character of height zero that means* $d(B) = d = d(\chi)$.

Definition 1.14. *Let* G *be a group. Let* p *be a prime number. Let* B *be a p-block of* G. *Let* g *be an element of* G. *A* ***defect group*** *which is denoted by* D *is a Sylow p-subgroup of the centralizer of* g *in* G *and the defect group* D *of p-block* B *is the Sylow p-subgroup of the order* $p^{d(B)}$.

Remark.
• *The p-block of defect zero is the p-block which has the identity subgroup as a defect group.*
• *The p-block of defect zero has an unique irreducible character with a degree which contains the whole p-part of the order of G.*

We will give some examples of the p-blocks of the group G with respect to the chosen prime number p.

Example 1.4. *Let $G = S_4$ and $p = 3$. The character table of S_4 is given below:*

conjugacy classes	(1)	(12)	(123)	(1234)	(12)(34)
$\|C(g)\|$	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Table 1.1: The character of S_4.

By the relation $\dfrac{|C(g)|\chi_i(g)}{\chi_i(1)}$ *mod p,* $\forall i = 1, ..., k(G), \forall\ g \in G$*. We can find the 3-blocks. If* $g = (123)$*. Then* $\dfrac{|C(123)|\chi_1((123))}{\chi_1(1)} = \dfrac{8 \cdot 1}{1}$ *mod* $3 = \bar{2}$, $\dfrac{|C(123)|\chi_2((123))}{\chi_2(1)} = \dfrac{8 \cdot 1}{1}$ *mod* $3 = \bar{2}$, $\dfrac{|C(123)|\chi_3((123))}{\chi_3(1)} = \dfrac{8 \cdot (-1)}{2}$ *mod* $3 = \bar{2}$, $\dfrac{|C(123)|\chi_4((123))}{\chi_4(1)} = \dfrac{8 \cdot 0}{3}$ *mod* $3 = \bar{0}$, $\dfrac{|C(123)|\chi_5((123))}{\chi_5(1)} = \dfrac{8 \cdot 0}{3}$ *mod* $3 = \bar{0}$*, and the same way for all* $g \in S_4$*. Then we have the new table*

$\dfrac{\|C(g)\|\chi(g)}{\chi(1)}$	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{0}$
χ_2	$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{0}$
χ_3	$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{0}$
χ_4	$\bar{1}$	$\bar{2}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
χ_5	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{2}$	$\bar{2}$

Table 1.2: The 3-blocks of S_4.

Thus S_4 has three 3-blocks. $B_0 = B_1 = \{\chi_1, \chi_2, \chi_3\}$, $B_2 = \{\chi_4\}$ *and* $B_3 = \{\chi_5\}$*. Where* $B_1 \dot{\cup} B_2 \dot{\cup} B_3 = Irr(S_4)$*, and* $B_i \cap B_j = \emptyset$, $\forall i \neq j$ *,*$i, j = 1, 2, 3$*. Therefore,*

$$\begin{aligned} 3^d\chi_1(1)_3 &= |G|_3 \\ 3^d(1 = 3^0)_3 &= (24)_3 \\ 3^d 3^0 &= 3 \\ 3^d &= 3. \end{aligned}$$

Then $d(\chi_1) = d(\chi_2) = d(\chi_3) = d = 1$.

$$3^d \chi_4(1)_3 = |G|_3$$
$$3^d(3)_3 = 3.$$

Then $d(\chi_4) = d(\chi_5) = d = 0$*, hence* $d(B_1) = max\ \{d(\chi_1), d(\chi_2), d(\chi_3)\} = 1$*,* $d(B_2) = max\ \{d(\chi_4)\} = 0$ *and* $d(B_3) = max\ \{d(\chi_5)\} = 0$*.* $\chi_1, \chi_2, \chi_3, \chi_4$ *and* χ_5 *of height zero. Moreover, the defect group is*

conjugacy class	(1)	(12)	(123)	(1234)	(12)(34)
$C_G(g)$	S_4	$\langle(12),(34)\rangle$	$\langle(123)\rangle$	$\langle(1234)\rangle$	H_1
$\|C_G(g)\|$	24	4	3	4	8
Sylow 3-subgroup(defect group)	$\langle(123)\rangle$	$\langle(1)\rangle$	$\langle(123)\rangle$	$\langle(1)\rangle$	$\langle(1)\rangle$
defect (3^a)	1	0	1	0	0

Table 1.3: The defect group of S_4, when $p = 3$.

where $H_1 = \{(1), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$*. If we take* $p = 2$*. Then we have*

$\frac{\|C(g)\|\chi(g)}{\chi(1)}$	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{1}$
χ_2	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{1}$
χ_3	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{1}$
χ_4	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{1}$
χ_5	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{1}$

Table 1.4: The 2-blocks of S_4.

Thus S_4 *has one 2-block only* $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$*. Therefore,*

$$2^d \chi_1(1)_2 = |G|_2$$
$$2^d(1 = 2^0)_2 = (24)_2$$
$$2^d 2^0 = 2^3$$
$$2^d = 2^3.$$

Then $d(\chi_1) = d(\chi_2) = d(\chi_4) = d(\chi_5) = d = 3$

$$2^d \chi_3(1)_2 = |G|_2$$
$$2^d(2)_2 = (24)_2$$
$$2^d 2 = 2^3.$$

Then $d(\chi_3) = d = 2$*, hence* $d(B_0) = max\ \{d(\chi_1), d(\chi_2), d(\chi_3), d(\chi_4), d(\chi_5)\} = 3$.
$h(\chi_3) = d(B_0) - d(\chi_3) = 3 - 2 = 1$*,* χ_1, χ_2, χ_4 *and* χ_5 *of height zero. Moreover, the defect group is*

conjugacy class	(1)	(12)	(123)	(1234)	(12)(34)
$C_G(g)$	S_4	$\langle(12),(34)\rangle$	$\langle(123)\rangle$	$\langle(1234)\rangle$	H_1
$\lvert C_G(g)\rvert$	24	4	3	4	8
Sylow 2-subgroup(defect group)	H_1	$\langle(12),(34)\rangle$	$\langle(1)\rangle$	$\langle(1)\rangle$	H_1
defect (2^a)	3	2	0	0	3

Table 1.5: The defect group of S_4, when $p = 2$.

Example 1.5. *Let $G = S_3$ and p =2. The character table of S_3 is given below:*

conjugacy class	(1)	(12)	(123)
$\lvert C(g)\rvert$	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 1.6: The character of S_3.

By the relation $\dfrac{|C(g)|\chi_i(g)}{\chi_i(1)}$ *mod p*, $\forall i = 1, ..., k(G), \forall\ g \in G$. *We can find the 2-blocks. If* $g = (12)$. *Then* $\dfrac{|C(12)|\chi_1((12))}{\chi_1(1)} = \dfrac{3 \cdot 1}{1}$ *mod* $2 = \bar{1}$, $\dfrac{|C(12)|\chi_2((12))}{\chi_2(1)} = \dfrac{3 \cdot (-1)}{1}$ *mod* $2 = \bar{1}$, $\dfrac{|C(12)|\chi_3((12))}{\chi_3(1)} = \dfrac{3 \cdot 0}{2}$ *mod* $2 = \bar{0}$, *and the same way for all* $g \in S_3$. *Then we have the new table*

$\dfrac{\lvert C(g)\rvert\chi(g)}{\chi(1)}$	(1)	(12)	(123)
χ_1	$\bar{1}$	$\bar{1}$	$\bar{0}$
χ_2	$\bar{1}$	$\bar{1}$	$\bar{0}$
χ_3	$\bar{1}$	$\bar{0}$	$\bar{1}$

Table 1.7: The 2-blocks of S_3.

Thus S_3 has two 2-blocks. $B_0 = B_1 = \{\chi_1, \chi_2\}$ *and* $B_2 = \{\chi_3\}$. *Therefore,*

$$\begin{aligned} 2^d\chi_1(1)_2 &= |G|_2 \\ 2^d(1 = 2^0)_2 &= (6)_2 \\ 2^d 2^0 &= 2 \\ 2^d &= 2. \end{aligned}$$

Then $d(\chi_1) = d = 1$.

$$\begin{aligned} 2^d\chi_2(1)_2 &= |G|_2 \\ 2^d(1 = 2^0)_2 &= (6)_2 \\ 2^d 2^0 &= 2 \\ 2^d &= 2. \end{aligned}$$

Then $d(\chi_2) = d = 1$.

$$\begin{aligned} 2^d \chi_3(1)_2 &= |G|_2 \\ 2^d (2)_2 &= (6)_2 \\ 2^d 2 &= 2 \\ 2^d &= 2. \end{aligned}$$

Then $d(\chi_3) = d = 0$, *hence* $d(B_1) = max\ \{d(\chi_1), d(\chi_2)\} = 1$ *and* $d(B_2) = max\ \{d(\chi_3)\} = 0$. $h(\chi_1) = d(B_1) - d(\chi_1) = 1 - 1 = 0$, $h(\chi_2) = d(B_1) - d(\chi_2) = 1 - 1 = 0$ *and* $h(\chi_3) = d(B_2) - d(\chi_3) = 0 - 0 = 0$, *thus* χ_1, χ_2 *and* χ_3 *of height zero. Moreover, the defect group is*

conjugacy class	(1)	(12)	(123)
$C_G(g)$	S_3	$\langle(12)\rangle$	$\langle(123)\rangle$
$\|C_G(g)\|$	6	2	3
Sylow 2-subgroup(defect group)	$\langle(12)\rangle$	$\langle(12)\rangle$	$\langle(1)\rangle$
defect (2^a)	1	1	0

Table 1.8: The defect group of S_3, when $p = 2$.

If we take $p = 3$. *Then we have*

$\frac{\|C(g)\|\chi(g)}{\chi(1)}$	(1)	(12)	(123)
χ_1	$\bar{1}$	$\bar{0}$	$\bar{2}$
χ_2	$\bar{1}$	$\bar{0}$	$\bar{2}$
χ_3	$\bar{1}$	$\bar{0}$	$\bar{2}$

Table 1.9: The 3-blocks of S_3.

Thus S_3 *has one 3-block only.* $B_0 = B_1 = \{\chi_1, \chi_2, \chi_3\}$. *Therefore,*

$$\begin{aligned} 3^d \chi_1(1)_3 &= |G|_3 \\ 3^d (1 = 3^0)_3 &= (6)_3 \\ 3^d 3^0 &= 3 \\ 3^d &= 3. \end{aligned}$$

Then $d(\chi_1) = d(\chi_2) = d(\chi_3) = d = 1$. *Hence* $d(B_0) = max\ \{d(\chi_1), d(\chi_2), d(\chi_3)\} = 1$. $h(\chi_1) = h(\chi_2) = h(\chi_3) = d(B_0) - d(\chi_1) = 1 - 1 = 0$, *thus* χ_1, χ_2 *and* χ_3 *of height zero. Moreover, the defect group is*

conjugacy class	(1)	(12)	(123)
$C_G(g)$	S_3	$\langle(12)\rangle$	$\langle(123)\rangle$
$\lvert C_G(g)\rvert$	6	2	3
Sylow 3-subgroup(defect group)	$\langle(123)\rangle$	$\langle(1)\rangle$	$\langle(123)\rangle$
defect (3^a)	1	0	1

Table 1.10: The defect group of S_3, when $p = 3$.

Example 1.6. *Let $G = GL(3,2)$ and $p = 7$. The character table of $GL(3,2)$ is given below:*

conjugacy class	1	2	3	4	7A	7B
$\lvert C(g)\rvert$	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_2	3	-1	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$
χ_3	3	-1	0	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$
χ_4	6	2	0	0	-1	-1
χ_5	7	-1	1	-1	0	0
χ_6	8	0	-1	0	1	1

Table 1.11: The character of $GL(3,2)$.

By the relation $\frac{|C(g)|\chi_i(g)}{\chi_i(1)}$ *mod p*, $\forall i = 1, ..., k(G), \forall\ g \in G$. *We can find the 7-blocks. If* $g \in 2$. *Then* $\frac{|C(g)|\chi_1(g)}{\chi_1(1)} = \frac{21 \cdot 1}{1}$ *mod* $7 = \bar{0}$, $\frac{|C(g)|\chi_2(g)}{\chi_2(1)} = \frac{21 \cdot (-1)}{3}$ *mod* $7 = \bar{0}$, $\frac{|C(g)|\chi_3(g)}{\chi_3(1)} = \frac{21 \cdot (-1)}{3}$ *mod* $7 = \bar{0}$, $\frac{|C(g)|\chi_4(g)}{\chi_4(1)} = \frac{21 \cdot 2}{6}$ *mod* $7 = \bar{0}$, $\frac{|C(g)|\chi_5(g)}{\chi_5(1)} = \frac{21 \cdot (-1)}{7}$ *mod* $7 = \bar{4}$, $\frac{|C(g)|\chi_6(g)}{\chi_6(1)} = \frac{21 \cdot 0}{8}$ *mod* $7 = \bar{0}$, *and the same way for all* $g \in GL(3,2)$. *Then we have the new table*

$\frac{\lvert C(g)\rvert\chi(g)}{\chi(1)}$	1	2	3	4	7A	7B
χ_1	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{3}$
χ_2	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{3}$
χ_3	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{3}$
χ_4	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{3}$
χ_5	$\bar{1}$	$\bar{4}$	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$
χ_6	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{3}$	$\bar{3}$

Table 1.12: The 7-blocks of $GL(3,2)$.

Thus $GL(3,2)$ *has two 7-blocks.* $B_0 = B_1 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_6\}$ *and* $B_2 = \{\chi_5\}$. *Therefore,*

$$7^d\chi_1(1)_7 = |G|_7$$
$$7^d(1 = 7^0)_7 = (168)_7$$
$$7^d7^0 = 7.$$

Then $d(\chi_1) = d(\chi_2) = d(\chi_3) = d(\chi_4) = d(\chi_6) = d = 1$.

$$7^d\chi_5(1)_7 = |G|_7$$
$$7^d 7 = 7.$$

Then $d(\chi_5) = d = 0$, *hence* $d(B_0) = d(B_1) =$ *max* $\{d(\chi_1), d(\chi_2), d(\chi_3), d(\chi_4), d(\chi_6)\} = 1$ *and* $d(B_2) =$ *max* $\{d(\chi_5)\} = 0$. $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$ *and* χ_6 *of height zero. Moreover, the defect group is*

conjugacy class	1	2	3	4	7A	7B
$C_G(g)$	$GL(3,2)$	Q_1	Q_2	Q_3	Q_4	Q_4
$\|C_G(g)\|$	168	8	3	4	7	7
Sylow 7-subgroup(defect group)	Q_4	$\langle I_3\rangle$	$\langle I_3\rangle$	$\langle I_3\rangle$	Q_4	Q_5
defect (7^a)	1	0	0	0	1	1

Table 1.13: The defect group of $GL(3,2)$, when $p = 7$.

Where $Q_1 \cong D_8$, $Q_2 \cong C_3$, $Q_3 \cong C_4$ *and* $Q_4 \cong C_7$.

Lemma 1.7. *Let p be a prime number. Let G be a p-group. Then G has one p-block only the principal one.*

Proof. Let G be a p-group, then $|G| = p^\alpha$, $\alpha \in \mathbb{N}$. By Definition 1.9 we have for all $\chi, \psi \in$ Irr(G) and $g \in G$,

$$\chi \sim \psi \leftrightarrow \frac{|C(g)|\chi(g)}{\chi(1)} \equiv_p \frac{|C(g)|\psi(g)}{\psi(1)}$$

hence we have only one orbit, which is clearly the principal p-block containing the trivial character. □

We give examples for this lemma.

Example 1.7. *Let* $G = D_8$ *and p =2. The character table of* D_8 *is given below:*

conjugacy class	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
$\|C(g)\|$	1	1	2	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 1.14: The character of D_8.

By the relation $\dfrac{|C(g)|\chi_i(g)}{\chi_i(1)}$ *mod p*, $\forall i, ..., k(G), \forall\ g \in G$. *We can find the 2-blocks. If* $g = a^2$. *Then* $\dfrac{|C(a^2)|\chi_1(a^2))}{\chi_1(1)} = \dfrac{1\cdot 1}{1}$ *mod* $2 = \bar{1}$, $\dfrac{|C(a^2)|\chi_2(a^2)}{\chi_2(1)} = \bar{1}$, $\dfrac{|C(a^2)|\chi_3(a^2))}{\chi_3(1)} = \bar{1}$, $\dfrac{|C(a^2)|\chi_4(a^2)}{\chi_4(1)} = \bar{1}$, $\dfrac{|C(a^2)|\chi_5(a^2)}{\chi_5(1)} = \bar{1}$, *and the same way for all* $g \in D_8$. *Then we have the new table*

$\frac{\lvert C(g)\rvert\chi(g)}{\chi(1)}$	1	a^2	$\{a, a^3\}$	$\{b, a^2b\}$	$\{ab, a^3b\}$
χ_1	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
χ_2	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
χ_3	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
χ_4	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
χ_5	$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$

Table 1.15: The 2-blocks of D_8.

Thus D_8 has one 2-block only $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$. Moreover,

$$\begin{aligned} 2^d \chi_1(1)_2 &= |G|_2 \\ 2^d(1 = 2^0)_2 &= (8)_2 \\ 2^d 2^0 &= 2^3. \end{aligned}$$

Then $d(\chi_1) = d(\chi_2) = d(\chi_3) = d(\chi_4) = d = 3$.

$$\begin{aligned} 2^d \chi_5(1)_2 &= |G|_2 \\ 2^d(2)_2 &= (8)_2 \\ 2^d 2 &= 2^3. \end{aligned}$$

Then $d(\chi_5) = d = 2$, hence $d(B_0) = max\ \{d(\chi_1), d(\chi_2), d(\chi_3), d(\chi_4), d(\chi_5)\} = 3$.
$h(\chi_5) = d(B_0) - d(\chi_5) = 3 - 2 = 1$, χ_1, χ_2, χ_3 and χ_4 of height zero.

Example 1.8. *Let $G = V_4 \cong C_2 \times C_2$ and $p = 2$. The character table of V_4 is given below:*

conjugacy class	1	a	b	ab
$\lvert C(g)\rvert$	1	1	1	1
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	1	-1	- 1
χ_4	1	-1	-1	1

Table 1.16: The character of V_4.

By the relation $\frac{|C(g)|\chi_i(g)}{\chi_i(1)}$ mod p, $\forall i, ..., k(G), \forall\ g \in G$. We can find the 2-blocks. If $g = a$.
Then $\frac{|C(a)|\chi_1(a)}{\chi_1(1)} = \frac{1 \cdot 1}{1}$ mod $2 = \bar{1}$, $\frac{|C(a)|\chi_2(a)}{\chi_2(1)} = \bar{1}$, $\frac{|C(a)|\chi_3(a)}{\chi_3(1)} = \bar{1}$, $\frac{|C(a)|\chi_4(a)}{\chi_4(1)} = \bar{1}$, and the same way for all $g \in V_4$. Then we have the new table

$\frac{\lvert C(g)\rvert \chi(g)}{\chi(1)}$	1	a	b	ab
χ_1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
χ_2	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
χ_3	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
χ_4	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$

Table 1.17: The 2-blocks of V_4.

Thus V_4 has one 2-block only $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4\}$.

