Signal Analysis (802321)

نموذج الحل

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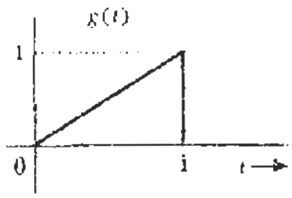
Final Exam
Second Semester
02-07-1431 H

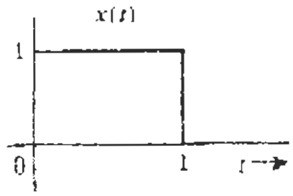
الاسم:
الرقم الجامعي:

Q1. For the signals $g(t)$ and $x(t)$ shown in the figure:

a) Define $g(t)$ mathematically.
b) Define $x(t)$ mathematically.
c) Find E_x (the energy of the signal $x(t)$).
d) Find the inner product of the two signals $g(t)$ and $x(t)$.
e) Are $g(t)$ and $x(t)$ orthogonal? Why?
f) Find the optimum value of c in the approximation $g(t) \approx c\, x(t)$ so that the error signal energy is minimum.
g) Sketch $g(-t + 4)$.
h) Sketch $x(t)\, u(t)$.
i) Sketch $x(t)\, \delta(t - ½)$.

10





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a) $g(t) = t \qquad 0 \le t \le 1$ (1)

b) $x(t) = 1 \qquad 0 \le t \le 1$ (1)

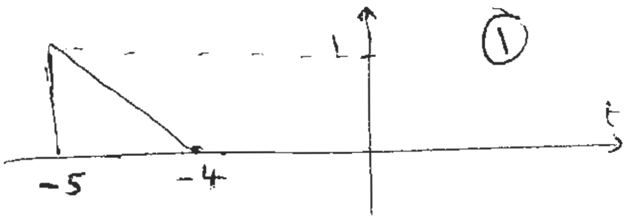
c) $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt$ (1/2) $= \int_0^1 (1)^2\,dt = [t]_0^1 = 1$ (1/2)

d) Inner product $= \int_{-\infty}^{\infty} g(t)\,x(t)\,dt$ (1/2) $= \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1$ (1/2) $= \frac{1}{2}$ (1/2)

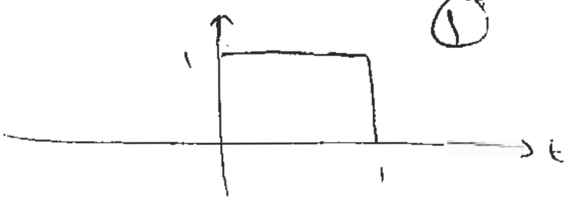
e) No, because the inner product is not zero. (1)

f) $c = \frac{1}{E_x}\int_{-\infty}^{\infty} g(t)\,x(t)\,dt$ (1) $= \frac{1}{1}\int_0^1 t\,dt = \frac{1}{2}$ (1/2)

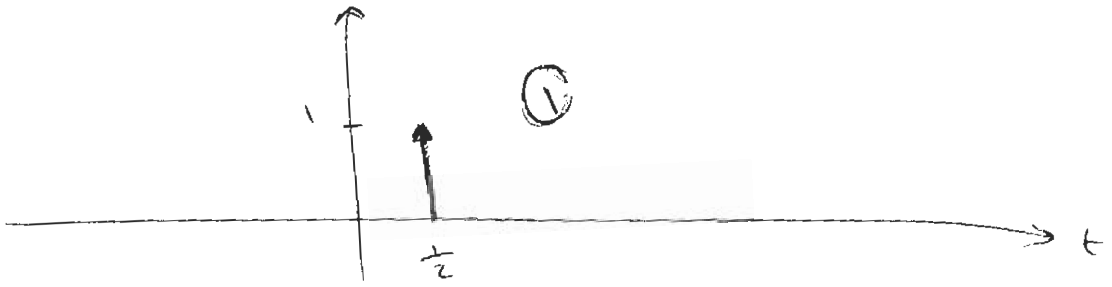
g) $g(-t+4)$ (1)



h) $x(t)\,u(t)$ (1)



i) $x(t)\,\delta(t-\frac{1}{2}) = x(\frac{1}{2})\,\delta(t-\frac{1}{2}) = 1 \cdot \delta(t-\frac{1}{2})$



Q2. In the Fourier series theory: (6)

a) Any periodic signal $x(t)$ can be represented as:

$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$ (1) $= \sum_{k=-\infty}^{\infty} a_k\, e^{jk\left(\frac{2\pi}{T}\right)t}$ (1/2)

In the series:

The term for $k = 0$ is called the ...... constant (1/2) ... component. ← or dc

The terms for $k = 1$, $k = -1$ have the frequency value of .. ω_0 (1/2) and are called the

.. first ... harmonic (1/2) components. ← (or fundamental components)

The terms for $k = 5$, $k = -5$ have the frequency value of $5\omega_0$ and are called the

fifth .... harmonic ..... components. (1/2)

(1/2)

b) The Fourier series coefficients a_k are calculated from the Fourier series *analysis equation* as:

$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ (1)

c) For $k = 0$, the analysis equation is simplified to:

$a_0 = \frac{1}{T}\int_T x(t)\, dt$ (1)

Q3. Determine the Fourier series coefficients for $x(t)$: (8)

$x(t) = ½ + 2 \cos \omega_0 t - 4 \cos 5\omega_0 t + \sin (2\omega_0 t + \pi/2)$

$$x(t) = \frac{1}{2} + \frac{2}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) - \frac{4}{2}\left(e^{j5\omega_0 t} + e^{-j5\omega_0 t}\right) + \frac{1}{2j}\left(e^{j(2\omega_0 t + \pi/2)} - e^{-j(2\omega_0 t + \pi/2)}\right)$$ (1)

$$x(t) = \frac{1}{2} + e^{j\omega_0 t} + e^{-j\omega_0 t} - 2e^{j5\omega_0 t} - 2e^{-j5\omega_0 t} + \frac{1}{2j} e^{j2\omega_0 t} \cdot e^{j\pi/2} - \frac{1}{2j} e^{-j2\omega_0 t} \cdot e^{-j\pi/2}$$

(1)

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j \sin\frac{\pi}{2} = j$$

$$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j \sin\frac{\pi}{2} = -j$$

$$\Rightarrow x(t) = \frac{1}{2} + e^{j\omega_0 t} + e^{-j\omega_0 t} - 2e^{j5\omega_0 t} - 2e^{-j5\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t}$$ (1)

$\Rightarrow$ $\boxed{a_0 = \frac{1}{2}}$ (1), $\boxed{a_1 = a_{-1} = 1}$ (1), $\boxed{a_2 = a_{-2} = \frac{1}{2}}$ (1), $\boxed{a_5 = a_{-5} = -2}$ (1)

For $k \neq \pm 1, \pm 2, \pm 5$, $a_k = 0$ (1)

Q4. Use the integration (or differentiation) property of continuous-time Fourier series to determine the Fourier series coefficients b_k of the periodic signal $x(t)$:

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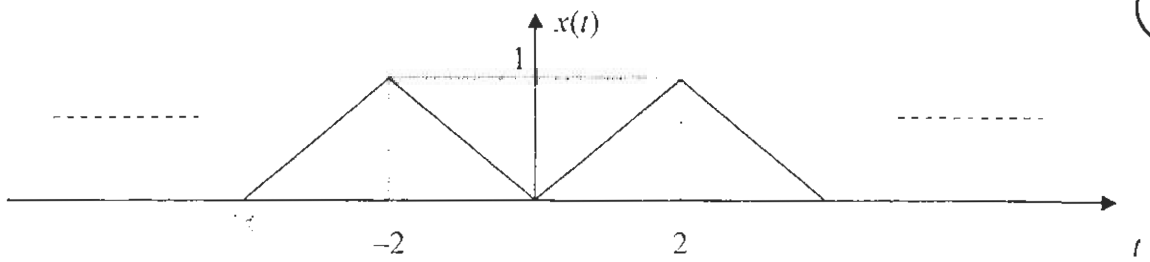
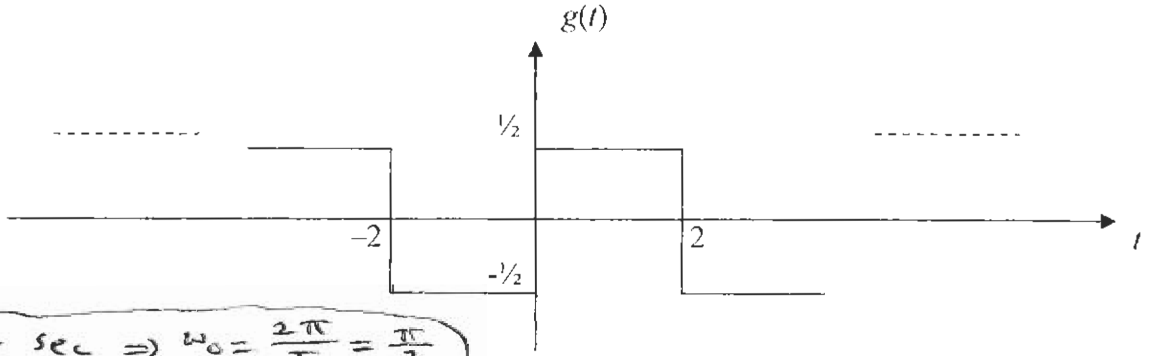


