Chapter 3

Vectors

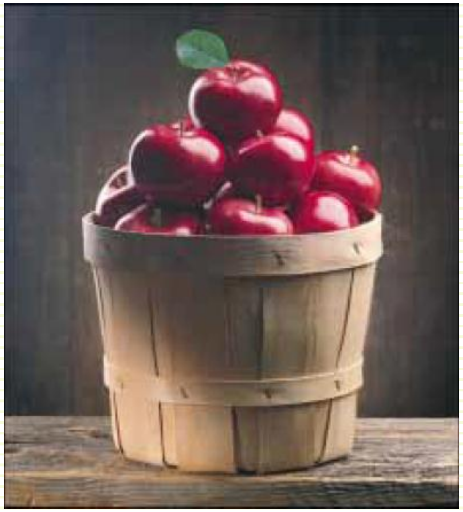
Units of Chapter 3: Vectors

- Vector and Scalar Quantities
- Adding Vectors: GRAPHICAL METHOD
- Components of a Vector and Unit
- Adding Vectors : COMPONENT METHOD
- Multiplying Vectors
- Problems

Learning goals of this chapter

- **On completing this chapter, the student will be able to :**
- Define the scalar and vector quantities.
- Adding and subtracting two vectors graphically and with mathematics.
- Analyze the vector to its components
- Multiplying the vector with constant scalar quantity and with another vector.
- Differentiate between the scalar and vector quantities.
- Differentiate between scalar product and vector product.
- Solve problems using vector methods.

Vector and Scalar Quantities



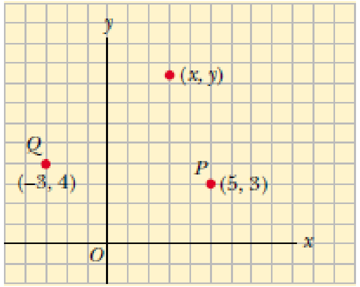
- A scalar quantity is specified by a single value with an appropriate unit and has no direction.

Examples of a scalar quantities:
1. Temperature
2. Speed
3. Distance
4. Volume
5. Mass,
6. Time intervals

- A vector quantity has both magnitude and direction.

Examples of a vector quantities:
1. Velocity
2. Displacement
3. Force

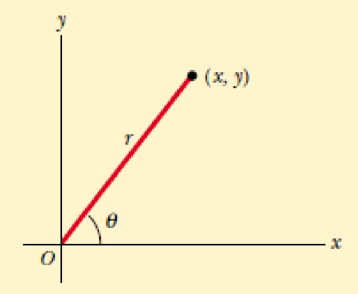
Vector and Scalar Quantities



- **Cartesian coordinates (*rectangular coordinates).***

In which horizontal and vertical axes intersect
at a point taken to be the origin.

- **polar coordinate system**



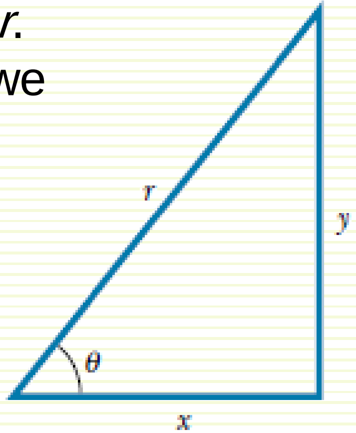
In this *polar coordinate system, r* is the distance from the origin to the point having Cartesian coordinates (*x*, *y*), and θ is the angle between *r* and a fixed axis. This fixed axis is usually the positive *x* axis, and is usually measured counterclockwise from it.

From the right triangle, we find that $\sin\theta = y/r$ and that $\cos\theta = x/r$. Therefore, starting with the plane polar coordinates of any point, we can obtain the Cartesian coordinates, using the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

Furthermore, the definitions of trigonometry tell us that

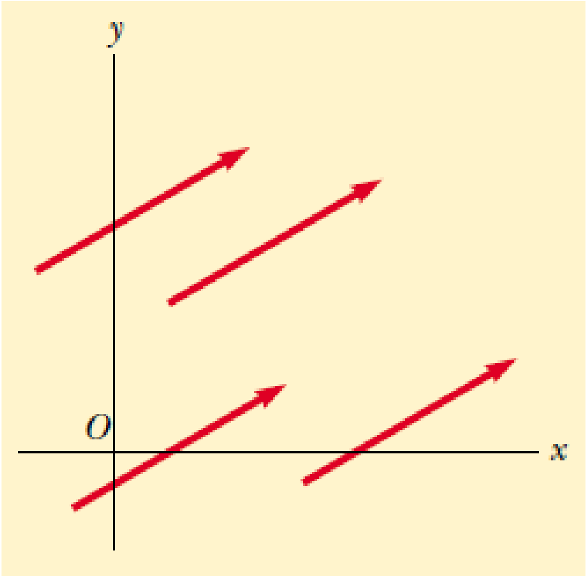
$$\tan\theta = \frac{y}{x} \qquad r = \sqrt{x^2 + y^2}$$



ADDING VECTORS: GRAPHICAL METHOD

➢ Equality of Two Vectors

Two vectors **A** and **B** may be defined to be equal if they have the same magnitude and point in the same direction along parallel lines.