Chapter 2

Relative Commutativity Degree

In this chapter, we will present three sections, in the first section; we will study the definition of a probability that two elements of a group commute, and state theories which give us an equivalent definition and upper boundary of this concept. In the second section, we will study some examples of the probability that two elements of a group commute, and in the third and last section, we will study the definition of a probability that a randomly chosen commutator is equal to a given element of a group G, state a theorem which gives us an equivalent definition via character theory with some examples, and mention some recent investigations into this concept. The basic references of this chapter are [2],[3],[5],[10],[11],[12],[13],[15],[17] and [24].

2.1 Probability of commuting elements in group theory

In this section, we will study the probability that two elements of a group commute.

Probabilistic method is the concept used to deal with random experiments whose outcomes can be predicted beforehand which statisticians employ to help them to know the extent of simple random representations. Let $X = \{A, M, ..., L\}$ *be a set of events, so the event A has a probability P(A), which is calculated by the probability function* $P : X \longrightarrow \mathbb{R}$ *given by the following equation:*

$$P(A) = \frac{\textit{Number of events classifiable as } A}{\textit{Total number of possible events}} \tag{2.1}$$

Taking into consideration that the previous equation has to fulfil three conditions. The first condition to be fulfilled is that the value of every probability has to be between zero, which indicates the impossibility of the event, and one, which indicates that the occurrence of the event is confirmed. The second condition is that the sum of all probabilities has to equal one. The third condition is for two disjoint subsets A and M of X, we have $P(A \cup M) = P(A) + P(M)$*. With the advancement of science it became possible to apply the concept of probability in several areas, including group theory.*
In this section, we will calculate the probability of commuting pairs of two subgroup elements of G, in the sense that, if H and K subgroups of G and from equation (2.1) then we have

$$\begin{aligned} P(H, K) &= \frac{\textit{Number of ordered pairs } (h, k) \in H \times K \textit{such that } hk = kh}{\textit{Total number of order pairs } (h, k) \in H \times K} \\ &= \frac{|\{(h, k) \in H \times K : hk = kh\}|}{|H \times K|} \end{aligned}$$

since $|H \times K| = |H||K|$ *and if* $hk = kh$ *that is mean* $[h, k] = 1_G$*. Therefore,*

$$P(H, K) = \frac{|\{(h, k) \in H \times K : [h, k] = 1_G\}|}{|H||K|} \tag{2.2}$$

we can generalize it as follows: if $n, m \in \mathbb{N}$ *then the probability that a randomly chosen commutator of weight* $n + m$ *of* $H \times K$ *is equal to the identity element of G*

$$P^{(n,m)}(H,K) = \frac{|\{(h_1,...,h_n,k_1,...,k_m) \in H^n \times K^m : [h_1,...,h_n,k_1,...,k_m] = 1_G\}|}{|H|^n|K|^m}, \quad (2.3)$$

where $[h_1,...,h_n,k_1,...,k_m] = h_1^{-1}\cdots h_n^{-1}k_1^{-1}\cdots k_m^{-1}h_1\cdots h_n k_1\cdots k_m$, $h_1,\cdots,h_n \in H$, $k_1,\cdots,k_m \in K$. *So that equation (2.2) is a special case when* $n = m = 1$ $(P^{(1,1)}(H,K) = P(H,K))$. *Now we can define the relative commutativity degree of the group G by the following definition.*

Definition 2.1. *Let G be a group. Let H and K be subgroups of G. We define* $d(H,K)$ *as the* ***relative commutativity degree*** *of H and K such that*

$$d(H,K) = P^{(1,1)}(H,K) = \frac{|\{(h,k) \in H \times K : [h,k] = 1_G\}|}{|H||K|} = \frac{\sum_{h\in H}|C_K(h)|}{|H||K|}.$$

Where $C_K(h)$ *is the centralizer of* $h \in H$ *in K.*

Remark.
1)If $H = K = G$, *then*

$$d(H,K) = d(G,G) = d(G) = P^{(1,1)}(G,G) = P(G)$$

such that

$$d(G) = \frac{|\{(x,y) \in G \times G : [x,y] = 1_G\}|}{|G|^2} \quad (2.4)$$

2)The generalization of d(G) is

$$d(H,G)^{(n)} = \frac{|\{(h_1,\ldots,h_n,x) \in H^n \times G : [h_1,\ldots,h_n,x] = 1_G\}|}{|H|^n|G|}.$$

3)Obviously if G abelian, then $d(H,K) = d(G) = 1$.

The following theorem gives us the equivalent definition of the probability of G by a number of conjugacy classes of G, as noted in [2],[3],[11],[12],[17, Lemma 1.1] and [24].

Theorem 2.1. *Let G be a group which acts on a finite set* $\{(x,y) \in G\times G : [x,y] = 1_G\}$ *by conjugation. Then*

$$d(G) = P(G) = \frac{|\{(x,y) \in G\times G : [x,y] = 1_G\}|}{|G|^2} = \frac{k(G)}{|G|}.$$

Proof. Let $C(x_1),...,C(x_{k(G)})$ be distinct conjugacy classes of G, where $x_i \in G$ for all $i = 1,...,k(G)$. From (2.4) we have

$$\begin{aligned}
d(G) &= \frac{|\{(x,y) \in G\times G : [x,y] = 1_G\}|}{|G|^2} \\
&= \frac{|\{(x,y) \in G\times G : xy = yx\}|}{|G|^2} \\
&= \frac{\sum_{x\in G}|C_G(x)|}{|G|^2} \\
&= \frac{\sum_{i=1}^{k(G)}\sum_{x\in C(x_i)}|C_G(x)|}{|G|^2} \\
&= \frac{\sum_{i=1}^{k(G)}[G : C_G(x_i)]|C_G(x_i)|}{|G|^2} \\
&= \frac{\sum_{i=1}^{k(G)}|G|}{|G|^2} \\
&= \frac{k(G)|G|}{|G|^2} \\
&= \frac{k(G)}{|G|}
\end{aligned}$$

thus

$$d(G) = \frac{k(G)}{|G|}.$$

□

In the same way we get the following corollary.

Corollary 2.1. *Let G be a group. Let H and K be subgroups of G. Then*

$$d(H, K) = \frac{|\{(h, k) \in H \times K : [h, k] = 1_G\}|}{|H||K|} = \frac{k_K(H)}{|H|}$$

where $k_K(H)$ is the number of K-cojugacy classes that constitute H.

The following lemma gives us the relation between the number of conjugacy classes of group G and the number of a conjugacy classes of group G/N such that N is a normal subgroup of G.

Lemma 2.1. *Let G be a group. Let N be a normal subgroup of G. Then*

$$k(G) \leq k(N)k(G/N).$$

Proof. The proof can be found in [11, Lemma 1, (ii)]. □

The following lemma gives us the relation between the probability of group G and the probability of group G/N such that N is a normal subgroup of G, as noted in [11, Lemma 2, (ii)] and [17, Lemma 1.4].

Lemma 2.2. *Let G be a group. Let N be a normal subgroup of G. Then*

$$d(G) \leq d(N)d(G/N).$$

In particulars we always have $d(G) \leq d(G/N)$.

Proof. By Lemma 2.1 we have

$$k(G) \leq k(N)k(G/N)$$

therefore

$$\frac{k(G)}{|G|} \leq \frac{1}{|G|}\frac{|N|}{|N|}k(N)k(G/N)$$

hence

$$\frac{k(G)}{|G|} \leq \frac{k(N)}{|N|}\frac{k(G/N)}{|G|/|N|}$$

by Theorem 2.1 we have

$$d(G) \leq d(N)d(G/N).$$

□

The following basic theorem will be helpful to prove the next theorem.

Theorem 2.2. *Let G be a group. If $G/Z(G)$ is cyclic, then G is an abelian group.*

Proof. Let $G/Z(G)$ be a cyclic group, then there is $x \in G$ such that $G/Z(G) = \langle xZ(G)\rangle$. Let $a, b \in G$, then for all $n, m \in \mathbb{Z}$ we have

$$aZ(G) = x^m Z(G) \qquad \text{and} \qquad bZ(G) = x^n Z(G),$$

hence

$$a = x^m c \qquad \text{and} \qquad b = x^n d,$$

for some $c, d \in Z(G)$, therefore,

$$\begin{aligned} ab &= (x^m c)(x^n d) = x^m(cx^n)d = x^m(x^n c)d = x^{m+n}cd = x^{n+m}cd \\ &= x^n(x^m d)c = x^n(dx^m)c = (x^n d)(x^m c) = ba. \end{aligned}$$

Since $c, d \in Z(G)$. Thus G is an abelian group. □

The following theorem gives us the upper boundary of the probability of a non-abelian group G, as in [2],[11],[13] and [24].

Theorem 2.3. *Let G be a non-abelian group. Then* $d(G) \leq \frac{5}{8}$.

Proof. Since G is disjoint union of conjugace classes then

$$|G| = |C(x_1)| + ... + |C(x_{k(G)})|,$$

for all $x_i \in G$, $i = 1, ..., k(G)$, if $x \in Z(G)$, then $|C(x)| = |\{y \in G : g^{-1}xg = y\}|$ $= |\{y \in G : x = y\}|\ |\{x\}| = 1$, therefore,

$$|G| = |Z(G)| + |C(x_1)| + ... + |C(x_n)|,$$

where $n = k(G) - |Z(G)|$. Since $C(x_i)$ is non-trivial, hence for all $i = 1, ..., n$, we have

$$2 \leq |C(x_i)|$$

then

$$2n \leq |C(x_1)| + ... + |C(x_n)|$$

since $|G| - |Z(G)| = |C(x_1)| + ... + |C(x_n)|$, hence

$$\begin{aligned} 2n &\leq |G| - |Z(G)| \\ n &\leq \frac{|G| - |Z(G)|}{2} \\ n + |Z(G)| &\leq \frac{|G| - |Z(G)|}{2} + |Z(G)| = \frac{|G| + |Z(G)|}{2} \\ k(G) &\leq \frac{|G| + |Z(G)|}{2} \end{aligned}$$

by Theorem 2.2 we have: if G is non-abelian, then the factor group $G/Z(G)$ is not a cyclic group, and by the fact that the smallest group is not cyclic it is of order 4, therefore $|Z(G)| \leq |G|/4$. Then

$$\begin{aligned} k(G) &\leq \frac{|G| + |Z(G)|}{2} \leq \frac{|G| + (|G|/4)}{2} = \frac{5|G|}{8} \\ k(G) &\leq \frac{5|G|}{8} \\ \frac{k(G)}{|G|} &\leq \frac{5}{8} \end{aligned}$$

by Theorem 2.1 we have

$$d(G) \leq \frac{5}{8}.$$

□

Remark. $d(G) = \frac{5}{8}$ *iff the factor group* $G/Z(G)$ *is isomorphic to the Klein four-group* V.

The following lemma gives us the upper boundary of the probability of a non-abelian p-group G, as in [12] and [18, Lemma 1.3].

Lemma 2.3. *Let G be a non-abelian p-group. Then*

$$d(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Proof. Since G is p-group, hence there is $n \in \mathbb{N}$ such that $|G| = p^n$ and $|Z(G)| = p^m$ where $m \leq n$, $m \in \mathbb{N}$ then $m \leq n-2$. Therefore,

$$\begin{aligned} d(G) &= \frac{\sum_{x\in G}|C_G(x)|}{|G|^2} \\ |G|^2 d(G) &= \sum_{x\in G}|C_G(x)| \\ (p^n)^2 d(G) &= \sum_{x\in Z(G)}|C_G(x)| + \sum_{x\in(G-Z(G))}|C_G(x)| \end{aligned}$$

since for all $x \in Z(G)$ we have $xg = gx$ for all $g \in G$, hence $C_G(x) = G$ for all $x \in Z(G)$, then $\sum_{x\in Z(G)}|C_G(x)| = \sum_{x\in Z(G)}|G| = |G||Z(G)| = p^n p^m$. Therefore,

$$\begin{aligned} &= p^n p^m + (|G| - |Z(G)|) \\ &= p^n p^m + (p^n - p^m) \\ &\leq p^n p^m + p^{n-1}(p^n - p^m) \\ &= p^{n+m} + p^{n-1}(p^n - p^m) \\ &= p^{n+m} + p^{2n-1} - p^{m+n-1} \\ &= p^{m+n-1}(p-1) + p^{2n-1} \\ since m \leq n-2 & \\ &\leq p^{n-2+n-1}(p-1) + p^{2n-1} \\ &= p^{2n-3}(p-1) + p^{2n-1} \\ &= p^{2n-3}(p-1) + p^{2n-1}p^{-2}p^2 \\ &= p^{2n-3}(p-1) + p^{2n-3}p^2 \\ &= p^{2n-3}(p^2+p-1) \\ p^{2n}d(G) &\leq p^{2n-3}(p^2+p-1) \\ d(G) &\leq p^{-3}(p^2+p-1) \end{aligned}$$

thus

$$d(G) \leq \frac{(p^2+p-1)}{p^3}.$$

□

The following theorem gives us the probability of the direct product of two groups, and we can apply that to more than two groups, provided that the direct product is limited, as noted in [12].

Theorem 2.4. *Let G and H be groups. Let $\mid G \mid= n, \mid H \mid= m$ and $g.c.d(n,m) = 1$ for all $n, m \in \mathbb{Z}^+$. Then $d(G \times H) = d(G) \cdot d(H)$.*

Proof. Since $\mid G \times H \mid=\mid G \mid \cdot \mid H \mid$, hence $(\mid G \times H \mid)^2 = (\mid G \mid)^2 \cdot (\mid H \mid)^2$. Then we have for all $(x, y) \in G \times H$,

$$\begin{aligned} C_{G\times H}((x,y)) &= \{(g,h) \in G \times H : (x,y)(g,h) = (g,h)(x,y)\} \\ &= \{(g,h) \in G \times H : (xg, yh) = (gx, hy)\} \\ &= \{g \in G : xg = gx\} \times \{h \in H : yh = hy\} \\ &= C_G(x) \times C_H(y). \end{aligned}$$

Hence $\mid C_{G\times H}((x,y)) \mid=\mid C_G(x) \times C_H(y) \mid=\mid C_G(x) \mid \cdot \mid C_H(y) \mid$, and by definition of $d(G \times H)$ we have

$$
\begin{aligned}
d(G \times H) &= |\{((x,y),(g,h)) \in (G \times H) \times (G \times H) : (x,y)(g,h) = (g,h)(x,y)\}| \\
&= \frac{1}{| G \times H |^2} \sum_{(x,y) \in G \times H} | C_{G \times H}((x,y)) | \\
&= \frac{1}{| G |^2 \cdot | H |^2} \sum_{x \in G, y \in H} | C_G(x) | \cdot | C_H(y) | \\
&= \frac{1}{| G |^2 | H |^2} \cdot \sum_{x \in G} \sum_{y \in H} | C_G(x) | \cdot | C_H(y) | \\
&= \frac{1}{| G |^2 \cdot | H |^2} \sum_{x \in G} | C_G(x) | \cdot \sum_{y \in H} | C_H(y) | \\
&= \left(\frac{1}{| G |^2} \sum_{x \in G} | C_G(x) | \right) \cdot \left(\frac{1}{| H |^2} \sum_{y \in H} | C_H(y) | \right) \\
&= d(G) \cdot d(H).
\end{aligned}
$$

□

2.2 Exampels

In this section, we will study some examples of calculating the probability of commuting pairs elements of G.

Example 2.1. *Let* $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$. *Then*

$$d(S_3) = P(S_3) = \frac{|\{(x,y) \in S_3 \times S_3 : [x,y] = (1)\}|}{|S_3|^2}$$

·	(1)	(12)	(13)	(23)	(123)	(132)
(1)	*	*	*	*	*	*
(12)	*	*				
(13)	*		*			
(23)	*			*		
(123)	*				*	*
(132)	*				*	*

Table 2.1: Commute Elements in S_3.

$\{(x,y) \in S_3 \times S_3 : [x,y] = (1)\} = \{((1),(1)), ((1),(12)), ((1),(13)), ((1),(23)), ((1),(123)),$ *((1),(132)) , ((12),(1)) , ((12), (12)) ,((13),(1)), ((13), (13)) ,((23),(1)) , ((23),(23)), ((123),(1)) ,((123),(123)),((123),(132)), ((132),(1)),((132),(123)),((132),(132))*$\}$. *Therefore,* $|\{(x,y) \in S_3 \times S_3 : [x,y] = (1)\}| = 18$. *So,* $P(S_3) = \frac{18}{36} = \frac{1}{2}$, *and by Theorem 2.1 we have*

$$P(S_3) = \frac{k(S_3)}{|S_3|} = \frac{3}{6} = \frac{1}{2}.$$

Example 2.2. *Let* $G = D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$. *Then*

·	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	*	*	*	*	*	*	*	*
a	*	*	*	*				
a^2	*	*	*	*	*	*	*	*
a^3	*	*	*	*				
b	*		*		*		*	
ab	*		*			*		*
a^2b	*		*		*		*	
a^3b	*		*			*		*

Table 2.2: Commute Elements in D_8.

$$d(D_8) = P(D_8) \ = \ \frac{|\{(x,y) \in D_8 \ \times D_8 \ : \ [x,y]=1 \ \}|}{|D_8|^2}$$

$\{(x,y) \in D_8 \times D_8 : [x,y] = 1\} = \{(1,1),(1,a),(1,a^2),(1,a^3),(1,b),(1,ab),(1,a^2b),(1,a^3b),(a,1),$ $(a^2,1),(a^3,1),(b,1),(ab,1),(a^2b,1),(a^3b,1),(b,b),(a,a^2),(a,a^3),(a,a),(a^2,a),(a^2,a^2),(a^2,a^3),$ $(a^2,b),(a^2,ab),(a^2,a^2b),(a^2,a^3b),(a^3,a),(a^3,a^2),(a^3,a^3),(b,a^2),(b,a^2b),(ab,a^2),(ab,ab),(ab,a^3b),$ $(a^2b,a^2),(a^2b,b),(a^2b,a^2b),(a^3b,a^2),(a^3b,ab),(a^3b,a^3b)\}$. Therefore, $|\{(x,y) \in D_8 \times D_8 : [x,y]=1 \}| = 40$. So, $P(D_8) = \frac{40}{64} = \frac{5}{8}$, and by Theorem 2.1 we have

$$P(D_8) = \frac{k(D_8)}{|D_8|} = \frac{5}{8}.$$

Satisfies the Remark $D_8/\langle a^2\rangle \cong V$ (mentioned earlier on page 21).

Example 2.3. *Let $G = S_4$. Then*

$$d(S_4) = P(S_4) \ = \ \frac{|\{(x,y) \in S_4 \times S_4 : [x,y] = (1)\}|}{|S_4|^2}$$

·	(1)	(12)	(13)	(14)	(23)	(24)	(34)	(123)	(132)	(124)	(142)	(134)	(143)	(234)	(243)	(12)(34)	(13)(24)	(14)(23)	(1234)	(1243)	(1324)	(1342)	(1423)	(1432)
(1)	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
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(1432)	*																*		*					*

Table 2.3: Commute Elements in S_4.