For example, all the vectors in the figure are equal even though they have different starting points

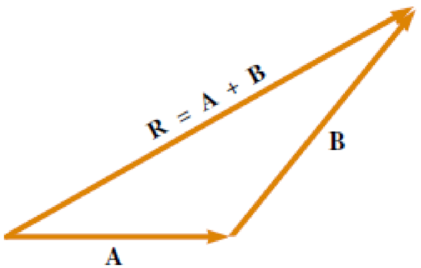
This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

That is, **A = B** only if $A = B$ *and* if **A** and **B** point in the same direction along parallel lines

ADDING VECTORS: GRAPHICAL METHOD

- Adding Vectors

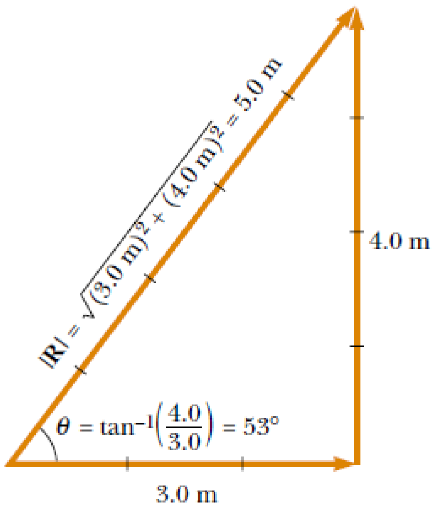
Triangle method of addition



In this procedure to add vector **B** to vector **A**, first draw vector **A**, with its magnitude represented by a convenient scale, on graph paper and then draw vector B to the same scale with its tail starting from the tip of **A**, as shown in the figure. The resultant vector **R = A** + **B** is the vector drawn from the tail of **A** to the tip of **B.**

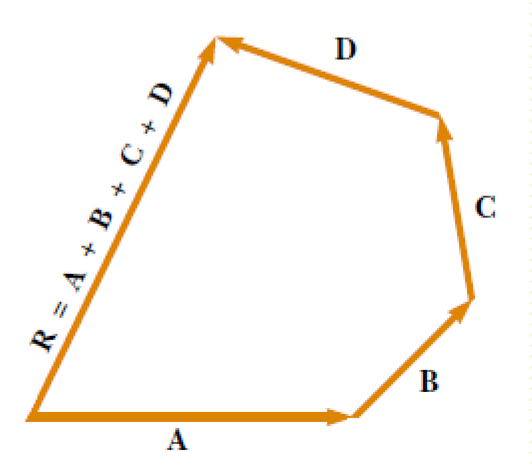
Example

if you walked 3.0 m toward the east and then 4.0 m toward the north, as shown in the figure, you would find yourself 5.0 m from where you started, measured at an angle of 53° north of east. Your total displacement is the vector sum of the individual displacements.



ADDING VECTORS: GRAPHICAL METHOD

A geometric construction can also be used to add more than two vectors. This is shown in the figure for the case of four vectors. The resultant vector



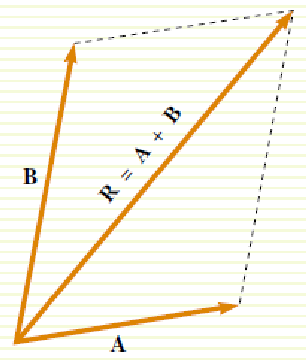
R = A + B + C + D

is the vector that completes the polygon. In other words, **R is the vector drawn from the tail of the first vector to the tip of the last vector.**

ADDING VECTORS: GRAPHICAL METHOD

Parallelogram rule of addition

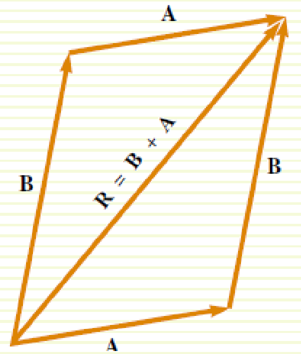
In this construction, the tails of the two vectors **A** and **B** are joined together and the resultant vector **R** is the diagonal of a parallelogram formed with **A** and **B** as two of its four sides as shown in the figure



Commutative law of addition

When two vectors are added, the sum is independent of the order of the addition.

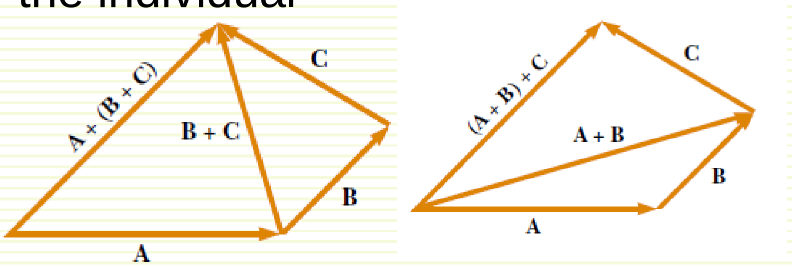
A + B = B + A



Associative law of addition

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together

A + (**B** + **C**) = (**A** + **B**) + **C**



Negative of a Vector

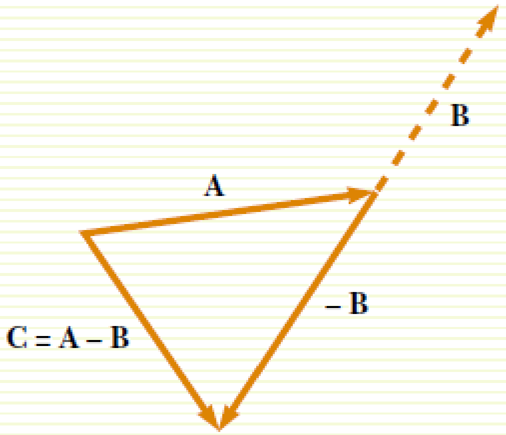
The negative of the vector **A** is defined as the vector that when added to **A** gives zero for the vector sum. That is, **A** + (- **A**) = 0. The vectors **A** and -**A** have the same magnitude but point in opposite directions

ADDING VECTORS: GRAPHICAL METHOD

Subtracting Vectors

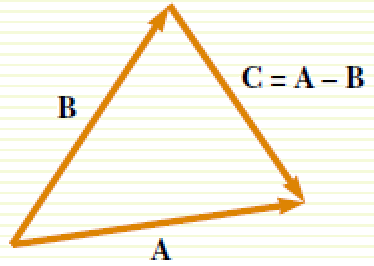
The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation **A** - **B** as vector - **B** added to vector **A**:

A - B = A + (- B)



The geometric construction for subtracting two vectors in this way is illustrated in the figure

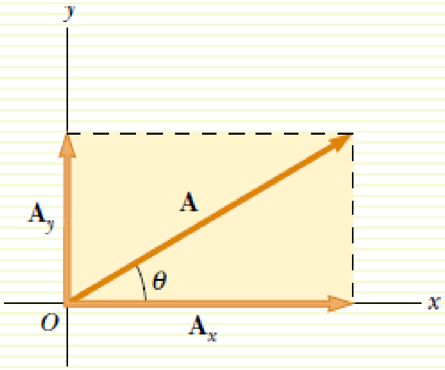
Another way of looking at vector subtraction is to note that the difference **A - B** between two vectors **A** and **B** is what you have to add to the second vector to obtain the first. In this case, the vector **A - B** points from the tip of the second vector to the tip of the first



COMPONENTS OF VECTORS

The graphical method, it is not very useful for vectors in three dimensions. Another way of adding vectors is the analytical method, involving the resolution of a vector into components with respect to a particular coordinate system.

It is shown that the three vectors form a right triangle and that **A** = **A**$_x$ + **A**$_y$ **.** (review the parallelogram rule)



The component A$_x$ represents the projection of **A** along the *x* axis.
The component A$_y$ represents the projection of **A** along the *y* axis

(Ax and Ay without the boldface notation).

From the figure and the definition of sine and cosine, we see that

$$A_x = A\cos\theta \qquad A_y = A\sin\theta \qquad A = \sqrt{A_x^2 + A_y^2} \qquad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