Therefore, $|\{(x, y) \in S_4 \times S_4 : [x,y] = (1)\}| = 120$. *So,* $P(S_4) = \frac{120}{576} = \frac{5}{24}$, *and by Theorem 2.1 we have*

$$P(S_4) = \frac{k(S_4)}{|S_4|} = \frac{5}{24}.$$

Example 2.4. *Let* $G = A_5$. *Then*

$$d(A_5) = P(A_5) = \frac{|\{(x, y) \in A_5 \times A_5 : [x,y] = (1)\}|}{|A_5|^2}$$

From Table 2.4 we have $|\{(x, y) \in A_5 \times A_5 : [x,y] = (1)\}| = 300$. *So,* $P(A_5) = \frac{300}{3600} = \frac{5}{60} = \frac{1}{12}$, *and by Theorem 2.1 we have*

$$P(A_5) = \frac{k(A_5)}{|A_5|} = \frac{5}{60} = \frac{1}{12}.$$

·	(1)	(123)	(132)	(124)	(142)	(125)	(152)	(134)	(143)	(135)	(153)	(145)	(154)	(234)	(243)	(235)	(253)	(245)	(254)	(345)	(354)	(12)(34)	(13)(24)	(14)(23)	(23)(45)	(24)(35)	(25)(34)	(13)(45)	(14)(35)	(15)(34)	(12)(45)	(14)(25)	(15(24)	(12(35)	(13(25)	(15)(23)	(12345)	(13524)	(14253)	(15432)	(12354)	(13425)	(15243)	(14532)	(12435)	(14523)	(13254)	(15342)	(12453)	(14325)	(15234)	(13542)	(12534)	(15423)	(13245)	(14352)	(12543)	(15324)	(14235)	(13452)
(1)	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
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Table 2.4: Commute Elements in A_5.

2.3 Probability of a commutator that is equal to a given element

In this section, we will study the probability that a randomly chosen commutator is equal to a given element of G.

Given two subgroups H and K of G and two natural numbers n and m, where $h \in H$ and $k \in K$. If the commutator $[h,k] = g$ such that $g \in G$ then the probability that a randomly chosen commutator of weight $n+m$ of $H \times K$ is equal to a given element of G is defined as follows:

$$P_g^{(n,m)}(H,K) = \frac{|\{(h_1,...,h_n,k_1,...,k_m) \in H^n \times K^m : [h_1,...,h_n,k_1,...,k_m] = g\}|}{|H|^n|K|^m} \tag{2.5}$$

*is clearly a generalization of $P^{(n,m)}(H,K)$, when $g = 1_G$. The case $n = m = 1$ is called **generalized commutativity degree of G** which $d_g(H,K) = P_g(H,K)$.*

The following proposition gives us the relation between the probability of a commutator that is equal to a given element g from G and the probability of a commutator that is equal to the inverse of g, as in [3, Proposition 2.4].

Proposition 2.1. *Let G be a group. Let H and K be subgroups of G. Then*

$$P_g^{(n,m)}(H,K) = P_{g^{-1}}^{(m,n)}(K,H).$$

Proof. From (2.5) we have

$$\begin{aligned} P_g^{(n,m)}(H,K) &= \frac{|\{(h_1,...,h_n,k_1,...,k_m) \in H^n \times K^m : [h_1,...,h_n,k_1,...,k_m] = g\}|}{|H|^n|K|^m} \\ &= \frac{|\{(h_1,...,h_n,k_1,...,k_m) \in H^n \times K^m : [h_1,...,h_n,k_1,...,k_m]^{-1} = g^{-1}\}|}{|H|^n|K|^m} \end{aligned}$$

by the commutator rules we get

$$\begin{aligned} &= \frac{|\{(k_m,...,k_1,h_n,...,h_1) \in K^m \times H^n : [k_m,...,k_1,h_n,...,h_1] = g^{-1}\}|}{|K|^m|H|^n} \\ &= P_{g^{-1}}^{(m,n)}(K,H). \end{aligned}$$

□

The following theorem gives us the significant restriction of the $P_g^{(n,m)}(H,K)$, as in [2, Theorem 1.1] and [3, Theorem 3.3].

Theorem 2.5. *Let G be a group. Let H and K be subgroups of G. Let p be the smallest prime divisor of $|G|$. Then*
(i) $P_g^{(n,m)}(H,K) \leq \frac{2p^n+p-2}{p^{m+n}}$;
(ii) $P_g^{(n,m)}(H,K) \geq \frac{(1-p)|Y_{H^n}|+p|H^n|}{|H^n||K^m|} - \frac{(|K|+p)|C_H(K)|^n}{|H^n||K^m|}$;
where $Y_{H^n} = \{[x_1,...,x_n] \in H^n : C_K([x_1,...,x_n]) = 1\}$.

Proof. The proof can be found in [3, Theorem 3.3].

□

The following corollary comes from achieving equality in Theorem 2.5 (i), as in [2, Corollary 1.2] and [3, Corollary 3.4].

Corollary 2.2. *If* $P_g^{(n,m)}(H,K) = \frac{2p^n + p - 2}{p^{m+n}}$ *and* $p \neq 2$. *Then*

$$[H : C_H(K)] \leq \frac{p \cdot p^{1/n}}{(p-2)^{1/n}}.$$

Proof. The proof can be found in [3, Corollary 3.4].

□

The following theorem comes from exercise (3.9) *page* 45 *in [15] which will be helpful to proof the next lemma. We shall study and investigate the coefficients* a_{ijv} *for* $i, j, v = 1, ..., k(G)$ *in more details in Chapter* 6*.*

Theorem 2.6. *Let* G *be a group. Let* $C(g_1), ..., C(g_{k(G)})$ *be the distinct conjugacy classes of* G *where* $g_i \in G$ *for all* $i = 1, \cdots, k(G)$*. Let* $g_i \in C(g_i)$ *be a representatives and let* a_{ijv} *be the integers defined by*

$$K_iK_j = \sum_{v=1}^{k(G)} a_{ijv}K_v.$$

Then

$$a_{ijv} = \frac{|C(g_i)||C(g_j)|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_v^{-1})}{\chi(1)}.$$

Proof. We have

$$K_iK_j = \sum_{t=1}^{k(G)} a_{ijt}K_t$$
$$\omega_\chi(K_iK_j) = \sum_{t=1}^{k(G)} \omega_\chi(a_{ijt}K_t)$$

since, ω_χ is an algebra homomorphism

$$\omega_\chi(K_i)\omega_\chi(K_j) = \sum_{t=1}^{k(G)} a_{ijt}\omega_\chi(K_t)$$

since, $\omega_\chi(K_l) = \frac{|C(g_l)|\chi(g_l)}{\chi(1)}$ for all $l = 1, ..., k(G)$, then for each $\chi \in Irr(G)$ we have

$$\frac{|C(g_i)|\chi(g_i)|C(g_j)|\chi(g_j)}{\chi(1)^2} = \sum_{t=1}^{k(G)} a_{ijt}\frac{|C(g_t)|\chi(g_t)}{\chi(1)}$$
$$\frac{|C(g_i)|\chi(g_i)|C(g_j)|\chi(g_j)}{\chi(1)} = \sum_{t=1}^{k(G)} a_{ijt}|C(g_t)|\chi(g_t)$$
$$\frac{|C(g_i)|\chi(g_i)|C(g_j)|\chi(g_j)\chi(g_v^{-1})}{\chi(1)} = \sum_{t=1}^{k(G)} a_{ijt}|C(g_t)|\chi(g_t)\chi(g_v^{-1}).$$

Summing over all $\chi \in Irr(G)$, we have

$$
\begin{aligned}
|C(g_i)||C(g_j)| \sum_{\chi \in Irr(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_v^{-1})}{\chi(1)} &= \sum_{\chi \in Irr(G)} \sum_{t=1}^{k(G)} a_{ijt}|C(g_t)|\chi(g_t)\chi(g_v^{-1}) \\
&= \sum_{t=1}^{k(G)} a_{ijt}|C(g_t)| \sum_{\chi \in Irr(G)} \chi(g_t)\chi(g_v^{-1})
\end{aligned}
$$

by the Second Orthogonality Relation Theorem

$$
\begin{aligned}
&= \sum_{t=v=1}^{k(G)} a_{ijv}|C(g_v)||C_G(g_v)| \\
&= a_{ijv}|G|
\end{aligned}
$$

thus

$$
a_{ijv} = \frac{|C(g_i)||C(g_j)|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_v^{-1})}{\chi(1)}.
$$

□

The following lemma comes from exercise (3.10)(*a*) *page* 45 *in [15] which will be helpful to proof the next lemma.*

Lemma 2.4. *Let G be a group and $g \in G$. Fix $x \in G$. Then $g \sim [x, y]$ for some $y \in G$ iff*

$$
\sum_{\chi \in Irr(G)} \frac{|\chi(x)|^2\chi(g^{-1})}{\chi(1)} \neq 0.
$$

Proof. Let $g \sim [x, y]$ for some $y \in G$. Then there is $h \in G$ such that $g^h = x^{-1}y^{-1}xy$, i.e., $xg^h \sim x$. By Theorem 2.6, if $x \in C(g_i)$ and $g \in C(g_j)$, then $a_{iji} \neq 0$ since $xg^h = x^y \in C(g_i)$. We have

$$
a_{iji} \neq 0.
$$

From Proposition 1.1(2) we have

$$
\frac{|C(g_i)||C(g_j)|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(x)\chi(g)\chi(x^{-1})}{\chi(1)} \neq 0
$$

since $\chi(g) = \chi(g^{-1})$ for all $g \in G$, then

$$
\sum_{\chi \in Irr(G)} \frac{|\chi(x)|^2\chi(g^{-1})}{\chi(1)} \neq 0.
$$

Conversely, let $\sum_{\chi \in Irr(G)} \frac{|\chi(x)|^2\chi(g^{-1})}{\chi(1)} \neq 0$, then $a_{iji} \neq 0$ where $x \in C(g_i)$ and $g \in C(g_j)$. So, for some $h, k, l \in G$ we have

$$
\begin{aligned}
x^h g^k = h^{-1}xhg^k &= x^l \\
xhg^k &= hx^l \\
xhg^k h^{-1} &= hx^l h^{-1} \\
xg^{kh^{-1}} &= x^{lh^{-1}}
\end{aligned}
$$

if we take $z = kh^{-1}$ and $y = lh^{-1}$, then

$$
\begin{aligned}
xg^z &= x^y \\
g^z &= x^{-1}x^y = [x, y]
\end{aligned}
$$

therefore, there is $z \in G$ such that $g^z = [x, y]$. Thus $g \sim [x, y]$.

□

The following lemma comes from exercise (3.10)(*b*) *page* 45 *in [15] which gives us the sum of* $\chi(g)/\chi(1)$ *is not equal to zero when the element* g *belongs to the derived group which will be helpful to proof the next theorem.*

Lemma 2.5. *Let* G *be a group and* $g \in G$. *Then* $g = [x, y]$ *for some* $x, y \in G$ *iff*

$$\sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)} \neq 0.$$

Proof. Let $g = [x, y]$ for some $x, y \in G$. By Lemma 2.4 we have

$$\sum_{\chi \in Irr(G)} \frac{|\chi(x)|^2 \chi(g^{-1})}{\chi(1)} \neq 0$$

by Theorem First Orthogonal Relation Theorem we have

$$\frac{1}{|G|} \sum_{z \in G} |\chi(z)|^2 = 1$$

then

$$\begin{aligned} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)} &= \sum_{\chi \in Irr(G)} \left(\frac{1}{|G|} \sum_{z \in G} |\chi(z)|^2 \right) \frac{\chi(g)}{\chi(1)} \\ &= \frac{1}{|G|} \sum_{z \in G} \sum_{\chi \in Irr(G)} \frac{|\chi(z)|^2 \chi(g)}{\chi(1)}. \end{aligned}$$

For each $z \in G$,

$$\sum_{\chi \in Irr(G)} \frac{|\chi(z)|^2 \chi(g)}{\chi(1)} = \frac{a_{ljl}|G|}{|C(g_l)||C(g_j)|}$$

where $z \in C(g_l)$ and $g \in C(g_j)$ is non-negative since by that if $K_i K_j = \sum_{v=1}^{k(G)} a_{ijk} K_v$, then the multiplication constants a_{ijv} are non-negative integers. Therefore,

$$\sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)} \neq 0.$$

Conversely, let $\sum_{\chi \in Irr(G)} \dfrac{\chi(g)}{\chi(1)} \neq 0$. Then

$$\frac{1}{|G|} \sum_{\chi \in Irr(G)} \sum_{z \in G} \frac{|\chi(z)|^2 \chi(g)}{\chi(1)} \neq 0.$$

So, $\sum_{\chi \in G} \dfrac{|\chi(w)|^2 \chi(g)}{\chi(1)} \neq 0$, for some $w \in G$. Then by Lemma 2.4 we have for some $y \in G$

$$g \sim [w, y]$$

then there is $h \in G$ such that

$$\begin{aligned} g^h &= [w, y] \\ g &= [w, y]^{h^{-1}} \\ g &= [w^{h^{-1}}, y^{h^{-1}}] \end{aligned}$$

so that $g = [x, y^{h^{-1}}]$ where $x = w^{h^{-1}}$.

□

The following theorem comes from exercise 3 page 183 in [5] which is the most important theorem that will help us convey the concept of probability and its calculation method via the character theory.

Theorem 2.7. *Let G be a group, and let's define the function*

$$\psi : G \longrightarrow \mathbb{C}$$

given by

$$\psi(g) = |\{(x, y) \in G \times G : [x, y] = g\}|,$$

for all $g \in G$. Then $\psi(g) = \sum_{i=1}^{k(G)} \frac{|G|}{\chi_i(1)}\chi_i(g)$.

Proof. Since $g \in G^{'}$ then by Lemma 2.5 we have $\sum_{i=1}^{k(G)} \frac{\chi_i(g)}{\chi_i(1)} \neq 0$, and since $|G| = |C_G(g)||C(g)|$, for all $g \in G$, hence we have

$$\psi(g) = \sum_{i=1}^{k(G)} \frac{|G|}{\chi_i(1)}\chi_i(g)$$
$$\psi(g) = \sum_{i=1}^{k(G)} \frac{|C_G(g)||C(g)|}{\chi_i(1)}\chi_i(g)$$
$$\psi(g) = |C_G(g)| \sum_{i=1}^{k(G)} \frac{|C(g)|}{\chi_i(1)}\chi_i(g)$$

since $\frac{|C(g)|}{\chi_i(1)}\chi_i(g) = \omega_{\chi_i}(K)$ for all $i = 1, ..., k(G)$, where K is class sum ($K = \sum_{g \in C(g)} g$). Then

$$\psi(g) = |C_G(g)| \sum_{i=1}^{k(G)} \omega_{\chi_i}(K)$$
$$\psi(g) = |C_G(g)| \sum_{i=1}^{k(G)} Z(\chi_i).$$

On the other hand, we have

$$\psi(g) = |\{(x, y) \in G \times G : [x, y] = g\}|$$

$$\psi(g) = |\{(x, y) \in G \times G : x^{-1}y^{-1}xy = g\}|$$

$$\psi(g) = |\{(x, y) \in G \times G : xy = yxg\}|$$

$$\psi(g) = |\{x \in G \text{ and } y \in C_G(x) : xy = yxg\}|$$

$$\psi(g) = |\{x \in G \text{ and } y \in C_G(x) : xy = xyg\}|$$

$$\psi(g) = |\{xy \in Fix_G(g) \text{ and } y \in C_G(x) : xy = xyg\}|$$

$$\psi(g) = |\{Fix_G(g) \times C_G(x)\}| = |C_G(x)| \cdot |Fix_G(g)|.$$

□

Corollary 2.3. *Let G be a group. Then the function*

$$\psi : G \longrightarrow \mathbb{C}$$

given by

$$\psi(g) = |\{(x, y) \in G \times G : [x, y] = g\}|,$$

is a character.

Proof. Since ψ is a class function and $\psi(g) \in \mathbb{N}$ for all $g \in G$. Thus, ψ is a character.

□

The following theorem gives us the method for calculating the probability of a commutator that is equal to a given element of G by the character theory, as in [2] and [3].

Theorem 2.8. *Let G be a group and $g \in G$. Then*

$$P_g(G) = \frac{1}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)}.$$

Proof. By definition of $P_g(G)$ we have

$$P_g(G) = \frac{|\{(x, y) \in G \times G : [x, y] = g\}|}{|G|^2},$$

and by Theorem 2.7 we have

$$P_g(G) = \frac{1}{|G|^2} \sum_{\chi \in Irr(G)} \frac{|G|}{\chi(1)} \chi(g),$$

thus

$$P_g(G) = \frac{1}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g)}{\chi(1)}.$$

□

The following are some examples in character theory which calculate $P_g(G)$ for all $g \in G$. Since the conjugate elements of a group G have the same character, then the conjugate elements have the same probability of a commutator that is equal to a given element from G.

Example 2.5. *Let $G = S_3$ and by Table 1.6 we have*
1) When $g = (1)$, then

$$P_{(1)}(S_3) = \frac{1}{|S_3|} \sum_{\chi \in Irr(S_3)} \frac{\chi((1))}{\chi((1))} = \frac{1}{|S_3|} k(S_3) = \frac{3}{6} = \frac{1}{2}.$$

2) When $g \in C((12))$, then

$$P_g(S_3) = \frac{1}{|S_3|} \sum_{\chi \in Irr(S_3)} \frac{\chi(g)}{\chi((1))} = \frac{1}{6}(1 - 1 + 0) = 0.$$

3) When $g \in C((123))$, then

$$P_g(S_3) = \frac{1}{|S_3|} \sum_{\chi \in Irr(S_3)} \frac{\chi(g)}{\chi((1))} = \frac{1}{6}(1 + 1 - \frac{1}{2}) = \frac{1}{4}.$$

Therefore, $\cup_{g \in S_3} P_g(S_3) = 1$.

Example 2.6. *Let $G = D_8$ and by Table 1.14 we have*

1) When $g = 1$, then

$$P_1(D_8) = \frac{1}{|D_8|} \sum_{\chi \in Irr(D_8)} \frac{\chi(1)}{\chi(1)} = \frac{1}{|D_8|} k(D_8) = \frac{5}{8}.$$

2) When $g \in C(a^2)$, then

$$P_g(D_8) = \frac{1}{|D_8|} \sum_{\chi \in Irr(D_8)} \frac{\chi(g)}{\chi(1)} = \frac{1}{8}(1+1+1+1-\frac{2}{2}) = \frac{3}{8}.$$

3) When $g \in C(b)$, then

$$P_g(D_8) = \frac{1}{|D_8|} \sum_{\chi \in Irr(D_8)} \frac{\chi(g)}{\chi(1)} = \frac{1}{8}(1-1+1-1+0) = 0.$$

4) When $g \in C(a)$, then

$$P_g(D_8) = \frac{1}{|D_8|} \sum_{\chi \in Irr(D_8)} \frac{\chi(g)}{\chi(1)} = \frac{1}{8}(1+1-1-1+0) = 0.$$

5) When $g \in C(ab)$, then

$$P_g(D_8) = \frac{1}{|D_8|} \sum_{\chi \in Irr(D_8)} \frac{\chi(g)}{\chi(1)} = \frac{1}{8}(1-1-1+1+0) = 0.$$

Therefore, $\cup_{g \in D_8} P_g(D_8) = 1$.