This process is called resolving **A** vector into its components

the quantities A$_x$ and A$_y$ are called the components of the vector **A**

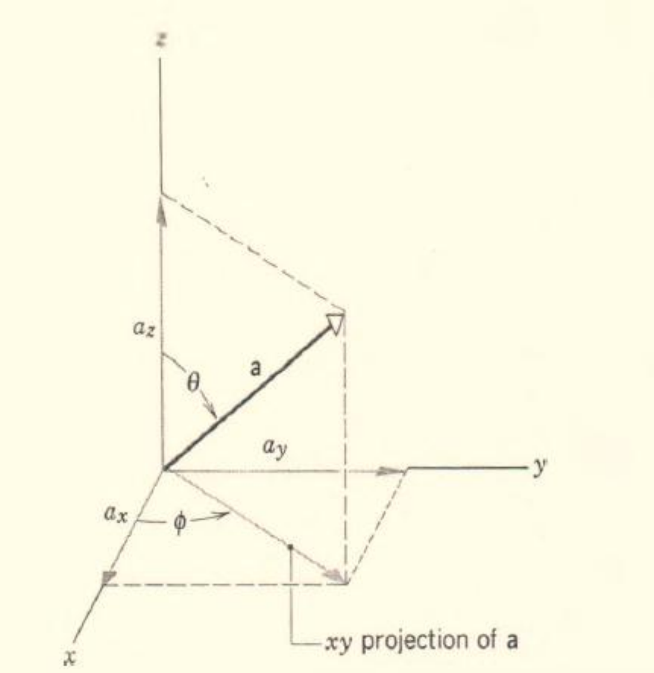
COMPONENTS OF VECTORS

In three dimensions the process works similarly: just draw perpendicular lines from the tip of the vector to the three coordinate axes X, *y,* and z

$$a_x = a \sin\theta \cos\phi,$$

$$a_y = a \sin\theta \sin\phi,$$

$$a_z = a \cos\theta.$$



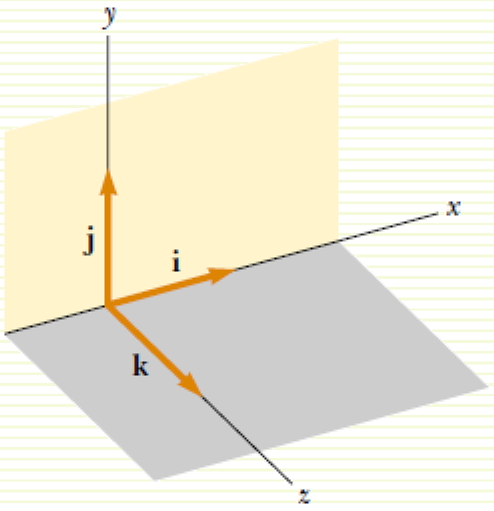
Unit Vectors

- A unit vector is a **dimensionless vector** having a magnitude of exactly 1, and used to specify a given direction and have no other physical significance.
- We shall use the symbols **i, j**, and **k** to represent unit vectors pointing in the positive x, y, and z directions, respectively
- The unit vectors i, j, and k form a set of mutually perpendicular vectors in a right-handed coordinate system

If **A** and **B** both have x, y, and z components, we express them in the form

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

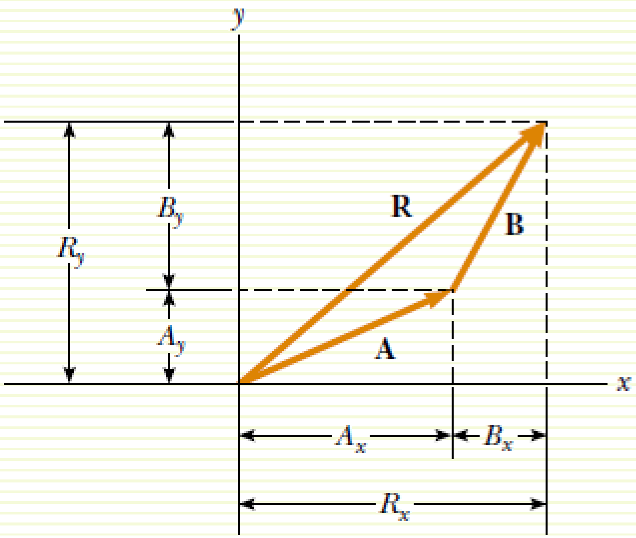
$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$



ADDING VECTORS: COMPONENT METHOD

If **A** and **B** both have x, y, and z components and we wish to add vector **B** to vector **A**,

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$



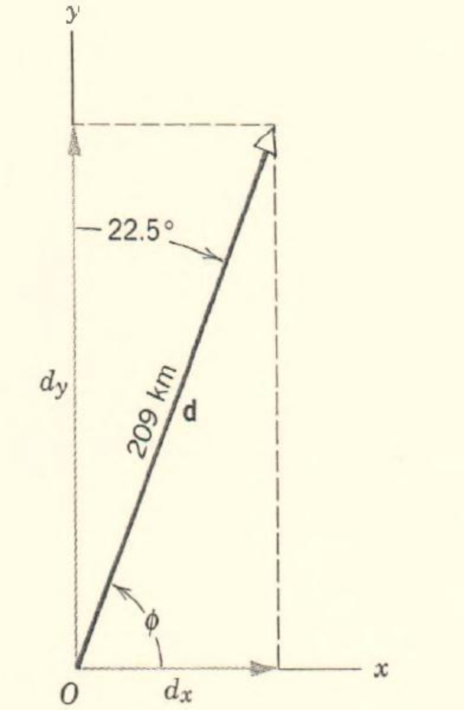
$$\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j}$$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$

$$R_x = A_x + B_x \qquad R_y = A_y + B_y$$

Problem 1

An airplane travels 209 km on a straight course making an angle of 22.5°east of due north. How far north and how far east did the plane travel from its starting point?



Solution

We choose the positive X direction to be east and the positive *y* direction to be north. Next, we draw a displacement vector from the origin (starting point), making an angle of 22.5° with the *y* axis (north) inclined along the positive X direction (east).

$$\phi = 90.0° - 22.5° = 67.5°,$$

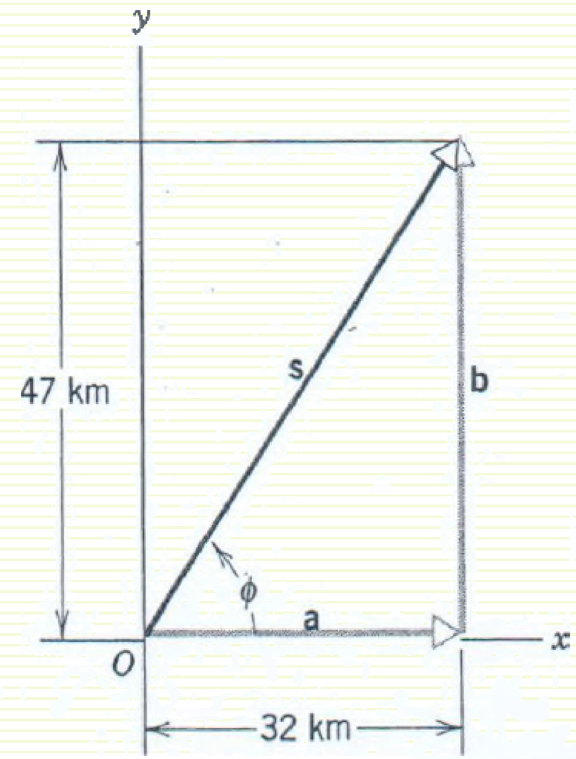
$$d_x = d \cos \phi = (209 \text{ km})(\cos 67.5°) = 80.0 \text{ km},$$

and

$$d_y = d \sin \phi = (209 \text{ km})(\sin 67.5°) = 193 \text{ km}.$$

Problem 2

An automobile travels due east on a level road for 32 km. It then turns due north at an intersection and travels 47 km before stopping. Find the resultant displacement of the car?



Solution

$$s_x = a_x + b_x = 32 \text{ km} + 0 = 32 \text{ km},$$

$$s_y = a_y + b_y = 0 + 47 \text{ km} = 47 \text{ km}.$$

The magnitude and direction ofs are then

$$s = \sqrt{s_x^2 + s_y^2} = \sqrt{(32 \text{ km})^2 + (47 \text{ km})^2} = 57 \text{ km},$$

$$\tan \phi = \frac{s_y}{s_x} = \frac{47 \text{ km}}{32 \text{ km}} = 1.47, \qquad \phi = \tan^{-1}(1.47) = 56°.$$

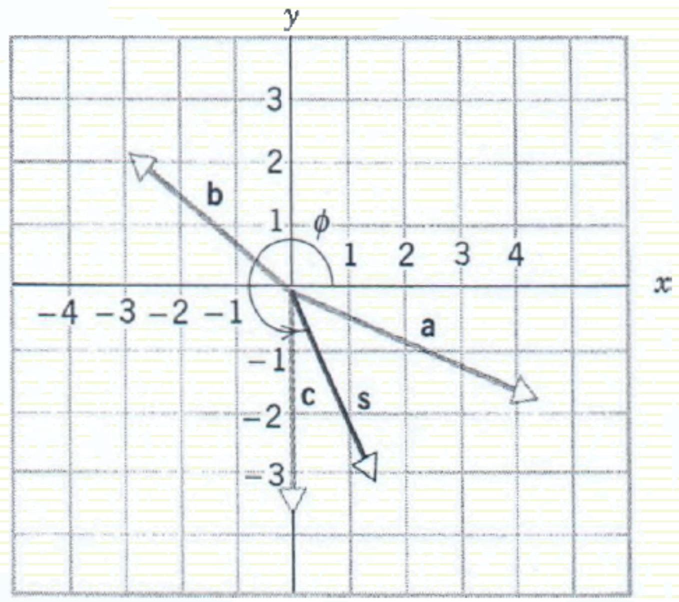
Problem 3

Three coplanar vectors are expressed with respect to a certain rectangular coordinate system as

$$\mathbf{a} = 4.3\mathbf{i} - 1.7\mathbf{j},$$

$$\mathbf{b} = -2.9\mathbf{i} + 2.2\mathbf{j},$$

$$\mathbf{c} = -3.6\mathbf{j},$$



in which the components are given in arbitrary units. Find the vector s which is the sum of these vectors?