Example 2.7. *Let $G = S_4$ and by Table 1.1 we have*

1) When $g = (1)$, then

$$P_{(1)}(S_4) = \frac{1}{|S_4|} \sum_{\chi \in Irr(S_4)} \frac{\chi((1))}{\chi((1))} = \frac{1}{|S_4|} k(S_4) = \frac{5}{24}.$$

2) When $g \in C((12))$, then

$$P_g(S_4) = \frac{1}{|S_4|} \sum_{\chi \in Irr(S_4)} \frac{\chi(g)}{\chi((1))} = \frac{1}{24}(1-1+0+\frac{1}{3}-\frac{1}{3}) = 0.$$

3) When $g \in C((123))$, then

$$P_g(S_4) = \frac{1}{|S_4|} \sum_{\chi \in Irr(S_4)} \frac{\chi(g)}{\chi((1))} = \frac{1}{24}(1+1-\frac{1}{2}+0+0) = \frac{1}{16}.$$

4) When $g \in C((1234))$, then

$$P_g(S_4) = \frac{1}{|S_4|} \sum_{\chi \in Irr(S_4)} \frac{\chi(g)}{\chi((1))} = \frac{1}{24}(1-1+0-\frac{1}{3}+\frac{1}{3}) = 0.$$

5) When $g \in C((12)(34))$, then

$$P_g(S_4) = \frac{1}{|S_4|} \sum_{\chi \in Irr(S_4)} \frac{\chi(g)}{\chi((1))} = \frac{1}{24}(1+1+1-\frac{1}{3}-\frac{1}{3}) = \frac{7}{72}.$$

Therefore, $\cup_{g \in S_4} P_g(S_4) = 1$.

Chapter 3

Relative Tensor Degree

In this chapter, we will present three sections; in the first section, we will introduce definitions of a compatiplity action and non-abelian tensor product of normal subgroups H and K of G, establish some basic properties that describe the main calculus rules in the non-abelian tensor product, explain the relation between the non-abelian tensor product group $H \otimes K$ and the derived group [H,K], and state the proposition which gives us the isomorphism between $H \otimes K$ and $K \otimes H$. In the second section, we will study the definitions of tensor centralizer and tensor center which have been developed in the previous section, the algebraic structures of these concepts, the definition of the relative tensor degree, and explain the relation between the tensor centralizer and the tensor degree, and in the third and last section, we will study the relation between the relative commutative degree and relative tensor degree. The basic references of this chapter are[4],[6],[19] and [23].

3.1 Compatiplity Action and Non-abelian Tensor Product

We will study the compatiplity action and non-abelian tensor product.

The following definition we will state the act compatibly as in [4], [6] and [19, Definition 1.2.1].

Definition 3.1. *Let G be a group. Let H and K be normal subgroups of G. We say that H and K* ***act compatibly*** *on each other if:*

$$\left(h_2^{k_1}\right)^{h_1} = \left((h_2^{h_1^{-1}})^{k_1}\right)^{h_1} \qquad \text{and} \qquad \left(k_2^{h_1}\right)^{k_1} = \left((k_2^{k_1^{-1}})^{h_1}\right)^{k_1},$$

for all $h_1, h_2 \in H$ and $k_1, k_2 \in K$. When the action is given by conjugation.

Example 3.1. *Let $G = D_8$. Let $H = \langle a \rangle$ and $K = \langle a^2, b \rangle$ be normal subgroups of D_8. Consider $h_1 = a, h_2 = a^2 \in H$ and $k_1 = a^2, k_2 = b \in K$. Then*

$$(a^{2^{a^2}})^a = ((a^{2^{a^3}})^{a^2})^a \quad \rightarrow \quad a^2 = a^2$$
$$(b^a)^{a^2} = ((b^{a^2})^a)^{a^2} \quad \rightarrow \quad a^2 b = a^2 b$$

the same way for all $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Thus, H and K act compatibly on each other.

Example 3.2. *Let $G = S_4$. Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ and $K = A_4$ be normal subgroups of S_4. Consider $h_1 = (13)(24), h_2 = (14)(23) \in H$ and $k_1 = (142), k_2 = (234) \in K$. Then*

$$((234)^{(13)(24)})^{(142)} \neq (((234)^{(124)})^{(13)(24)})^{(142)} \quad \rightarrow \quad (124) \neq (132).$$

Thus, H and K do not act compatibly on each other.

We will state the non-abelian tensor product as in [4] page 1, [6] in page 178 and [19, Definition 1.2.4].

Definition 3.2. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Let $h \in H$ and $k \in K$. We define the group $H \otimes K$ as the* ***non-abelian tensor product*** *of H and K generated by the symbols $h \otimes k$ such that*

$$h_1 h_2 \otimes k_1 = (h_2^{h_1} \otimes k_1^{h_1})(h_1 \otimes k_1) \qquad \text{and} \qquad h_1 \otimes k_1 k_2 = (h_1 \otimes k_1)(h_1^{k_1} \otimes k_2^{k_1}),$$

for all $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

The following proposition gives us the action of normal subgroups on the non-abelian tensor product in group G, as in [16, Proposition 1] and [6, Proposition 1.2.6].

Proposition 3.1. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Then H and K act on the non-abelian tensor product group $H \otimes K$ by:*

$$(h' \otimes k)^h = (h')^h \otimes (k)^h \qquad \text{and} \qquad (h \otimes k')^k = (h)^k \otimes (k')^k,$$

for all $h, h' \in H$ and $k, k' \in K$.

Proof. Let $h, h' \in H$ and $k, k' \in K$. Then

$$(h')^h \otimes (k)^h = (1_H \cdot h')^h \otimes (k)^h = 1_H^h \cdot h'^h \otimes k^h.$$

by Definition 3.2

$$\begin{aligned} &= \big(((h'^h)^{1_H})^h \otimes ((k^h)^{1_H})^h\big)\big(1_H^h \otimes k^h\big) \\ &= \big((h')^{1_H h} \otimes (k)^{1_H h}\big)\big(1_H^h \otimes k^h\big) \\ &= \big((h')^{1_H} \otimes (k)^{1_H}\big)^h \big(1_H \otimes k\big)^h \\ &= \big[\big((h')^{1_H} \otimes (k)^{1_H}\big)\big(1_H \otimes k\big)\big]^h \end{aligned}$$

by Definition 3.2

$$\begin{aligned} &= \big[(1_H h' \otimes k)\big]^h \\ &= (h' \otimes k)^h. \end{aligned}$$

Thus, $(h' \otimes k)^h = (h')^h \otimes (k)^h$, and the same way to prove $(h \otimes k')^k = (h)^k \otimes (k')^k$. □

The following lemmas describe the main calculus rules in $H \otimes K$, as in [6, Proposition 3.], [19, Proposition 1.2.8] and [23] page 1.

Lemma 3.1. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h \in H$ and $k \in K$. Then*

$$1_H \otimes k = h \otimes 1_K = 1_{H \otimes K}$$

in the non-abelian tensor product group $H \otimes K$.

Proof. Let $h, h_1 \in H$ and $k, k_1 \in K$.

$$\begin{aligned}
\text{If } k = k_1 = 1_K, \text{ then } (h \otimes kk_1) &= (h \otimes k)(h \otimes k_1)^k \\
(h \otimes 1_K) &= (h \otimes 1_K)(h \otimes 1_K)^{1_K} \\
(h \otimes 1_K) &= (h \otimes 1_K)(h \otimes 1_K) \\
&\xrightarrow{(h \otimes 1_K)^{-1}} \\
1_{H \otimes K} &= (h \otimes 1_K) \to (1) \\
\text{and if } h = h_1 = 1_H, \text{ then } (hh_1 \otimes k) &= (h_1 \otimes k)^h (h \otimes k) \\
(1_H \otimes k) &= (1_H \otimes k)^{1_H} (1_H \otimes k) \\
(1_H \otimes k) &= (1_H \otimes k)(1_H \otimes k) \\
&\xrightarrow{(1_H \otimes k)^{-1}} \\
1_{H \otimes K} &= (1_H \otimes k) \to (2) \\
\text{by (1) and (2) we have} \\
1_H \otimes k &= h \otimes 1_K = 1_{H \otimes K}.
\end{aligned}$$

□

Lemma 3.2. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h \in H$ and $k \in K$. Then*

$$(h \otimes k)^{-1} = (h^{-1} \otimes k)^h = (h \otimes k^{-1})^k$$

in the non-abelian tensor product group $H \otimes K$.

Proof. Let $h, h_1 \in H$ and $k, k_1 \in K$.

$$\begin{aligned}
\text{If } k_1 = k^{-1}, \text{ then } (h \otimes kk_1) &= (h \otimes k)(h \otimes k_1)^k \\
(h \otimes 1_K) &= (h \otimes k)(h \otimes k^{-1})^k \\
1_{H \otimes K} &= (h \otimes k)(h \otimes k^{-1})^k \\
&\xrightarrow{(h \otimes k)^{-1}} \\
(h \otimes k)^{-1} &= 1_{H \otimes K}(h \otimes k^{-1})^k \\
(h \otimes k)^{-1} &= (h \otimes k^{-1})^k \to (1) \\
\text{and if } h_1 = h^{-1}, \text{ then } (hh_1 \otimes k) &= (h_1 \otimes k)^h (h \otimes k) \\
(1_H \otimes k) &= (h^{-1} \otimes k)^h (h \otimes k) \\
1_{H \otimes K} &= (h^{-1} \otimes k)^h (h \otimes k) \\
&\xleftarrow{(h \otimes k)^{-1}} \\
(h \otimes k)^{-1} &= (h^{-1} \otimes k)^h \to (2) \\
\text{by (1) and (2) we have} \\
(h \otimes k)^{-1} &= (h^{-1} \otimes k)^h = (h \otimes k^{-1})^k.
\end{aligned}$$

□

Lemma 3.3. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h, h_1 \in H$ and $k, k_1 \in K$. Then*

$$(h \otimes k)(h_1 \otimes k_1)^{hk} = (h_1 \otimes k_1)^{kh}(h \otimes k)$$

in the non-abelian tensor product group $H \otimes K$.

Proof. by Definition 3.2 of $hh_1 \otimes kk_1$ we have

$$\begin{aligned} hh_1 \otimes kk_1 &= (h_1 \otimes kk_1)^h (h \otimes kk_1) \\ &= \big((h_1 \otimes k)(h_1 \otimes k_1)^k\big)^h \big((h \otimes k)(h \otimes k_1)^k\big) \\ &= (h_1 \otimes k)^h (h_1 \otimes k_1)^{kh} (h \otimes k)(h \otimes k_1)^k \to (1) \\ \text{and} & \\ hh_1 \otimes kk_1 &= (hh_1 \otimes k)(hh_1 \otimes k_1)^k \\ &= \big((h_1 \otimes k)^h (h \otimes k)\big)\big((h_1 \otimes k_1)^h (h \otimes k_1)\big)^k \\ &= (h_1 \otimes k)^h (h \otimes k)(h_1 \otimes k_1)^{hk} (h \otimes k_1)^k \to (2) \end{aligned}$$

by (1) and (2) we have

$$(h_1 \otimes k)^h (h_1 \otimes k_1)^{kh} (h \otimes k)(h \otimes k_1)^k = (h_1 \otimes k)^h (h \otimes k)(h_1 \otimes k_1)^{hk} (h \otimes k_1)^k$$

$$\xrightarrow{\big((h_1 \otimes k)^h\big)^{-1}}$$
$$\xleftarrow{\big((h \otimes k_1)^k\big)^{-1}}$$

thus

$$(h \otimes k)(h_1 \otimes k_1)^{hk} = (h_1 \otimes k_1)^{kh} (h \otimes k).$$

□

Lemma 3.4. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h, h_2 \in H$ and $k, k_2 \in K$. Then*

$$(h_2 \otimes k_2)^{h \otimes k} = (h_2 \otimes k_2)^{[h,k]}$$

in the non-abelian tensor product group $H \otimes K$.

Proof. By Lemma 3.3 we have for all h,h_1 ∈ H and k,k_1∈ K

$$(h_1 \otimes k_1)^{kh} (h \otimes k) = (h \otimes k)(h_1 \otimes k_1)^{hk}$$

$$\xrightarrow{(h \otimes k)^{-1}}$$

$$(h \otimes k)^{-1} (h_1 \otimes k_1)^{kh} (h \otimes k) = (h_1 \otimes k_1)^{hk}$$

$$\xrightarrow{khh^{-1}k^{-1} = 1}$$

$$\begin{gathered} (h \otimes k)^{-1} (h_1 \otimes k_1)^{kh} (h \otimes k) = ((h_1 \otimes k_1))^{khh^{-1}k^{-1}hk} \\ (h \otimes k)^{-1} (h_1^{kh} \otimes k_1^{kh})(h \otimes k) = ((h_1^{kh} \otimes k_1^{kh}))^{h^{-1}k^{-1}hk} \\ ((h_1^{kh} \otimes k_1^{kh}))^{(h \otimes k)} = ((h_1^{kh} \otimes k_1^{kh}))^{[h,k]} \end{gathered}$$

and let $h_2 = h_1^{kh}, k_2 = k_1^{kh}$, thus

$$((h_2 \otimes k_2))^{(h \otimes k)} = ((h_2 \otimes k_2))^{[h,k]}.$$

□

Lemma 3.5. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h \in H$ and $k, k_1 \in K$. Then*

$$\big(h(h^{-1})^k\big)\otimes k_1 = (h \otimes k)\,\big((h \otimes k)^{-1}\big)^{k_1}$$

in the non-abelian tensor product group $H \otimes K$.

Proof.

$$\begin{aligned}
(h(h^{-1})^k) \otimes k_1 &= ((h^{-1})^k \otimes k_1)^h (h \otimes k_1) \\
&= ((h^{-1})^k \otimes (k_1)^{k^{-1}k})^h (h \otimes k_1) \\
&= ((h^{-1}) \otimes (k_1)^{k^{-1}})^{kh} (h \otimes k_1) \\
&= (h^{-1} \otimes k^{-1}k_1k)^{kh} (h \otimes k_1) \\
&= \big[(h^{-1} \otimes k^{-1})(h^{-1} \otimes k_1k)^{k^{-1}}\big]^{kh} (h \otimes k_1) \\
&= (h^{-1} \otimes k^{-1})^{kh} (h^{-1} \otimes k_1k)^h (h \otimes k_1) \\
&= ((h^{-1} \otimes k^{-1})^k)^h (h^{-1} \otimes k_1k)^h (h \otimes k_1)
\end{aligned}$$

by Lemma 3.2

$$\begin{aligned}
&= ((h \otimes k)^{h^{-1}})^h (h^{-1} \otimes k_1k)^h (h \otimes k_1) \\
&= (h \otimes k)\,\big[(h^{-1} \otimes k_1)(h^{-1} \otimes k)^{k_1}\big]^h (h \otimes k_1) \\
&= (h \otimes k)(h^{-1} \otimes k_1)^h (h^{-1} \otimes k)^{k_1h} (h \otimes k_1)
\end{aligned}$$

by Lemma 3.2

$$= (h \otimes k)(h \otimes k_1)^{-1} (h^{-1} \otimes k)^{k_1h} (h \otimes k_1)$$

by Lemma 3.3

$$\begin{aligned}
&= (h \otimes k)(h^{-1} \otimes k)^{hk_1} (h \otimes k_1)^{-1} (h \otimes k_1) \\
&= (h \otimes k)(h^{-1} \otimes k)^{hk_1} \\
&= (h \otimes k)((h^{-1} \otimes k)^h)^{k_1}
\end{aligned}$$

by Lemma 3.2

$$= ((h \otimes k)(h \otimes k)^{-1})^{k_1}.$$

□

Lemma 3.6. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. If $h, h_1 \in H$ and $k, k_1 \in K$. Then*

$$\big[h \otimes k, h_1 \otimes k_1\big] = (h(h^{-1})^k) \otimes (k_1^{h_1} k_1^{-1})$$

in the non-abelian tensor product group $H \otimes K$.

Proof. If we take ($k^h = hkh^{-1}$ and $[h, k] = hkh^{-1}k^{-1}$), and since by Lemma 3.5 we have

$$\begin{aligned}
(h(h^{-1})^k) \otimes (k_1^{h_1} k_1^{-1}) &= (h \otimes k)((h \otimes k)^{-1})^{k_1^{h_1} k_1^{-1}} \\
&= (h \otimes k)((h \otimes k)^{-1})^{h_1k_1h_1^{-1}k_1^{-1}} \\
&= (h \otimes k)((h \otimes k)^{-1})^{[h_1,k_1]}
\end{aligned}$$

by Lemma 3.4

$$\begin{aligned}
&= (h \otimes k)((h \otimes k)^{-1})^{(h_1 \otimes k_1)} \\
&= (h \otimes k)(h \otimes k)^{-(h_1 \otimes k_1)} \\
&= (h \otimes k)((h \otimes k)^{(h_1 \otimes k_1)})^{-1} \\
&= (h \otimes k)((h_1 \otimes k_1)(h \otimes k)(h_1 \otimes k_1)^{-1})^{-1} \\
&= (h \otimes k)(h_1 \otimes k_1)(h \otimes k)^{-1}(h_1 \otimes k_1)^{-1} \\
&= [h \otimes k, h_1 \otimes k_1].
\end{aligned}$$

□

The following theorem gives us the relation between the non-abelian tensor product group $H \otimes K$ and the derived group $[H,K]$.

Theorem 3.1. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Let $h \in H$ and $k \in K$. The map*

$$\kappa_{H,K} : H \otimes K \longrightarrow [H,K]$$

given by

$$\kappa_{H,K}(h \otimes k) = [h,k] = h^{-1}k^{-1}hk = h^{-1}h^{k}.$$

Defines a group epimorphism, whose kernal ker $\kappa_{H,K}$ = J(G,H,K) is central in $H \otimes K$.

Proof. The well-definedness of $\kappa_{H,K}$ is clear. To prove that $\kappa_{H,K}$ is a group homomorphism, it suffices to show that

$$\begin{aligned}
(i)\ \kappa_{H,K}(h_2^{h_1} \otimes k_1^{h_1})\ \kappa_{H,K}(h_1 \otimes k_1) &= (h_2^{h_1})^{-1}(h_2^{h_1})^{k_1^{h_1}}(h_1^{-1}h_1^{k_1}) \\
&= (h_1^{-1}h_2h_1)^{-1}(h_2^{h_1})^{k_1^{h_1}}(h_1^{-1}h_1^{k_1}) \\
&= (h_1^{-1}h_2^{-1}h_1)(h_2^{h_1})^{h_1^{-1}k_1h_1}(h_1^{-1}h_1^{k_1}) \\
&= (h_1^{-1}h_2^{-1}h_1)(h_2)^{h_1h_1^{-1}k_1h_1}(h_1^{-1}h_1^{k_1}) \\
&= h_1^{-1}h_2^{-1}h_1(h_2)^{k_1h_1}h_1^{-1}h_1^{k_1} \\
&= h_1^{-1}h_2^{-1}((h_2)^{k_1h_1})^{h_1^{-1}}h_1^{k_1} \\
&= h_1^{-1}h_2^{-1}(h_2)^{k_1h_1h_1^{-1}}h_1^{k_1} \\
&= h_1^{-1}h_2^{-1}{h_2}^{k_1}h_1^{k_1} \\
&= (h_2h_1)^{-1}(h_2h_1)^{k_1} \\
&= \kappa_{H,K}(h_2h_1 \otimes k_1) \\
&= \kappa_{H,K}(h_1h_2 \otimes k_1).
\end{aligned}$$

$$\begin{aligned}
(ii)\ \kappa_{H,K}(h_1 \otimes k_1)\ \kappa_{H,K}(h_1^{k_1} \otimes k_2^{k_1}) &= (h_1^{-1}h_1^{k_1})(h_1^{k_1})^{-1}(h_1^{k_1})^{k_2^{k_1}} \\
&= h_1^{-1}h_1^{k_1}(h_1^{k_1})^{-1}(h_1^{k_1})^{k_2^{k_1}} \\
&= h_1^{-1}(h_1^{k_1})^{k_2^{k_1}} \\
&= h_1^{-1}(h_1^{k_1})^{k_1^{-1}k_2k_1} \\
&= h_1^{-1}(h_1)^{k_1k_1^{-1}k_2k_1} \\
&= h_1^{-1}(h_1)^{k_2k_1} \\
&= \kappa_{H,K}(h_1 \otimes k_2k_1) \\
&= \kappa_{H,K}(h_1 \otimes k_1k_2).
\end{aligned}$$

Thus, $\kappa_{H,K}$ is a group homomorphism. Since for all $h^{-1}h^{k} \in [H,K]$ there is $h \otimes k \in H \otimes K$. Hence, $\kappa_{H,K}$ is a group epimorphism. Furthermore,

$$\begin{aligned}
ker\ \kappa_{H,K} &= \{h \otimes k \in H \otimes K : \kappa_{H,K}(h \otimes k) = 1_G\} \\
&= \{h \otimes k \in H \otimes K : [h,k] = h^{-1}k^{-1}hk = 1_G\} \\
&= \{h \otimes k \in H \otimes K : hk = kh, \forall h \in H, \forall k \in K\} \\
&= Z(H \otimes K).
\end{aligned}$$

□

The following proposition gives us the isomorphism between $H \otimes K$ and $K \otimes H$, as in [19, Proposition 1.2.7].