Solution

$$s_x = a_x + b_x + c_x = 4.3 - 2.9 + 0 = 1.4,$$

$$s_y = a_y + b_y + c_y = -1.7 + 2.2 - 3.6 = -3.1.$$

$$\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{j} = 1.4\mathbf{i} - 3.1\mathbf{j}.$$

$$\phi = \tan^{-1}(-3.1/1.4) = 294°.$$

Multiplying Vectors

There are three ways in which vectors can be multiplied, but none is exactly like the usual algebraic multiplication

➢ Multiplying a Vector by a Scalar

If we multiply a vector by a scalar *s*, we get a new vector. Its magnitude is the product of the magnitude of the vector and the absolute value of *s*

For example, the vector 5A is five times as long as A and points in the same direction as A; the vector $-\frac{1}{3}$ A is one-third the length of A and points in the direction opposite A.

➢ Multiplying a Vector by a Vector

There are two ways to multiply a vector by a vector: one way produces a scalar (called the ***scalar product***), and the other produces a new vector (called the ***vector product***).

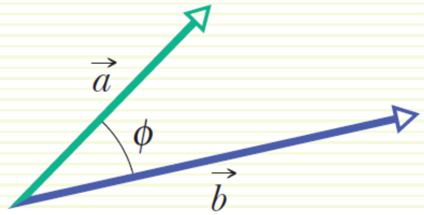
Multiplying Vectors

➢ The Scalar Product

The **scalar product** (also known as the **dot product)** $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \cdot \vec{b}$

And defined to be

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$



where a is the magnitude of $\vec{a}$, b is the magnitude of $\vec{b}$.

And Φ is the angle between the two vectors.

The **commutative** law applies to a scalar product, so we can write

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

When two vectors are in unit-vector notation, we write their dot product as

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Multiplying Vectors

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z.$$

- **The Vector Product**

The **vector product** of $\vec{a}$ and $\vec{b}$, written $\vec{a} \times \vec{b}$, produces a third vector $\vec{c}$

The magnitude of the third vector is

$$c = ab \sin \phi,$$

The direction of $\vec{c}$ is perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$.

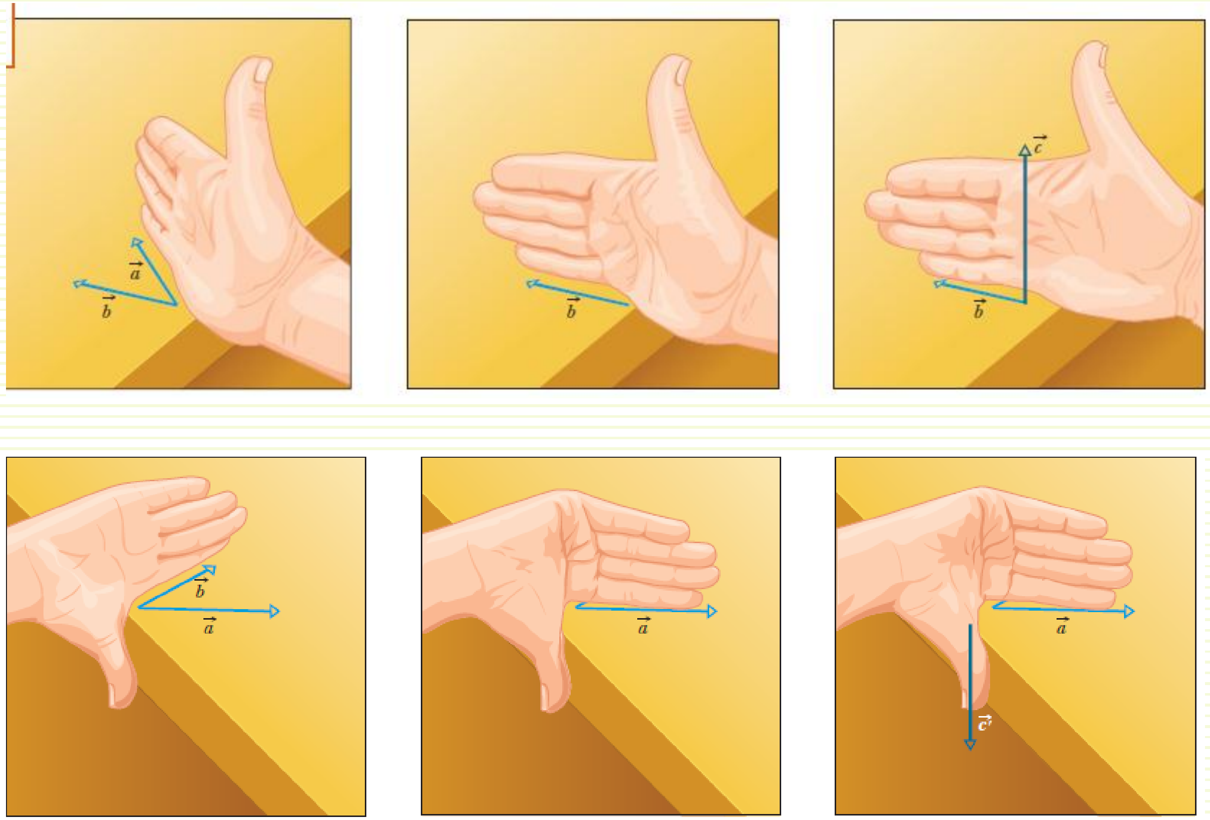
where ϕ is the *smaller* of the two angles between $\vec{a}$ and $\vec{b}$. (You must use the smaller of the two angles between the vectors because $\sin \phi$ and $\sin(360° - \phi)$ differ in algebraic sign.) Because of the notation, $\vec{a} \times \vec{b}$ is also known as the **cross product**, and in speech it is "a cross b."