Proposition 3.2. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Then $H \otimes K \cong K \otimes H$.*

Proof. Suppose the map $\alpha : H \otimes K \to K \otimes H$ given by $\alpha(h \otimes k) = (k \otimes h)^{-1}$. The well-definedness of α is clear. α is a group homomorphism since

$$\begin{aligned}\alpha(h_1h_2 \otimes k) &= (k \otimes h_1h_2)^{-1} = \Big[(k \otimes h_1)(k^{h_1} \otimes h_2^{h_1})\Big]^{-1} \\ &= (k^{h_1} \otimes h_2^{h_1})^{-1}(k \otimes h_1)^{-1} = \alpha(h_2^{h_1} \otimes k^{h_1})\alpha(h_1 \otimes k)\end{aligned}$$

and

$$\begin{aligned}\alpha(h \otimes k_1k_2) &= (k_1k_2 \otimes h)^{-1} = \Big[(k_2^{k_1} \otimes h^{k_1})(k_1 \otimes h)\Big]^{-1} \\ &= (k_1 \otimes h)^{-1}(k_2^{k_1} \otimes h^{k_1})^{-1} = \alpha(h \otimes k_1)\alpha(h^{k_1} \otimes k_2^{k_1}),\end{aligned}$$

for all $h, h_1, h_2 \in H$ and $k, k_1, k_2 \in K$. α is an injective since if $\alpha(h_1 \otimes k_1)$ is equal to $\alpha(h_2 \otimes k_2)$ in $K \otimes H$, then $(k_1 \otimes h_1)^{-1}$ is equal to $(k_2 \otimes h_2)^{-1}$, thus, $h_1 \otimes k_1$ is equal to $h_2 \otimes k_2$ in $H \otimes K$, and α is a surjective since for all $(k \otimes h)^{-1} \in K \otimes H$ there is $(h \otimes k) \in H \otimes K$. Then α is a group isomorphism. Thus, $H \otimes K \cong K \otimes H$.

□

Special case. *If $H = K = G$. We denote by $G \otimes G$ the non-abelian tensor square of G. With considering the map $\kappa_{G,G} : G \otimes G \to [G, G] = G'$, given by $\kappa_{G,G}(x \otimes y) = [x, y]$. for all $x, y \in G$ such that $\kappa_{G,G}$ is a group epimorphism, and ker $\kappa_{G,G} = J(G, G, G) = J(G)$.*

3.2 Tensor Centralizer and Relative Tensor Degree

We will study the tensor centralizer and the relative tensor degree, and explain the relation between them.

We will state the tensor centralizer as in [23].

Definition 3.3. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. We define the set $C_K^{\otimes}(H)$ to be the **tensor centralizer** of H with respect to K such that*

$$C_K^{\otimes}(H) = \{k \in K : h \otimes k = 1_{H \otimes K}, \forall h \in H\}.$$

We see that $C_K^{\otimes}(H) = \cap_{h \in H} C_K^{\otimes}(h)$.

We will state the tensor center as in [23].

Definition 3.4. *Let G be a group which acts on itself compatibly. We define the set $Z^{\otimes}(G)$ to be the **tensor center** of G such that*

$$Z^{\otimes}(G) = \{g \in G : x \otimes g = 1_{G \otimes G}, \forall x \in G\}.$$

We see that $Z^{\otimes}(G) = C_G^{\otimes}(G) = \cap_{x \in G} C_G^{\otimes}(x)$.

We will explain below some algebraic structures for some concepts, as in [4] page 2.

Lemma 3.7. *Let G be a group. Then $C_G^{\otimes}(x)$ is a subgroup of G for all $x \in G$.*

Proof. The subgroup conditions are verified as follows: $C_G^{\otimes}(x)$ is not an empty set since by Lemma 3.1 we have $x \otimes 1_G = 1_{G\otimes G}$, hence $1_G \in C_G^{\otimes}(x)$ for all $x \in G$, and consider any two elements g_1 and $g_2 \in C_G^{\otimes}(x)$. Then

$$x \otimes g_1 g_2 = (x \otimes g_1)(x \otimes g_2)^{g_1} = (1_{G\otimes G})(1_{G\otimes G})^{g_1} = 1_{G\otimes G}$$

hence $g_1 g_2 \in C_G^{\otimes}(x)$. Thus $C_G^{\otimes}(x)$ is a subgroup of G. □

Lemma 3.8. *Let G be a group. Then $Z^{\otimes}(G)$ is a subgroup of G.*

Proof. Since $Z^{\otimes}(G)$ is equal to $\cap_{x\in G} C_G^{\otimes}(x)$, and $C_G^{\otimes}(x)$ is a subgroup of G, for all $x \in G$. Then by the fact that intersection of subgroups is a subgroup hence, $\cap_{x\in G} C_G^{\otimes}(x)$ is a subgroup of G, thus $Z^{\otimes}(G)$ is a subgroup of G. □

Lemma 3.9. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Then $C_K^{\otimes}(H)$ is a subgroup of K.*

Proof. The subgroup conditions are verified as follows: $C_K^{\otimes}(h)$ is not an empty set since by Lemma 3.1 we have $h \otimes 1_K = 1_{H\otimes K}$, hence $1_K \in C_K^{\otimes}(h)$ for all $h \in H$, and consider any two elements k_1 and $k_2 \in C_K^{\otimes}(h)$. Then

$$h \otimes k_1 k_2 = (h \otimes k_1)(h \otimes k_2)^{k_1} = (1_{H\otimes K})(1_{H\otimes K})^{k_1} = 1_{H\otimes K}$$

hence $k_1 k_2 \in C_K^{\otimes}(h)$ for all $h \in H$. Thus, $C_K^{\otimes}(h)$ is a subgroup of K for all $h \in H$, then $\cap_{h\in H} C_K^{\otimes}(h)$ is a subgroup of K. But $\cap_{h\in H} C_K^{\otimes}(h)$ is equal to $C_K^{\otimes}(H)$, thus $C_K^{\otimes}(H)$ is a subgroup of K.

□

Lemma 3.10. *Let G be a group. Let H and K be normal subgroups of G. Then $C_K^{\otimes}(h) \lhd C_K(h)$ for all $h \in H$.*

Proof. Let $\psi : C_K(h) \longrightarrow J(G, H, K) = \ ker\, \kappa_{H,K}$ given by $\psi(k) = h \otimes k$, where $h \otimes k \in J(G, H, K)$. ψ is a group homomorphism since for all $k, k' \in C_K(h)$ we have

$$\begin{aligned} \psi(kk') &= h \otimes kk' \\ &= (h \otimes k)(h \otimes k')^k \\ &= (h \otimes k)(h \otimes k') \\ &= \psi(k)\psi(k') \end{aligned}$$

then

$$\begin{aligned} ker(\psi) &= \{k \in C_K(h) : \psi(k) = h \otimes k = 1_{J(G,H,K)} = 1_{H\otimes K}\} \\ &= \{k \in C_K(h) \subseteq K : h \otimes k = 1_{H\otimes K}\} \\ &= \{k \in K : h \otimes k = 1_{H\otimes K}\} \\ &= C_K^{\otimes}(h). \end{aligned}$$

Therefore, by the fact that if $\psi : G_1 \longrightarrow G_2$ is a group homomorphism then ker $(\psi) \lhd G_1$. Thus, $C_K^{\otimes}(h) \lhd C_K(h)$ for all $h \in H$.

□

We will state the relative tensor degree as in [23] page 3.

Definition 3.5. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. We define* $d^{\otimes}(H,K)$ *as the* ***relative tensor degree*** *of H and K such that*

$$d^{\otimes}(H,K) = \frac{|\{(h,k) \in H \times K : h \otimes k = 1_{H\otimes K}\}|}{|H||K|} = \frac{\sum_{h\in H} |C_K^{\otimes}(h)|}{|H||K|}.$$

Remark. *If* $H = K = G$*, then* $d^{\otimes}(G,G) = d^{\otimes}(G)$ *of* G.

The following lemma deals with different aspects and correlates the tensor centralizers with the tensor degree, as in [4, Lemma 2.1.] and [23, Lemma 2.2.].

Lemma 3.11. *Let G be a group. Let H and K be normal subgroups of G. Then*

$$d^{\otimes}(H,K) = \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \frac{|C_K^{\otimes}(h_i)|}{|C_K(h_i)|}.$$

If $G = HK$*, then* $\dfrac{C_K(h_i)}{C_K^{\otimes}(h_i)}$ *is isomorphic to a subgroup of J(G,H,K) and* $\dfrac{|C_K(h_i)|}{|C_K^{\otimes}(h_i)|} \leq |J(G,H,K)|$ *for all* $i = 1, ..., k_K(H)$.

Proof. Since H is a normal subgroup of G, we consider the distinct K-conjugacy classes $C(h_1), ..., C(h_{k_K(H)})$ where $h_i \in H$ for all $i = 1, ..., k_K(H)$, that constitute H. Therefore,

$$\begin{aligned} d^{\otimes}(H,K) &= \frac{\sum_{h\in H} |C_K^{\otimes}(h)|}{|H||K|} \\ &= \frac{\sum_{i=1}^{k_K(H)} \sum_{h\in C(h_i)} |C_K^{\otimes}(h)|}{|H||K|} \\ &= \frac{\sum_{i=1}^{k_K(H)} |K : C_K(h_i)||C_K^{\otimes}(h_i)|}{|H||K|} \\ &= \frac{1}{|H|} \sum_{i=1}^{k_K(H)} \frac{|C_K^{\otimes}(h_i)|}{|C_K(h_i)|}. \end{aligned}$$

Let $G = HK$. Consider the map

$$\psi : \frac{C_K(h_i)}{C_K^{\otimes}(h_i)} \longrightarrow J(G,H,K)$$

given by

$$\psi(kC_K^{\otimes}(h_i)) = h_i \otimes k.$$

Where $h_i \otimes k \in J(G,H,K)$, for all $i = 1, .., k_K(H)$. ψ is a group homomorphism since

$$\begin{aligned} \psi(k_1k_2C_K^{\otimes}(h_i)) &= h_i \otimes k_1k_2 = (h_i \otimes k_1)(h_i \otimes k_2)^{k_1} \\ &= (h_i \otimes k_1)(1_{H\otimes K})^{k_1} = (h_i \otimes k_1)(1_{H\otimes K}) \\ &= (h_i \otimes k_1)(h_i \otimes k_2) = \psi(k_1C_K^{\otimes}(h_i))\psi(k_2C_K^{\otimes}(h_i)), \end{aligned}$$

for all $k_1, k_2 \in C_K(h_i)$. ψ is a group monomorphism since

$$\begin{aligned} ker\psi &= \{kC_K^{\otimes}(h_i) : \psi(kC_K^{\otimes}(h_i)) = 1_{H\otimes K}\} \\ &= \{kC_K^{\otimes}(h_i) : h_i \otimes k = 1_{H\otimes K}\} \\ &= \{1_K C_K^{\otimes}(h_i)\} = C_K^{\otimes}(h_i), \end{aligned}$$

hence $\dfrac{C_K(h_i)}{C_K^{\otimes}(h_i)}$ is isomorphic to a subgroup of J(G,H,K) and $\dfrac{\mid C_K(h_i) \mid}{\mid C_K^{\otimes}(h_i) \mid} \leq \mid J(G,H,K) \mid$ for all $i = 1, ..., k_K(H)$.

□

The following lemma gives us the relative tensor degree of group G. It will be 1 when the tensor center of G is G itself, and vice versa as well.

Lemma 3.12. *$d^{\otimes}(G) = 1$ iff $Z^{\otimes}(G) = G$.*

The following theorem gives us the relative tensor degree of the direct product of two groups, and we can apply them to more than two groups, provided that the direct product is limited.

Theorem 3.2. *Let G and H be groups. Let $\mid G \mid= n$, $\mid H \mid= m$ and $g.c.d(n,m) = 1$ for all $n, m \in \mathbb{Z}^+$. Then $d^{\otimes}(G \times H) = d^{\otimes}(G) \cdot d^{\otimes}(H)$,*

Proof. Since $\mid G \times H \mid=\mid G \mid \cdot \mid H \mid$, hence $(\mid G \times H \mid)^2 = (\mid G \mid)^2 \cdot (\mid H \mid)^2$. Then we have for all $(x, y) \in G \times H$,

$$\begin{aligned} C_{G\times H}^{\otimes}((x,y)) &= \{(g,h) \in G \times H : (x,y) \otimes (g,h) = 1_{(G\times H)\otimes(G\times H)}\} \\ &= \{(g,h) \in G \times H : (x \otimes g, y \otimes h) = 1_{(G\times H)\otimes(G\times H)} = (1_{G\otimes G}, 1_{H\otimes H})\} \\ &= \{g \in G : x \otimes g = 1_{G\otimes G}\} \times \{h \in H : y \otimes h = 1_{H\otimes H}\} \\ &= C_G^{\otimes}(x) \times C_H^{\otimes}(y). \end{aligned}$$

Hence $\mid C_{G\times H}^{\otimes}((x,y)) \mid=\mid C_G^{\otimes}(x) \times C_H^{\otimes}(y) \mid=\mid C_G^{\otimes}(x) \mid \cdot \mid C_H^{\otimes}(y) \mid$, and by definition of $d^{\otimes}(G\times H)$ we have

$$\begin{aligned} d^{\otimes}(G \times H) &= \frac{1}{\mid G \times H \mid^2} \sum_{(x,y)\in G\times H} \mid C_{G\times H}^{\otimes}((x,y)) \mid \\ &= \frac{1}{\mid G \mid^2 \cdot \mid H \mid^2} \sum_{x\in G, y\in H} \mid C_G^{\otimes}(x) \mid \cdot \mid C_H^{\otimes}(y) \mid \\ &= \frac{1}{\mid G \mid^2 \cdot \mid H \mid^2} \sum_{x\in G} \sum_{y\in H} \mid C_G^{\otimes}(x) \mid \cdot \mid C_H^{\otimes}(y) \mid \\ &= \frac{1}{\mid G \mid^2 \cdot \mid H \mid^2} \sum_{x\in G} \mid C_G^{\otimes}(x) \mid \cdot \sum_{y\in H} \mid C_H^{\otimes}(y) \mid \\ &= \left(\frac{1}{\mid G \mid^2} \sum_{x\in G} \mid C_G^{\otimes}(x) \mid \right) \cdot \left(\frac{1}{\mid H \mid^2} \sum_{y\in H} \mid C_H^{\otimes}(y) \mid \right) \\ &= d^{\otimes}(G) \cdot d^{\otimes}(H). \end{aligned}$$

□

3.3 Results and Boundary of Relative Tensor Degree

In this section, we will study the relation between the relative commutativity degree and the relative tensor degree.

The following theories and definition give us the relation between the notion of relative tensor degree with that of relative commutativity degree.

Theorem 3.3. *Let G be a group. Let H and K be normal subgroups of G. Let p be the smallest prime divisor of $|G|$. Then*

$$\frac{d(H,K)}{|J(G,H,K)|} + \frac{|C_K^{\otimes}(H)|}{|H|}\left(1 - \frac{1}{|J(G,H,K)|}\right) \leq d^{\otimes}(H,K)$$

and

$$d^{\otimes}(H,K) \leq d(H,K) - (1 - \frac{1}{p})\left(\frac{|C_K(H)| - |C_K^{\otimes}(H)|}{|H|}\right).$$

Proof. The proof can be found in [4, Theorem 1.1]. □

The following theorem describes a special case of Theorem 3.3 when $H = K = G$.

Theorem 3.4. *Let G be a group. Let p be the smallest prime divisor of $|G|$. Then*

$$\frac{d(G)}{|J_2(G)|} + \frac{|Z^{\otimes}(G)|}{|G|}\left(1 - \frac{1}{|J_2(G)|}\right) \leq d^{\otimes}(G)$$

and

$$d^{\otimes}(G) \leq d(G) - (1 - \frac{1}{p})\left(\frac{|Z(G)| - |Z^{\otimes}(G)|}{|G|}\right).$$

Proof. The proof can be found in [23, Theorem 2.3].

□

The following theorem is a consequence of Theorem 3.3, as in [4, Theorem 1.1.].

Theorem 3.5. *Let G be a group. Let H and K be normal subgroups of G such that $G = HK$. Then*

$$\frac{d(H,K)}{\mid J(G,H,K) \mid} \leq d^{\otimes}(H,K) \leq d(H,K).$$

If J(G,H,K) is trivial, then $d^{\otimes}(H,K) = d(H,K)$.

Proof. From Lemma 3.11 we have

$$d^{\otimes}(H,K) = \frac{1}{\mid H \mid} \textstyle\sum_{i=1}^{k_K(H)} \frac{\mid C_K^{\otimes}(h_i) \mid}{\mid C_K(h_i) \mid}$$

and for all $i = 1, ..., k_K(H)$, we have

$$\frac{\mid C_K(h_i) \mid}{\mid C_K^{\otimes}(h_i) \mid} \leq \mid J(G,H,K) \mid$$

hence

$$\frac{| C_K^{\otimes}(h_i) |}{| C_K(h_i) |} \geq \frac{1}{| J(G,H,K) |}$$

then

$$\begin{aligned}
d^{\otimes}(H,K) &= \frac{1}{| H |} \sum_{i=1}^{k_K(H)} \frac{| C_K^{\otimes}(h_i) |}{| C_K(h_i) |} \\
&\geq \frac{1}{| H |} \sum_{i=1}^{k_K(H)} \frac{1}{| J(G,H,K) |} \\
&= \frac{1}{| H |} \cdot \frac{1}{| J(G,H,K) |} \sum_{i=1}^{k_K(H)} 1 \\
&= \frac{1}{| H |} \cdot \frac{1}{| J(G,H,K) |} k_K(H) \\
&= \frac{k_K(H)}{| H |} \cdot \frac{1}{| J(G,H,K) |}
\end{aligned}$$

from definition of d(H,K) we have

$$= \frac{d(H,K)}{| J(G,H,K) |}$$

thus

$$d^{\otimes}(H,K) \geq \frac{d(H,K)}{| J(G,H,K) |}. \to (1)$$

Since the map $\psi : C_K(h_i) \to J(G,H,K)$ is a group homomorphism, which ker $(\psi) = C_K^{\otimes}(h_i)$ for all $i = 1, ..., k_K(H)$. But ker $(\psi) \leq C_K(h_i)$, hence $| \, ker(\psi)| \leq |C_K(h_i)|$, then $\frac{| \, ker \, (\psi)|}{|C_K(h_i)|} \leq 1$. Again from Lemma 3.11 we have

$$\begin{aligned}
d^{\otimes}(H,K) &= \frac{1}{| H |} \sum_{i=1}^{k_K(H)} \frac{| C_K^{\otimes}(h_i) |}{| C_K(h_i) |} \\
&\leq \frac{1}{| H |} \sum_{i=1}^{k_K(H)} 1 \\
&= \frac{k_K(H)}{| H |} \\
&= d(H,K).
\end{aligned}$$

Thus

$$d^{\otimes}(H,K) \leq d(H,K). \to (2)$$

From (1) and (2) we have

$$\frac{d(H,K)}{| J(G,H,K) |} \leq d^{\otimes}(H,K) \leq d(H,K).$$

It is easy to see if J(G,H,K) is trivial (i.e. $J(G,H,K) = 1_{H \otimes K}$) then $| \, J(G,H,K) \, |= 1$. Since

$$\frac{d(H,K)}{| J(G,H,K) |} \leq d^{\otimes}(H,K) \leq d(H,K)$$

hance

$$d(H,K) \leq d^{\otimes}(H,K) \leq d(H,K)$$

thus

$$d^{\otimes}(H,K) = d(H,K).$$

□

The following corollary gives us the bounded when the group is abelian.

Corollary 3.1. *Let G be an abelian group. Then*

$$\frac{1}{|G|} + \frac{|G|-1}{|G||G \otimes G|} \leq d^{\otimes}(G) \leq \frac{1}{p} + \frac{p-1}{p|G|}$$

where p is the smallest prime divisor of $|G|$.

Proof. Since G is an abelian group, then $Z(G)$ = G and each conjugacy class is a set containing one element ($k(G) = |G|$), then d(G)= $\frac{k(G)}{|G|} = \frac{|G|}{|G|} = 1$, $Z^{\otimes}(G)$ is trivial, and $J_2(G) = ker(\kappa_{G \otimes G}) = \{x \otimes y \in G \otimes G : \kappa_{G \otimes G}(x \otimes y) = 1_G\} = \{x \otimes y \in G \otimes G : [x, y] = 1_G\}$, but G is an abelian, hence $J_2(G) = \{x \otimes y \in G \otimes G\} = G \otimes G$. From Theorem 3.4 we have

$$\frac{d(G)}{|J_2(G)|} + \frac{|Z^{\otimes}(G)|}{|G|}\left(1 - \frac{1}{|J_2(G)|}\right) \leq d^{\otimes}(G) \leq d(G) - (1 - \frac{1}{p})\left(\frac{|Z(G)| - |Z^{\otimes}(G)|}{|G|}\right)$$
$$\frac{1}{|G \otimes G|} + \frac{1}{|G|}\left(1 - \frac{1}{|G \otimes G|}\right) \leq d^{\otimes}(G) \leq 1 - (1 - \frac{1}{p})\left(1 - \frac{1}{|G|}\right)$$
$$\frac{1}{|G \otimes G|} + \frac{1}{|G|} - \frac{1}{|G||G \otimes G|} \leq d^{\otimes}(G) \leq 1 - \left[1 - \frac{1}{|G|} - \frac{1}{p} + \frac{1}{p|G|}\right]$$
$$\frac{1}{|G|} + \frac{|G|-1}{|G||G \otimes G|} \leq d^{\otimes}(G) \leq \frac{1}{|G|} + \frac{1}{p} - \frac{1}{p|G|}$$
$$\frac{1}{|G|} + \frac{|G|-1}{|G||G \otimes G|} \leq d^{\otimes}(G) \leq \frac{1}{p} + \frac{p-1}{p|G|}.$$

□

We will state the left unidegree as in [23] page 6.

Definition 3.6. *Let G be a group. If $d^{\otimes}(G) = d(G)$, then we call G a* ***left unidegree****.*

Chapter 4

Relative Exterior Degree

In this chapter, we will present three sections; in the first section, we will define a non-abelian exterior product of normal subgroups H and K of G, and explain the relation between the non-abelian exterior product group $H \wedge K$ and the derived group [H,K]. In the second section, we will study the definitions of exterior centralizer and exterior center which paved by the constructing of the previous section, algebraic structures of these concepts along with the previous concepts, and the definition of the relative exterior degree. In the third and last section, we will study the general relation among the relative commutative degree, relative tensor degree and relative exterior degree. The basic references of this chapter are [4],[19],[22] and [23].

4.1 Non-abelian Exterior Product

We will study the non-abelian exterior product.

The following definition we will state the non-abelian exterior product as in [4].

Definition 4.1. *Let G be a group. Let H and K be normal subgroups of G. We define the group $H \wedge K$ to be the* ***non-abelian exterior product*** *of H and K such that*

$$H \wedge K = \frac{(H \otimes K)}{\triangledown(H \cap K)},$$

where

$$\triangledown(H \cap K) = \langle g \otimes g : g \in H \cap K \rangle,$$

and

$$H \wedge K = \langle h \wedge k : h, k \in H \cap K \rangle = \langle (h \otimes k) \triangledown (H \cap K) : h, k \in H \cap K \rangle.$$

If $G = H = K$, and if all actions by conjugation, then we denote by $G \wedge G$ the ***non-abelian exterior square*** *of G.*

Remark. $\triangledown(H \cap K)$ *is a central subgroup of $H \otimes K$.*

The following theorem gives us the relation between the non-abelian exterior group $H \wedge K$ and the derived group $[H, K]$.

Theorem 4.1. *Let G be a group. Let H and K be normal subgroups of G. Let $h \in H$ and $k \in K$. The map*

$$\kappa'_{H,K} : H \wedge K \longmapsto [H,K]$$

given by

$$\kappa'_{H,K}(h \wedge k) = [h,k] = h^{-1}h^{k}$$

defines a group epimorphism, whose kernal ker $\kappa'_{H,K}$ = M(G,H,K), which calls M(G,H,K) the Schur multiplier of the triple (G,H,K). If G = H = K, then we call M(G,G,G) = M(G) the Schur multiplier of G, and M(G)= $H_2(G,\mathbb{Z})$ is the second integer homology group of G.

Proof. The well-definedness of $\kappa'_{H,K}$ is clear. To prove that $\kappa'_{H,K}$ is a group homomorphism, it suffices to show that

$$\begin{aligned}
(i)\ \kappa'_{H,K}(h_2^{h_1} \wedge k^{h_1})\ \kappa'_{H,K}(h_1 \wedge k) &= (h_2^{h_1})^{-1}((h_2^{h_1})^{k})^{h_1}h_1^{-1}h_1^{k} \\
&= (h_1^{-1}h_2h_1)^{-1}((h_2^{h_1})^{h_1^{-1}kh_1}h_1^{-1}h_1^{k} \\
&= h_1^{-1}h_2^{-1}h_1(h_2)^{h_1h_1^{-1}kh_1}h_1^{-1}h_1^{k} \\
&= (h_2h_1)^{-1}h_1(h_2)^{kh_1}h_1^{-1}h_1^{k} \\
&= (h_2h_1)^{-1}((h_2)^{kh_1})^{h_1^{-1}}h_1^{k} \\
&= (h_2h_1)^{-1}(h_2)^{kh_1h_1^{-1}}h_1^{k} \\
&= (h_2h_1)^{-1}(h_2)^{k}h_1^{k} \\
&= (h_2h_1)^{-1}(h_2h_1)^{k} \\
&= \kappa'_{H,K}(h_2h_1 \wedge k) \\
&= \kappa'_{H,K}(h_1h_2 \wedge k).
\end{aligned}$$

$$\begin{aligned}
(ii)\ \kappa'_{H,K}(h \wedge k_1)\ \kappa'_{H,K}(h^{k_1} \wedge k_2^{k_1}) &= h^{-1}h^{k_1}(h^{k_1})^{-1}((h^{k_1})^{k_2})^{k_1} \\
&= h^{-1}((h^{k_1})^{k_2})^{k_1} \\
&= h^{-1}((h)^{k_1})^{k_1^{-1}k_2k_1} \\
&= h^{-1}(h)^{k_1k_1^{-1}k_2k_1} \\
&= h^{-1}(h)^{k_2k_1} \\
&= \kappa'_{H,K}(h \wedge k_2k_1) \\
&= \kappa'_{H,K}(h \wedge k_1k_2).
\end{aligned}$$

Thus, $\kappa'_{H,K}$ is a group homomorphism. Since for all $h^{-1}h^{k} \in [H,K]$ there is $h \wedge k \in H \wedge K$. Hence, $\kappa'_{H,K}$ is a group epimorphism.

□

We have $\kappa_{H,K}$ ($\kappa'_{H,K}$) as a group epimorphism function, α_1 (α_2) is inclusion function, which means monmorphism. Furthermore, Im (α_1)= ker $\kappa_{H,K} = J_2(G,H,K)$ (Im (α_2)= ker $\kappa'_{H,K} = H_2(G,H,K)$), then by definition of short exact sequence we have

$$0 \longrightarrow J_2(G,H,K) \xrightarrow{\alpha_1} H \otimes K \xrightarrow{\kappa_{H,K}} [H,K] \longrightarrow 0$$
$$0 \longrightarrow H_2(G,H,K) \xrightarrow{\alpha_2} H \wedge K \xrightarrow{\kappa'_{H,K}} [H,K] \longrightarrow 0$$

and since, $\theta : \{0\} \rightarrow \{1\}$ and θ^{-1} as isomorphism, thus

$$1 \longrightarrow J_2(G,H,K) \xrightarrow{\alpha_1} H \otimes K \xrightarrow{\kappa_{H,K}} [H,K] \longrightarrow 1$$
$$1 \longrightarrow H_2(G,H,K) \xrightarrow{\alpha_2} H \wedge K \xrightarrow{\kappa'_{H,K}} [H,K] \longrightarrow 1$$

when $(G = H = K)$, then

$$1 \longrightarrow J_2(G) \xrightarrow{\alpha_1} G \otimes G \xrightarrow{\kappa_{H,K}} G' \longrightarrow 1$$
$$1 \longrightarrow H_2(G) \xrightarrow{\alpha_2} G \wedge G \xrightarrow{\kappa'_{H,K}} G' \longrightarrow 1$$

G $\otimes$ G is finite if G is finite. We can see that the structure of G is influenced by that of G $\otimes$ G and viceversa, as in [4] pages 1 and 2, and [23] page 2.

The following lemma gives us the relation $(h \wedge k) = (k \wedge h)^{-1}$, when $h, k \in H \cap K$ in a non-abelian exterior product group $H \wedge K$.

Lemma 4.1. *Let G be a group. Let H and K be normal subgroups of G. Let $H \wedge K$ be the non-abelian exterior product group. Let $h \in H$ and $k \in K$. If $h, k \in H \cap K$. Then $(h \wedge k) = (k \wedge h)^{-1}$.*

Proof. Since $kh \wedge kh = 1_{H \wedge K}$, then we have

$$\begin{aligned} 1_{H \wedge K} &= kh \wedge kh = (h \wedge kh)^k (k \wedge kh) = \big((h \wedge k)(h \wedge h)^k\big)^k \big((k \wedge k)(k \wedge h)^k\big) \\ &= \big((h \wedge k)(1_{H \wedge K})^k\big)^k \big((1_{H \wedge K})(k \wedge h)^k\big) = (h \wedge k)^k (k \wedge h)^k. \end{aligned}$$

The remaining part of the statement follows easily.

□

4.2 Exterior Centralizer and Relative Exterior Degree

In this section, we will study the exterior centralizer and the relative exterior degree.

We will state the exterior centralizer as in [4] page 3.

Definition 4.2. *Let G be a group. Let H and K be normal subgroups of G. We define the set $C_K^{\wedge}(H)$ as the* ***exterior centralizer*** *of H with respect to K such that*

$$C_K^{\wedge}(H) = \{k \in K : h \wedge k = 1_{H \wedge K}, \forall h \in H\}.$$

We see that $C_K^{\wedge}(H) = \cap_{h \in H} C_K^{\wedge}(h)$.

We will state the exterior center as in [4] page 3.

Definition 4.3. *Let G be a group. We define the set $Z^{\wedge}(G)$ as the* ***exterior center*** *of G such that*

$$Z^{\wedge}(G) = \{g \in G : x \wedge g = 1_{G \wedge G}, \forall x \in G\}.$$

We see that $Z^{\wedge}(G) = C_G^{\wedge}(G) = \cap_{x \in G} C_G^{\wedge}(x)$.

We will explain below some algebraic structures for some concepts, as in [4] page 3.

Lemma 4.2. *Let G be a group. Then $C_G^{\wedge}(x)$ is a subgroup of G for all $x \in G$.*

Proof. The subgroup conditions are verified as follows: $C_G^{\wedge}(x)$ is not an empty set since we have $x \wedge 1_G = 1_{G\wedge G}$, hence $1_G \in C_G^{\wedge}(x)$ for all $x \in G$, and consider any two elements g_1 and $g_2 \in C_G^{\wedge}(x)$. Then

$$x \wedge g_1 g_2 = (x \wedge g_1)(x^{g_1} \wedge g_2^{g_1}) = (x \wedge g_1)(x \wedge g_2)^{g_1} = (1_{G\wedge G})(1_{G\wedge G})^{g_1} = 1_{G\wedge G}$$

hence $g_1 g_2 \in C_G^{\wedge}(x)$. Thus $C_G^{\wedge}(x)$ is a subgroup of G.

□

Lemma 4.3. *Let G be a group. Then $Z^{\wedge}(G)$ is a subgroup of G.*

Proof. Since $Z^{\wedge}(G)$ is equal to $\cap_{x\in G} C_G^{\wedge}(x)$, and $C_G^{\wedge}(x)$ is a subgroup of G, for all $x \in G$. Then by the fact that intersection of subgroups is a subgroup hence, $\cap_{x\in G} C_G^{\wedge}(x)$ is a subgroup of G, thus $Z^{\wedge}(G)$ is a subgroup of G.

□

Lemma 4.4. *Let G be a group. Let H and K be normal subgroups of G. Then $C_K^{\wedge}(H)$ is a subgroup of K.*

Proof. The subgroup conditions are verified as follows: $C_K^{\wedge}(h)$ is not an empty set since we have $h \wedge 1_K = 1_{H\wedge K}$, hence $1_K \in C_K^{\wedge}(h)$ for all $h \in H$, and consider any two elements k_1 and $k_2 \in C_K^{\wedge}(h)$. Then

$$h \wedge k_1 k_2 = (h \wedge k_1)(h \wedge k_2)^{k_1} = (1_{H\wedge K})(1_{H\wedge K})^{k_1} = 1_{H\wedge K}$$

hence $k_1 k_2 \in C_K^{\wedge}(h)$ for all $h \in H$. Thus, $C_K^{\wedge}(h)$ is a subgroup of K for all $h \in H$, then $\cap_{h\in H} C_K^{\wedge}(h)$ is a subgroup of K. But $\cap_{h\in H} C_K^{\wedge}(h)$ is equal to $C_K^{\wedge}(H)$, thus $C_K^{\wedge}(H)$ is a subgroup of K.

□

Proposition 4.1. *Let G be a group. Let H and K be normal subgroups of G which act on each other compatibly. Then*

$$C_K^{\otimes}(h) \subseteq C_K^{\wedge}(h) \subseteq C_K(h),$$

for all $h \in H$.

Proof. For all $k \in C_K^{\otimes}(h)$ we have

$$h \wedge k = (h \otimes k) \bigtriangledown (H \cap K) = 1_{H\otimes K} \bigtriangledown (H \cap K) = \bigtriangledown(H \cap K) = 1_{H\wedge K}$$

Then $k \in C_K^{\wedge}(h)$. Therefore, $C_K^{\otimes}(h) \subseteq C_K^{\wedge}(h)$. Suppose the map

$$\psi : C_K(h) \longrightarrow M(G, H, K) = ker\ \kappa'_{H,K}$$

given by

$$\psi(k) = h \wedge k$$

where $h \wedge k \in M(G, H, K)$, ψ is a group homomorphism since for all $k, k' \in C_K(h)$ we have

$$\begin{aligned}
\psi(kk') &= h \wedge kk' \\
&= (h \otimes kk') \bigtriangledown (H \cap K) \\
&= (h \otimes k)(h \otimes k')^{k} \bigtriangledown (H \cap K) \\
&= (h \otimes k)(h \otimes k') \bigtriangledown (H \cap K) \\
&= (h \wedge k)(h \wedge k') \\
&= \psi(k)\psi(k').
\end{aligned}$$

$C_K^{\wedge}(h)$ is a subset of $C_K(h)$ since

$$\begin{aligned} ker(\psi) &= \{k \in C_K(h) : \psi(k) = h \wedge k = 1_{M(G,H,K)} = 1_{H \wedge K}\} \\ &= \{k \in C_K(h) \subseteq K : h \wedge k = 1_{H \wedge K}\} \\ &= \{k \in K : h \wedge k = 1_{H \wedge K}\} \\ &= C_K^{\wedge}(h). \end{aligned}$$

But $ker(\psi)$ is a subgroup of $C_K(h)$, then $C_K^{\wedge}(h)$ is a subgroup of $C_K(h)$ hence $C_K^{\wedge}(h) \subseteq C_K(h)$. Thus for all $h \in H$ we have

$$C_K^{\otimes}(h) \subseteq C_K^{\wedge}(h) \subseteq C_K(h).$$

□

Remark. *If $G = H = K$, then $C_G^{\otimes}(x) \subseteq C_G^{\wedge}(x) \subseteq C_G(x)$ for all $x \in G$.*

Proposition 4.2. *Let G be a group. Then*

$$Z^{\otimes}(G) \subseteq Z^{\wedge}(G) \subseteq Z(G).$$

Proof. Since for all $x \in G$ and by Proposition 4.1 when $G = H = K$, hence

$$C_G^{\otimes}(x) \subseteq C_G^{\wedge}(x) \subseteq C_G(x)$$

then

$$\cap_{x \in G} C_G^{\otimes}(x) \subseteq \cap_{x \in G} C_G^{\wedge}(x) \subseteq \cap_{x \in G} C_G(x)$$

thus

$$Z^{\otimes}(G) \subseteq Z^{\wedge}(G) \subseteq Z(G).$$

□

Lemma 4.5. *Let G be a group. Let H and K be normal subgroups of G. Then $C_K^{\wedge}(h) \triangleleft C_K(h)$ for all $h \in H$.*

Proof. Let $\psi : C_K(h) \longrightarrow M(G,H,K) = ker\, \kappa'_{H,K}$ given by $\psi(k) = h \wedge k$, where $h \wedge k \in M(G,H,K)$. ψ is a group homomorphism since for all $k, k' \in C_K(h)$ we have

$$\begin{aligned} \psi(kk') &= h \wedge kk' \\ &= (h \otimes kk') \bigtriangledown (H \cap K) \\ &= (h \otimes k)(h \otimes k')^k \bigtriangledown (H \cap K) \\ &= (h \otimes k)(h \otimes k') \bigtriangledown (H \cap K) \\ &= (h \wedge k)(h \wedge k') \\ &= \psi(k)\psi(k'). \end{aligned}$$

Then

$$\begin{aligned} ker(\psi) &= \{k \in C_K(h) : \psi(k) = h \wedge k = 1_{M(G,H,K)} = 1_{H \wedge K}\} \\ &= \{k \in C_K(h) \subseteq K : h \wedge k = 1_{H \wedge K}\} \\ &= \{k \in K : h \wedge k = 1_{H \wedge K}\} \\ &= C_K^{\wedge}(h). \end{aligned}$$

Therefore, by the fact that if $\psi : G_1 \longrightarrow G_2$ is a group homomorphism then ker $(\psi) \triangleleft G_1$. Thus, $C_K^{\wedge}(h) \triangleleft C_K(h)$ for all $h \in H$.