Multiplying Vectors

The right-hand rule

To determine the direction of the vector produced from the vector product we must use the **right-hand rule as in the figure.**

The order of the vector multiplication is important. So, the commutative law does not apply to a vector product



(b)

Multiplying Vectors

In unit-vector notation, we write

$$\vec{a} \times \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0, \qquad \hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}.$$

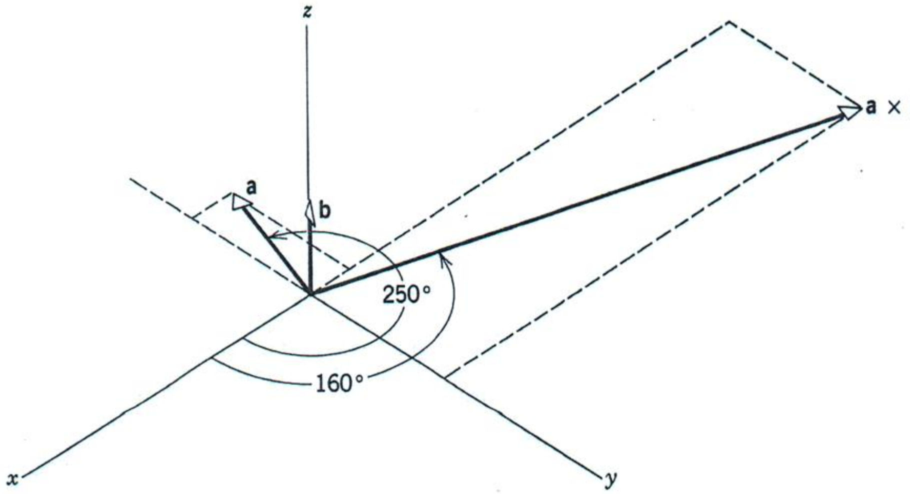
Or we can write

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$$

Problem 4

A certain vector **a** in the XY plane is directed 250° counterclockwise from the positive x axis and has magnitude 7.4 units. Vector **b** has magnitude 5.0 units and is directed parallel to the z axis. Calculate (a) **a .b** and (*b*) (**a** x **b**).



Solution (a) $\mathbf{a}\cdot\mathbf{b} = ab\cos\phi = ab\cos 90^\circ = (7.4)(5.0)(0) = 0,$

(b) The magnitude of the vector product is $|\mathbf{a}\times\mathbf{b}| = ab\sin\phi = (7.4)(5.0)\sin 90^\circ = 37.0$

The *direction* of the vector product is perpendicular to the plane formed by a and b.

$$a_x = 7.4\cos 250^\circ = -2.5, \quad a_y = 7.4\sin 250^\circ = -7.0, \quad a_z = 0;$$

$$b_x = 0, \quad b_y = 0, \quad b_z = 5.0.$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ -2.5 & -7.5 & 0 \\ 0 & 0 & 5.0 \end{vmatrix} = [(-7.0)(5.0) - (0)(0)]\mathbf{i} + [(0)(0) - (-2.5)(5.0)]\mathbf{j} + [(-2.5)(0) - (-7.0)(0)]\mathbf{k} = -35\mathbf{i} + 13\mathbf{j}.$$

Ch. 3 Summary

Scalars and Vectors *Scalars*, such as temperature, have magnitude only. They are specified by a number with a unit (10°C) and obey the rules of arithmetic and ordinary algebra. *Vectors*, such as displacement, have both magnitude and direction (5 m, north) and obey the rules of vector algebra.