□

We will state the relative exterior degree as in [4] page 3.

Definition 4.4. *Let G be a group. Let H and K be normal subgroups of G. We define* $d^{\wedge}(H,K)$ *as the* ***relative exterior degree*** *of H and K such that*

$$d^{\wedge}(H,K) = \frac{\mid \{(h,k) \in H \times K : h \wedge k = 1_{H \wedge K}\} \mid}{\mid H \mid\mid K \mid} = \frac{\sum_{h \in H} \mid C_K^{\wedge}(h) \mid}{\mid H \mid\mid K \mid}.$$

Remark. *If* $G = H = K$, *then* $d^{\wedge}(G,G) = d^{\wedge}(G)$ *of G.*

The following lemma deals with different aspects and correlates the exterior centralizers with the exterior degree.

Lemma 4.6. *Let G be a group. Let H and K be normal subgroups of G. Then*

$$d^{\wedge}(H,K) = \frac{1}{\mid H \mid} \sum_{i=1}^{k_K(H)} \frac{\mid C_K^{\wedge}(h_i) \mid}{\mid C_K(h_i) \mid}.$$

If $G = HK$, *then* $\frac{C_K(h_i)}{C_K^{\wedge}(h_i)}$ *is isomorphic to a subgroup of M(G,H,K) and* $\frac{\mid C_K(h_i) \mid}{\mid C_K^{\wedge}(h_i) \mid} \leq \mid M(G,H,K) \mid$ *for all* $i = 1, ..., k_K(H)$.

Proof. Such as the method to proof of Lemma 3.11, page 43 in this work. □

The following lemma gives us the relative exterior degree of the group G. It will be 1 when the exterior center of G is G itself, and vice versa as well, as in [4] and [23] page 2.

Lemma 4.7. $d^{\wedge}(G) = 1$ *iff* $G = Z^{\wedge}(G)$.

The following theorem gives us the relative exterior degree of the direct product of two groups, and we can apply them to more than two groups, provided that the direct product is limited, as in [22, Lemma 2.10].

Theorem 4.2. *Let G and H be groups. Let* $\mid G \mid= n$, $\mid H \mid= m$ *and* $g.c.d(n,m) = 1$ *for all* $n, m \in \mathbb{Z}^+$. *Then* $d^{\wedge}(G \times H) = d^{\wedge}(G) \cdot d^{\wedge}(H)$,

Proof. Since $\mid G \times H \mid=\mid G \mid \cdot \mid H \mid$, hence $(\mid G \times H \mid)^2 = (\mid G \mid)^2 \cdot (\mid H \mid)^2$. Then we have for all $(x,y) \in G \times H$.

$$\begin{aligned} C_{G \times H}^{\wedge}((x,y)) &= \{(g,h) \in G \times H : (x,y) \wedge (g,h) = 1_{(G \times H) \wedge (G \times H)}\} \\ &= \{(g,h) \in G \times H : (x \wedge g, y \wedge h) = 1_{(G \times H) \wedge (G \times H)} = (1_{G \wedge G}, 1_{H \wedge H})\} \\ &= \{g \in G : x \wedge g = 1_{G \wedge G}\} \times \{h \in H : y \wedge h = 1_{H \wedge H}\} \\ &= C_G^{\wedge}(x) \times C_H^{\wedge}(y). \end{aligned}$$

Hence $\mid C_{G \times H}^{\wedge}((x,y)) \mid=\mid C_G^{\wedge}(x) \times C_H^{\wedge}(y) \mid=\mid C_G^{\wedge}(x) \mid \cdot \mid C_H^{\wedge}(y) \mid$, and by definition of $d^{\wedge}(G \times H)$ we have

$$\begin{aligned}
d^\wedge(G\times H) &= \frac{1}{\mid G\times H\mid^2}\sum_{(x,y)\in G\times H}\mid C^\wedge_{G\times H}((x,y))\mid \\
&= \frac{1}{\mid G\mid^2\cdot\mid H\mid^2}\sum_{x\in G, y\in H}\mid C^\wedge_G(x)\mid\cdot\mid C^\wedge_H(y)\mid \\
&= \frac{1}{\mid G\mid^2\cdot\mid H\mid^2}\sum_{x\in G}\sum_{y\in H}\mid C^\wedge_G(x)\mid\cdot\mid C^\wedge_H(y)\mid \\
&= \frac{1}{\mid G\mid^2\cdot\mid H\mid^2}\sum_{x\in G}\mid C^\wedge_G(x)\mid\cdot\sum_{y\in H}\mid C^\wedge_H(y)\mid \\
&= \left(\frac{1}{\mid G\mid^2}\sum_{x\in G}\mid C^\wedge_G(x)\mid\right)\cdot\left(\frac{1}{\mid H\mid^2}\sum_{y\in H}\mid C^\wedge_H(y)\mid\right) \\
&= d^\wedge(G)\cdot d^\wedge(H).
\end{aligned}$$

□

4.3 Results of Relative Exterior Degree

In this section, we will study the general relation among the relative commutativity degree, the relative tensor degree and relative exterior degree.

The following theorem gives us the fundamental relation between commutativity, tensor and exterior degrees.

Theorem 4.3. *Let G be a group. Let H and K be normal subgroups of G. Then*

$$d^\otimes(H,K)\le d^\wedge(H,K)\le d(H,K).$$

If J(G, H, K) is trivial, then $d^\otimes(H,K)=d^\wedge(H,K)=d(H,K)$.

Proof. Since we have

$\mathrm{d}^\otimes(H,K)=\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C^\otimes_K(h_i)|}{|C_K(h_i)|}$, $d^\wedge(H,K)=\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C^\wedge_K(h_i)|}{|C_K(h_i)|}$,

$d(H,K)=\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C_K(h_i)|}{|C_K(h_i)|}$.

By Proposition 4.1, for all $h_i\in H, i=1,...,k_K(H)$ we have

$$C^\otimes_K(h_i)\subseteq C^\wedge_K(h_i)\subseteq C_K(h_i)$$

hence

$$|C^\otimes_K(h_i)|\le|C^\wedge_K(h_i)|\le|C_K(h_i)|.$$

Thus

$$\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C^\otimes_K(h_i)|}{|C_K(h_i)|}\le\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C^\wedge_K(h_i)|}{|C_K(h_i)|}\le\frac{1}{|H|}\sum_{i=1}^{k_K(H)}\frac{|C_K(h_i)|}{|C_K(h_i)|}$$

$$d^\otimes(H,K)\le d^\wedge(H,K)\le d(H,K).$$

If J(G, H, K) is trivial, then by Theorem 3.5 we have

$$d(H,K) \leq d^{\otimes}(H,K) \leq d(H,K)$$

then

$$d(H,K) = d^{\otimes}(H,K)$$

thus

$$d^{\otimes}(H,K) = d^{\wedge}(H,K) = d(H,K).$$

□

Example 4.1. *Let $G = D_8$. Then*

$$d^{\otimes}(D_8) \leq d^{\wedge}(D_8) \leq d(D_8).$$

$$\frac{5}{16} \leq \frac{5}{8} \leq \frac{5}{8}.$$

Theorem 4.4. *Let G be a group. Then*

$$\frac{d(G)}{|M(G)|} + \frac{|Z^{\wedge}(G)|}{|G|}\left(1 - \frac{1}{|M(G)|}\right) \leq d^{\wedge}(G) \leq d(G) - \left(\frac{p-1}{p}\right)\left(\frac{|Z(G)| - |Z^{\wedge}(G)|}{|G|}\right).$$

Proof. The proof can be found in [22, Theorem 2.3.].

□

We will state the unicentral and right unidegree as in [23] page 6.

Definition 4.5. *Let G be a group. If $Z^{\wedge}(G) = Z(G)$, then we call G an* ***unicentral****.*

Definition 4.6. *Let G be a group. If $d^{\wedge}(G) = d(G)$, then we call G a* ***right unidegree****.*

Corollary 4.1. *Let G be a group. If G is right unidegree, then G is unicentral ($Z^{\wedge}(G) = Z(G)$).*

Proof. Let G be a right unidegree. Since $Z^{\otimes}(G) \subseteq Z^{\wedge}(G) \subseteq Z(G)$, and by Theorem 4.4 we have

$$d^{\wedge}(G) \leq d(G) - (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\wedge}(G) \mid}{\mid G \mid})$$

since $d^{\wedge}(G) = d(G)$, then

$$d(G) \leq d(G) - (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\wedge}(G) \mid}{\mid G \mid})$$

$$0 \leq -(\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\wedge}(G) \mid}{\mid G \mid})$$

$$\xrightarrow{(-1)}$$

$$0 \geq (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\wedge}(G) \mid}{\mid G \mid})$$

$$\xrightarrow{(\frac{p}{p-1}) \mid G \mid}$$

$$0 \geq\mid Z(G) \mid - \mid Z^{\wedge}(G) \mid$$

$$\mid Z^{\wedge}(G) \mid\geq\mid Z(G) \mid .$$

But $Z^{\wedge}(G) \subseteq Z(G)$. Thus, $Z^{\wedge}(G) = Z(G)$.

□

Corollary 4.2. *Let G be a group. If G is left unidegree, then* $Z^{\otimes}(G) = Z(G)$.

Proof. Let G be a left unidegree, then $d^{\otimes}(G) = d(G)$. From Theorem 3.4 we have

$$d^{\otimes}(G) \leq d(G) - (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid}{\mid G \mid})$$

since $d^{\otimes}(G) = d(G)$, then

$$d(G) \leq d(G) - (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid}{\mid G \mid})$$

$$0 \leq -(\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid}{\mid G \mid})$$

$\xrightarrow{(-1)}$

$$0 \geq (\frac{p-1}{p})(\frac{\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid}{\mid G \mid})$$

$$0 \geq \frac{(p-1)(\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid)}{p \mid G \mid}$$

$\xrightarrow{(p \mid G \mid)}$

$$0 \geq (p-1)(\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid)$$

$\xrightarrow{\frac{1}{p-1}}$

$$0 \geq\mid Z(G) \mid - \mid Z^{\otimes}(G) \mid$$

$$\mid Z^{\otimes}(G) \mid\geq\mid Z(G) \mid .$$

But $Z^{\otimes}(G) \subseteq Z(G)$. Thus, $Z^{\otimes}(G) = Z(G)$.

□

Remark. *If G is left and right unidegree, then* $Z^{\wedge}(G) = Z^{\otimes}(G) = Z(G)$.

Chapter 5

Probabilistic Methods In Block Theory

In this chapter, we will present three sections; in the first and second section, we will study the notion of a probability of p-block and give examples, and in the third and last section, we will study few facts about the probability of irreducible ordinary character χ and of B_0, by Brauer-Feit theorem as in [25, Theorem 2.4]. we will show the relation between the probability of the principal p-block in group G and the order of Irr(G), and some current conjectures in this concept. The basic references of this chapter are [1] and [25].

5.1 Probability of p-blocks

In this section, we will study the definition of a probability of the p-block.

Let G be a group and p be a prime number, then from the definition of p-block one will have an equivalent relation. By dividing the irreducible ordinary characters to t of equivalent classes in which $B_i \cap B_j = \emptyset$ and $Irr(G) = B_1 \cup ... \cup B_t$ for $i, j = 1, ..., t$. Using the equation (2.1), one can calculate the probability of selecting the irreducible ordinary characters χ from $Irr(G)$. Moreover, based on the definition of p-block, χ belongs to a specific p-block based on the choice of p, which is given the symbol B, so that $\chi \in B$. Therefore, one can calculate the probability of the irreducible ordinary character χ using the equation (2.1) along with p-block's definition; considering $Irr(G)$ as the sample space and presuming that the event containing B is the $Irr(B)$, as follows:

$$P(\chi) = \begin{cases} \dfrac{k(B)}{k(G)} & \textit{if } \chi \in B, \\ 0 & \textit{if } \chi \notin B. \end{cases}$$

Since it is assumed that $\chi \notin B$, then the occurrence probability of χ in p-block B is impossible, so its probability value equals zero. Therefore, we have a probability value equal to one which is certain to occur, and this is achieved in one case only, when the group G has an unique p-block, which is called the principle p-block B_0, so that $k(B_0) = k(G)$. Therefore, for all $\chi \in Irr(G)$ we have

$$P(\chi) = \frac{k(B_0)}{k(G)} = \frac{k(G)}{k(G)} = 1.$$

Similarly, we can define the probability of the p-block B of the group G of the mathematical definition of probability and we will get $P(B) = \dfrac{k(B)}{k(G)}$.

Since B is an equivalent class, then $B_i \cap B_j = \emptyset$, for all $i \neq j$ and from the characteristics of probability we find that $P(B_i \cup B_j) = P(B_i) + P(B_j)$ and based on that we can say that

$$\begin{aligned} P(Irr(G)) &= P(B_1 \cup ... \cup B_t) \\ &= P(B_1) + ... + P(B_t) \\ &= \frac{k(B_1)}{k(G)} + ... + \frac{k(B_t)}{k(G)} \\ &= \frac{k(B_1) + ... + k(B_t)}{k(G)} \\ &= 1, \end{aligned}$$

and

$$P(B_i \cap B_j) = \begin{cases} P(B_i) & \text{if } i = j, \\ P(\emptyset) = 0 & \text{if } i \neq j. \end{cases}$$

If $\chi \in B$, then $P(\chi) = P(B)$ and if $\chi \notin B$, that means $P(\chi) = 0$. Then we can write

$$\sum_{\chi \in Irr(G)} P(\chi) = \sum_{i=1}^{t} \sum_{\chi \in Irr(B_i)} P(B_i) = 1.$$

Now we can define the probability of the p-block B of the group G by the following definition, as in [1, Definition 3.1.].

Definition 5.1. *Let G be a group. Let p be a prime number. Let S^* be a set of all p-blocks of G of order t, $t \in \mathbb{N}$. Let B be a p-block of G with defect group D. The* ***probability of the p-block*** *B of the group G is the real number*

$$P(B) = \frac{k(B)}{k(G)}$$

where $0 \leq P(B) \leq 1$, for all $B \in S^*$ and $\sum_{i=1}^{t} P(B_i) = 1$.

5.2 Examples

In this section, we will give examples for calculating a probability of the p-blocks.

We will give some examples of the probability of the p-block B of the group G with respect to the prime number p.

Example 5.1. *Let $G = S_3$.*
(1) When $p = 2$. Then S_3 has two 2-blocks $B_0 = B_1 = \{\chi_1, \chi_2\}$ and $B_2 = \{\chi_3\}$. Therefore,

$$P(B_1) = \frac{k(B_1)}{k(G)} = \frac{2}{3} \text{ and } P(B_2) = \frac{k(B_2)}{k(G)} = \frac{1}{3}.$$

(2) When $p = 3$. Then S_3 has an unique 3-block $B_0 = \{\chi_1, \chi_2, \chi_3\}$. Therefore,

$$P(B_0) = \frac{k(B_0)}{k(G)} = \frac{3}{3} = 1.$$

Example 5.2. *Let $G = D_8$ and $p = 2$. Then D_8 has an unique 2-block $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$. Therefore,*

$$P(B_0) = \frac{k(B_0)}{k(G)} = \frac{5}{5} = 1.$$

Example 5.3. *Let $G = V \cong C_2 \times C_2$ and $p = 2$. Then V has an unique 2-block $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Therefore,*

$$P(B_0) = \frac{k(B_0)}{k(G)} = \frac{4}{4} = 1.$$

Example 5.4. *Let $G = S_4$.*
(1) When $p = 2$. Then S_4 has an unique 2-block $B_0 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$. Therefore,

$$P(B_0) = \frac{k(B_0)}{k(G)} = \frac{5}{5} = 1.$$

(2) When $p = 3$. Then S_4 has three 3-blocks $B_0 = B_1 = \{\chi_1, \chi_2, \chi_3\}$, $B_2 = \{\chi_4\}$ and $B_3 = \{\chi_5\}$. Therefore,

$$P(B_1) = \frac{k(B_1)}{k(G)} = \frac{3}{5}, P(B_2) = \frac{k(B_2)}{k(G)} = \frac{1}{5} \text{ and } P(B_3) = \frac{k(B_3)}{k(G)} = \frac{1}{5}.$$

Example 5.5. *Let $G = GL(3,2)$ and $p = 7$. Then $GL(3,2)$ has two 7-blocks $B_0 = B_1 = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_6\}$ and $B_2 = \{\chi_5\}$. Therefore,*

$$P(B_1) = \frac{k(B_1)}{k(G)} = \frac{5}{6} \text{ and } P(B_2) = \frac{k(B_2)}{k(G)} = \frac{1}{6}.$$

5.3 Relations with some current conjectures

In this section, we will study facts and some current conjectures in this concept.

Write S^* to mean the set of all p-blocks of the group G relative to a fixed prime number p. Then the correspondence for all $i, j = 1, ..., t$.

$$\begin{array}{ccc} B_i & \longleftrightarrow & P(B_i) \\ \in S^* & & \in [0,1] \end{array}$$

which $0 \leq P(B_i) \leq 1$ and $\sum_{i=1}^{t} P(B_i) = 1$, in this direction which gives us the opportunity to construct finite probability space which is associated to the sample space consisting of all irreducible ordinary characters of the group G. The following theorem is an observation in this direction, which gives us the probability of the irreducible ordinary character χ which will be 1 when G has an unique p-block, and vice versa as well.

Theorem 5.1. *Let G be a group. Let p be a prime number. Then G has an unique p-block iff $P(\chi) = 1$ for all $\chi \in Irr(G)$.*

Proof. Assume that G has an unique p-block B, then B has all irreducible ordinary characters of G, hence $k(B) = k(G)$. By Definition 5.1 we have

$$P(B) = \frac{k(B)}{k(G)} = \frac{k(G)}{k(G)} = 1.$$

By the definition of the probability of irreducible ordinary character we have

$$P(\chi) = P(B) = 1,$$

for all $\chi \in B$, since B has all irreducible ordinary characters of G, hence

$$P(\chi) = 1,$$

for all $\chi \in Irr(G)$. On the other hand, assume $P(\chi) = 1$, for all $\chi \in Irr(G)$ and since $P(\chi) = P(B)$, for all $\chi \in Irr(B)$ we have

$$P(B_i) = 1,$$

for all $B_i \in S^*$ and $i = 1, ..., t$. By Definition 5.1 we have

$$P(B_i) = \frac{k(B_i)}{k(G)} = 1.$$

That means $k(B_i) = k(G)$ hence B_i has all irreducible ordinary character of G for all $i = 1, ..., t$. But $B_i \cap B_j = \{\emptyset\}$ for all $i \neq j$ (such that $i, j = 1, ..., t$.) then $B_1 = ... = B_t$. Thus G has an unique p-block the principle one.

□

The following corollary gives us the probability of the irreducible ordinary character in group G which will be 1 when group G has a p-subgroup Q in which the centralizer of Q is Q itself.

Corollary 5.1. *Let G be a group. Let p be a prime number. Let Q be a p-subgroup of G. If $C_G(Q) = Q$, then $P(\chi) = 1$ for all $\chi \in Irr(G)$.*

Proof. The proof can be found in [1, Corollary 3.5.].

□

Example 5.6. *Let $G = S_3$ and $p = 3$. A_3 is the 3-subgroup of S_3 such that $C_{S_3}(A_3) = A_3$. Since S_3 has an unique 3-block the principle one, hence for all $\chi \in Irr(G)$ we have*

$$P(\chi) = \frac{k(B_0)}{k(G)} = \frac{k(G)}{k(G)} = 1.$$

Example 5.7. *Let $G = S_4$ and $p = 2$. $H_1 = \{(1), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$ is the 2-subgroup of S_4 such that $C_{S_4}(H_1) = H_1$. Since S_4 has an unique 2-block the principle one, hence for all $\chi \in Irr(G)$ we have*

$$P(\chi) = \frac{k(B_0)}{k(G)} = \frac{k(G)}{k(G)} = 1.$$

Remark. *The p-block B of defect zero has an unique irreducible character, then the probability of this p-block B will be given by $\frac{1}{k(G)}$, and will be denoted by ξ (i.e. $\xi = \frac{1}{k(G)}$).*

The following theorem will be helpful to prove most of the theories in this subject.

Theorem 5.2 (Brauer-Feit)**.** *Let G be a group. Let p be a prime number. Let B be a p-block of G. Let $d(B)$ be a defect number of B. Then we have $k(B) \leq 1 + \frac{1}{4}p^{2d(B)}$. If B contains an irreducible character of positive height, then even $k(B) \leq 1/2\ p^{2d(B)-2}$.*

Proof. The proof can be found in [25, Theorem 2.4]. □

The following proposition and corollaries give us the relation between the probability of the p-block in group G and the order of Irr(G).

Proposition 5.1. *Let G be a group. Let p be a prime number. Let B be a p-block of G with defect group D, $|D| = p^d(B)$. Then $k(G) < \frac{p^{2d(B)}}{P(B) - \xi}$.*

Proof. By Theorem 5.2 we have

$$k(B) \leq 1 + \frac{1}{4}p^{2d(B)}$$
$$\frac{k(B)}{k(G)} \leq \frac{1 + \frac{1}{4}p^{2d(B)}}{k(G)}$$

by Definition 5.1 we have

$$P(B) \leq \frac{1}{k(G)} + \frac{p^{2d(B)}}{4k(G)}$$

since $\xi = \frac{1}{k(G)}$

$$P(B) \leq \xi + \frac{p^{2d(B)}}{4k(G)} < \xi + \frac{p^{2d(B)}}{k(G)}$$
$$P(B) - \xi < \frac{p^{2d(B)}}{k(G)}$$
$$\frac{P(B) - \xi}{p^{2d(B)}} < \frac{1}{k(G)}$$
$$k(G) < \frac{p^{2d(B)}}{P(B) - \xi}.$$

□

Corollary 5.2. *[1, Corollary 3.7.] Let G be a group. Let p be a prime number. Let B be a p-block of G. Then the probability of any B of G with defect group D, $|D| = p^d(B)$ satisfies $P(B) < \xi(1 + p^{2d(B)})$.*

Proof. By Theorem 5.2 we have

$$k(B) \leq 1 + \frac{1}{4}p^{2d(B)}$$
$$\frac{k(B)}{k(G)} \leq \frac{1 + \frac{1}{4}p^{2d(B)}}{k(G)}$$

by Definition 5.1 we have

$$P(B) \leq \frac{1}{k(G)} + \frac{p^{2d(B)}}{4k(G)} < \frac{1}{k(G)} + \frac{p^{2d(B)}}{k(G)}$$

since $\xi = \dfrac{1}{k(G)}$, then

$$\begin{aligned} P(B) &< \xi + \xi(p^{2d(B)}) = \xi(1 + p^{2d(B)}) \\ P(B) &< \xi(1 + p^{2d(B)}). \end{aligned}$$

□

Corollary 5.3. *[1, Corollary 3.8.] Let G be a group. Let p be a prime number. Let B be a p-block of G with defect group D, $|D| = p^d(B)$. If B contains an irreducible character of positive height then $k(G) < \dfrac{p^{2d(B)-2}}{P(B)}$.*

Proof. By Definition 5.1 we have

$$P(B) = \frac{k(B)}{k(G)}$$

by Theorem 5.2 we have

$$\begin{aligned} P(B) &= \frac{k(B)}{k(G)} \leq \frac{p^{2d(B)-2}}{2k(G)} \\ P(B) &\leq \frac{p^{2d(B)-2}}{2k(G)} < \frac{p^{2d(B)-2}}{k(G)} \\ P(B) &< \frac{p^{2d(B)-2}}{k(G)} \end{aligned}$$

thus

$$k(G) < \frac{p^{2d(B)-2}}{P(B)}.$$

□

The following conjecture is Brauer $k(B)$-conjecture.

Conjecture 5.1. *[1, Conjecture 3.9.] Let G be a group. Let p be a prime number. Let B be a p-block of G with defect group D. Then $P(G) \cdot [G : D] \cdot P(B) \leq 1$.*

Example 5.8. *Let $G = S_3$ and $p = 2$. Then*

$$\begin{aligned} P(S_3) \cdot [S_3 : \langle(12)\rangle] \cdot P(B_1) &= \frac{1}{2} \cdot \frac{6}{2} \cdot \frac{2}{3} = 1 \leq 1. \\ P(S_3) \cdot [S_3 : \langle(1)\rangle] \cdot P(B_2) &= \frac{1}{2} \cdot \frac{6}{1} \cdot \frac{1}{3} = 1 \leq 1. \end{aligned}$$

Example 5.9. *Let $G = S_3$ and $p = 3$. Then*

$$P(S_3) \cdot [S_3 : A_3] \cdot P(B_0) = \frac{1}{2} \cdot \frac{6}{3} \cdot \frac{3}{3} = 1 \leq 1.$$

Example 5.10. *Let $G = GL(3, 2)$ and $p = 7$. Then*

$$P(GL(3,2)) \cdot [GL(3,2) : Q_4] \cdot P(B_1) = \frac{6}{168} \cdot \frac{168}{7} \cdot \frac{5}{6} = \frac{5}{7} \leq 1.$$
$$P(GL(3,2)) \cdot [GL(3,2) : I_3] \cdot P(B_2) = \frac{6}{168} . \frac{168}{1} \cdot \frac{1}{6} = 1 \leq 1.$$

Example 5.11. *Let $G = S_4$ and $p = 2$. Then*

$$P(S_4) \cdot [S_4 : H_1] \cdot P(B_0) = \frac{5}{24} \cdot \frac{24}{8} \cdot \frac{5}{5} = \frac{5}{8} \leq 1.$$

Example 5.12. *Let $G = S_4$ and $p = 3$. Then*

$$P(S_4) \cdot [S_4 : \langle(123)\rangle] \cdot P(B_1) = \frac{5}{24} \cdot \frac{24}{3} \cdot \frac{3}{5} = 1 \leq 1.$$
$$P(S_4) \cdot [S_4 : \langle(1)\rangle] \cdot P(B_2) = \frac{5}{24} \cdot \frac{24}{1} \cdot \frac{1}{5} = 1 \leq 1.$$
$$P(S_4) \cdot [S_4 : \langle(1)\rangle] \cdot P(B_3) = \frac{5}{24} \cdot \frac{24}{1} \cdot \frac{1}{5} = 1 \leq 1.$$

Example 5.13. *Let $G = D_8$ and $p = 2$. Then*

$$P(D_8) \cdot [D_8 : D_8] \cdot P(B_0) = \frac{5}{8} \cdot \frac{8}{8} \cdot \frac{5}{5} = \frac{5}{8} \leq 1.$$

Example 5.14. *Let $G = V$ and $p = 2$. Then*

$$P(V) \cdot [V : V] \cdot P(B_0) = \frac{4}{4} \cdot \frac{4}{4} \cdot \frac{4}{4} = 1 \leq 1.$$

The following corollary gives us the probability of the principal p-block in group G which will be 1 when the group G is a p-group or G has a p-subgroup Q in which the centralizer of Q is Q itself.

Corollary 5.4. *[1, Corollary 3.10.] Let G be a group. Let p be a prime number. If G is a p-group or G has a p-subgroup Q such that $C_G(Q) = Q$. Then $P(B_0) = 1$.*

Proof. Let G be a p-group, then from Lemma 1.7 G has one p-block only the principal one. Therefore, $P(B_0) = 1$. If G has a p-subgroup Q such that $C_G(Q) = Q$, then from Corollary 5.1 we have $P(\chi) = 1$ for all $\chi \in \text{Irr(G)}$. Therefore, from Theorem 5.1 G has an unique p-block B_0, thus $P(B_0) = 1$.

□

Example 5.15. *See Examples 5.6 and 5.7, on page 60.*

Example 5.16. *Let $G = D_8$. Let $p = 2$. Then D_8 has an unique 2-block the principle one, hence*

$$P(B_0) = \frac{k(B_0)}{k(G)} = \frac{k(G)}{k(G)} = 1.$$

Chapter 6

The Structure Constants and Probabilistic Methods

In this chapter, we will present three sections; in the first section, we will study the definition of algebra over a field F and from it we derive the definition of structure constants of this concept. In the second section, we will study the definition of the group algebra G over F which is denoted by $F[G]$ with mention the basis for it, the definition of the center of $F[G]$ which is denoted by $Z(F[G])$, and the important theorem in this section which explains the basis for $Z(F[G])$, and in the third and last section, we will present theories and examples that demonstrate the theorem of the probability that two elements of a finite group commute by the concept of structure constants. The basic references of this chapter are [5], [15] and [20].

6.1 The Structure Constants of an Algebra

We will state the algebra over a field F, as in [5] page 114, [20] page 155 and [15, Definition 1.1].

Definition 6.1. *Let $(A, +, \times)$ be an* ***algebra over a field*** *F, denoted by F-algebra. This means that A is*
(1) $(A, +, \times)$ is a ring.
(2) $(A, +)$ is a vector space over F.
(3) $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for any $\lambda \in F$ and $a, b \in A$.

Let us assume that A is finite dimensional over the field F. Consider a basis of A such that $\beta = \{x_1, ..., x_n\}$ when $n = dim_F(A) \in \mathbb{N}$. This means that β spans A and β is linearly independent over F.

Since the product $x_i x_j$ is a well-defined element in the algebra A, we have the motivation of the following definition.

Definition 6.2. *The* ***structure constants*** *of A with respect to the basis $\beta = \{x_1, ..., x_n\}$ are the scalers $a_{ijv} \in F$ with $i, j, v \in \{1, ..., n\}$ which are defined by the product:*

$$x_i x_j = \sum_{v=1}^{v=n} a_{ijv} x_v.$$

Remark.
(1) The number of a_{ijv} is n^3 scalers.
(2)All a_{ijv} $(i, j, v = 1, \cdots, k(G))$ belonging to the field F.
(3) The structure constant of A are with respect to the basis β.

(4) The knowledge of the structure constants of the algebra A with respect to the basis β completely determines the multiplication in the algebra A with coefficients from the field F.

6.2 Center of Group algebra $Z(F[G])$

In the following definition, we will state the group algebra, as in [15] page 2 and [20] page 261.

Definition 6.3. *The* ***group algebra*** *of a group G over a field F, denoted by $F[G]$, is the F-algebra whose elements are all possible finite sums of the type $\sum_{g\in G} r_g g$, $g \in G$, $r_g \in F$, the operations being defined by the formulas:*

$$r + r' = \sum_{g\in G} r_g g + \sum_{g\in G} r'_g g = \sum_{g\in G} (r_g + r'_g) g,$$

$$rr' = \left(\sum_{g\in G} r_g g\right)\left(\sum_{g'\in G} r'_{g'} g'\right) = \sum_{g\in G}\sum_{g'\in G} (r_g r'_{g'}) gg' = \sum_{q\in G}\left(\sum_{k\in G} (r_k r_{qk^{-1}})\right) q,$$

for all $r, r' \in F[G]$.

The following lemma gives us the basis of $F[G]$, as in [15] page 2.

Lemma 6.1. *Let G be a group. Let F be a field. The group algebra $F[G]$ is an algebra over F with basis G.*

We will define the center of a group algebra $F[G]$.

Definition 6.4. *Let G be a group. Let F be a field. Let $F[G]$ be the group algebra of G over F. The* ***center of the group algebra*** *$F[G]$ is the set of elements that commutes with every element of $F[G]$, which is denoted by $Z(F[G])$. In symbols*

$$Z(F[G]) = \{z \in F[G] : qz = zq \text{ for all } q \in F[G]\}.$$

The following theorem gives us the basis of $Z(F[G])$, as in [15, Theorem 2.4].

Theorem 6.1. *Let G be a group with distinct conjugacy classes $C(g_1), \cdots, C(g_{k(G)})$ where $g_i \in G$ for all $i = 1, \cdots, k(G)$. Let F be a field. Let $K_i = \sum_{g\in C(g_i)} g \in F[G]$, for all $i = 1, \cdots, k(G)$. Then $K_1, \cdots, K_{k(G)}$ forms basis for $Z(F[G])$.*

Proof. It is clear that the $K_1, \cdots, K_{k(G)}$ lies in $Z(F[G])$. $K_1, \cdots, K_{k(G)}$ are linearly independent over F since $C(g_i) \cap C(g_j) = \emptyset$ for all $i \neq j$, such that $i, j = 1, \cdots, k(G)$. If $z = \sum_{g\in G} a_g g \in Z(F[G])$ and $h \in G$, we have

$$\text{h z} = \text{z h}$$
$$\text{z} = \text{h}^{-1} z h$$
$$\sum_{g\in G} a_g g = \sum_{g\in G} a_g g^h$$

by comparing the coefficients of g^h on both sides, we obtain

$$a_{g^h} = a_g$$

that means the coefficients a_g have the constant value a_i for all $g \in C_i$, then

$$z = \sum_{i=1}^{k(G)} a_i K_i$$

thus

$$K_1, \cdots, K_{k(G)} \text{ span } Z(F[G]).$$

□

Remark.
(1) The elements K_i $(1 \leq i \leq k(G))$ are called class sums.
(2) We shall use the multiplication $K_i \cdot K_j$ to define a_{ijv}.
(3)We shall use the matrix $[a_{ijv}]$ which represents the multiplication table of the basis $\{a_{ijv}\}_{i=1}^{i=k(G)}$.

6.3 Application of Probability Methods with examples

In this section, we present a new application of this direction by using the structure constant. We use matrices and the sum over the required constants. We give theories and examples of our discovery.

Example 6.1. *When $G = S_3$, we have*

$$d(S_3) = P(S_3) = \frac{\sum_{i=1}^{k(S_3)} \sum_{j=1}^{k(S_3)} \sum_{v=1}^{k(S_3)} a_{ijv}}{|S_3|^2} = \frac{1}{2}.$$

With matrix:

$$[a_{ijv}] = \begin{pmatrix} \underset{111}{a} & \underset{112}{a} & \underset{113}{a} & \underset{121}{a} & \underset{122}{a} & \underset{123}{a} & \underset{131}{a} & \underset{132}{a} & \underset{133}{a} \\ \underset{211}{a} & \underset{212}{a} & \underset{213}{a} & \underset{221}{a} & \underset{222}{a} & \underset{223}{a} & \underset{231}{a} & \underset{232}{a} & \underset{233}{a} \\ \underset{311}{a} & \underset{312}{a} & \underset{313}{a} & \underset{321}{a} & \underset{322}{a} & \underset{323}{a} & \underset{331}{a} & \underset{332}{a} & \underset{333}{a} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Example 6.2. *When $G = D_8$, we have*

$$d(D_8) = P(D_8) = \frac{\sum_{i=1}^{k(D_8)} \sum_{j=1}^{k(D_8)} \sum_{v=1}^{k(D_8)} a_{ijv}}{|D_8|^2} = \frac{5}{8}.$$

With matrix:

$$[a_{ijv}] = \begin{pmatrix} \underset{111}{a} & \underset{112}{a} & \cdots & \underset{155}{a} \\ \underset{211}{a} & \underset{212}{a} & \cdots & \underset{255}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{511}{a} & \underset{512}{a} & \cdots & \underset{555}{a} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

Example 6.3. *When $G = S_4$, we have*

$$d(S_4) = P(S_4) = \frac{\sum_{i=1}^{k(S_4)} \sum_{j=1}^{k(S_4)} \sum_{v=1}^{k(S_4)} a_{ijv}}{|S_4|^2} = \frac{5}{24}.$$

With matrix:

$$[a_{ijv}] = \begin{pmatrix} \underset{111}{a} & \underset{112}{a} & \cdots & \underset{155}{a} \\ \underset{211}{a} & \underset{212}{a} & \cdots & \underset{255}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{511}{a} & \underset{512}{a} & \cdots & \underset{555}{a} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 6 & 0 & 2 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 3 & 0 & 8 & 0 & 8 & 4 & 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 3 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 4 & 6 & 0 & 2 & 3 & 0 \end{pmatrix}$$

Recall that, for the group algebra $F[G]$, the definition of the structure constants a_{ijv} comes from the multiplication $K_i \cdot K_j = \sum_{v=1}^{k(G)} a_{ijv} K_v$.

The following theorem is the main discovery and contribution of the thesis.

Theorem 6.2. *Let G be a finite group. Let $k(G)$ be the number of conjugacy classes of G. Then*

$$\sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv} = k(G)|G|.$$

Proof. Let $C(x_1), ..., C(x_{k(G)})$ be the distinct conjugacy classes of G where $x_i \in G$ for all $i = 1, \cdots, k(G)$. Then by using the definition a_{ijv}, we have

$$\begin{aligned} \sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv} &= |\{(x, y) \in G \times G : [x, y] = 1_G\}| \\ &= |\{(x, y) \in G \times G : xy = yx\}| \\ &= \sum_{x \in G} |C_G(x)| = \sum_{i=1}^{k(G)} \sum_{x \in C(x_i)} |C_G(x)| \\ &= \sum_{i=1}^{k(G)} [G : C_G(x_i)] |C_G(x_i)| \\ &= \sum_{i=1}^{k(G)} |G| = k(G)|G| \end{aligned}$$

thus

$$\sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv} = k(G)|G|.$$

□

Our next main theorem is the following:

Theorem 6.3. *Let G be a finite group. Then*

$$d(G) = P(G) = \frac{\sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv}}{|G|^2}.$$

Proof. By Theorem 2.1 and Theorem 6.2 we have

$$d(G) = P(G) = \frac{k(G)}{|G|} = \frac{k(G)|G|}{|G|^2} = \frac{\sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv}}{|G|^2},$$

thus

$$d(G) = P(G) = \frac{\sum_{i=1}^{k(G)} \sum_{j=1}^{k(G)} \sum_{v=1}^{k(G)} a_{ijv}}{|G|^2}.$$

□

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