Adding Vectors Geometrically Two vectors $\vec{a}$ and $\vec{b}$ may be added geometrically by drawing them to a common scale and placing them head to tail. The vector connecting the tail of the first to the head of the second is the vector sum $\vec{s}$. To subtract $\vec{b}$ from $\vec{a}$, reverse the direction of $\vec{b}$ to get $-\vec{b}$; then add $-\vec{b}$ to $\vec{a}$. Vector addition is commutative and obeys the associative law.

Components of a Vector The (scalar) *components* a_x and a_y of any two-dimensional vector $\vec{a}$ along the coordinate axes are found by dropping perpendicular lines from the ends of $\vec{a}$ onto the coordinate axes. The components are given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta,$$

where θ is the angle between the positive direction of the x axis and the direction of $\vec{a}$. The algebraic sign of a component indicates its direction along the associated axis. Given its components, we can find the magnitude and orientation of the vector $\vec{a}$ with

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}.$$

Unit-Vector Notation *Unit vectors* $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$ have magnitudes of unity and are directed in the positive directions of the x, y, and z axes, respectively, in a right-handed coordinate system. We can write a vector $\vec{a}$ in terms of unit vectors as

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$

in which $a_x\hat{\mathrm{i}}$, $a_y\hat{\mathrm{j}}$, and $a_z\hat{\mathrm{k}}$ are the **vector components** of $\vec{a}$ and a_x, a_y, and a_z are its **scalar components.**

Adding Vectors in Component Form To add vectors in component form, we use the rules

$$r_x = a_x + b_x \quad r_y = a_y + b_y \quad r_z = a_z + b_z.$$

Here $\vec{a}$ and $\vec{b}$ are the vectors to be added, and $\vec{r}$ is the vector sum. Note that we add components axis by axis.

Ch. 3 Summary

Product of a Scalar and a Vector The product of a scalar s and a vector $\vec{v}$ is a new vector whose magnitude is sv and whose direction is the same as that of $\vec{v}$ if s is positive, and opposite that of $\vec{v}$ if s is negative. To divide $\vec{v}$ by s, multiply $\vec{v}$ by $1/s$.

The Scalar Product The **scalar** (or **dot**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\cdot\vec{b}$ and is the *scalar* quantity given by

$$\vec{a}\cdot\vec{b} = ab\cos\phi, \tag{3-20}$$

in which ϕ is the angle between the directions of $\vec{a}$ and $\vec{b}$. A scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector. In unit-vector notation,

$$\vec{a}\cdot\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\cdot(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \tag{3-22}$$

which may be expanded according to the distributive law. Note that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

The Vector Product The **vector** (or **cross**) **product** of two vectors $\vec{a}$ and $\vec{b}$ is written $\vec{a}\times\vec{b}$ and is a *vector* $\vec{c}$ whose magnitude c is given by

$$c = ab\sin\phi, \tag{3-27}$$

in which ϕ is the smaller of the angles between the directions of $\vec{a}$ and $\vec{b}$. The direction of $\vec{c}$ is perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$ and is given by a right-hand rule, as shown in Fig. 3-19. Note that $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$. In unit-vector notation,

$$\vec{a}\times\vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}})\times(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \tag{3-29}$$

which we may expand with the distributive law.

Homework of Ch. 3

1- A person walks in the following pattern: 3.1 km north, then 2.4 km west, and finally 5.2 km south. (a) Construct the vector diagram that represents this motion. (b) How far and in what direction would a bird fly in a straight line to arrive at the same final point?

2- Two vectors are given by

$$\vec{a} = (4.0\text{ m})\hat{\text{i}} - (3.0\text{ m})\hat{\text{j}} + (1.0\text{ m})\hat{\text{k}}$$

and $$\vec{b} = (-1.0\text{ m})\hat{\text{i}} + (1.0\text{ m})\hat{\text{j}} + (4.0\text{ m})\hat{\text{k}}.$$

In unit-vector notation, find (a) $\vec{a} + \vec{b}$, (b) $\vec{a} - \vec{b}$, and (c) a third vector $\vec{c}$ such that $\vec{a} - \vec{b} + \vec{c} = 0$.

3- Three vectors are given by $\vec{a} = 3.0\hat{\text{i}} + 3.0\hat{\text{j}} - 2.0\hat{\text{k}}$, $\vec{b} = -1.0\hat{\text{i}} - 4.0\hat{\text{j}} + 2.0\hat{\text{k}}$, and $\vec{c} = 2.0\hat{\text{i}} + 2.0\hat{\text{j}} + 1.0\hat{\text{k}}$. Find (a) $\vec{a} \cdot (\vec{b} \times \vec{c})$, (b) $\vec{a} \cdot (\vec{b} + \vec{c})$, and (c) $\vec{a} \times (\vec{b} + \vec{c})$.