Note that the Fourier series coefficients of the square signal $g(t)$ shown below are:

$$a_k = \begin{cases} \dfrac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} & , for\ k \neq 0 \\ \dfrac{1}{2} & , for\ k = 0 \end{cases}$$



$T = 4$ sec $\Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$

$$x(t) = \begin{cases} \frac{1}{2}t & , 0 \le t \le 2 \\ -\frac{1}{2}t & , -2 \le t \le 0 \end{cases} \quad (1) \qquad g(t) = \begin{cases} \frac{1}{2} & , 0 \le t \le 2 \\ -\frac{1}{2} & , -2 \le t \le 0 \end{cases} \quad (1)$$

$$x(t) = \int_T g(t)\,dt \quad \text{or} \quad g(t) = \frac{dx(t)}{dt} \quad (1)$$

Integration property:

if $g(t) \xleftrightarrow{FS} a_k$

$\Rightarrow x(t) \longleftrightarrow b_k = \frac{1}{jk\omega_0} a_k$ (1)

$\uparrow$

$\int_T g(t)\,dt$

(1)

$$\Rightarrow b_k = \frac{1}{jk\omega_0} \cdot \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} = \frac{1}{jk(\frac{\pi}{2})} \cdot \frac{\sin(k\pi/2)}{k\pi} e^{-jk\frac{\pi}{2}} = \frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\frac{\pi}{2}}, \text{ for } k \neq 0$$

$$b_0 = \frac{1}{T}\int_{-2}^{2} x(t)\,dt = \frac{1}{4}\int_{-2}^{0} -\frac{1}{2}t\,dt + \frac{1}{4}\int_{0}^{2} \frac{1}{2}t\,dt$$

$$= -\frac{1}{8}\left[\frac{t^2}{2}\right]_{-2}^{0} + \frac{1}{8}\left[\frac{t^2}{2}\right]_{0}^{2}$$

$$= -\frac{1}{8}(0-2) + \frac{1}{8}(2-0)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad 1$$

$$\Rightarrow b_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{2\sin(k\pi/2)}{j(k\pi)^2} e^{-jk(\pi/2)} & k \neq 0 \end{cases}$$

Q5. The continuous-time Fourier transform pair of equations are: 2
(i.e. the *inverse Fourier transform* $x(t)$ and the *Fourier transform* $X(j\omega)$).

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \quad 1$$

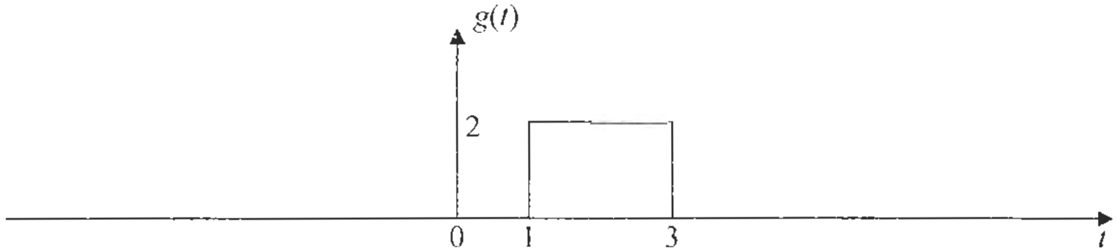
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad 1$$

Q6.

a) Determine and sketch the Fourier transform $X(j\omega)$ for each of the following signals: 18

5 1) $x(t) = e^{-at}u(t), \quad a > 0$
3 2) $x(t) = \delta(t)$
3 3) $x(t) = \cos \omega_0 t$
4 4) $x(t) = \begin{cases} 1 & ,|t|<1 \\ 0 & ,|t|>1 \end{cases}$

3 b) Determine the Fourier transform $G(j\omega)$ for the signal $g(t)$ shown below by using properties of continuous-time Fourier transform.



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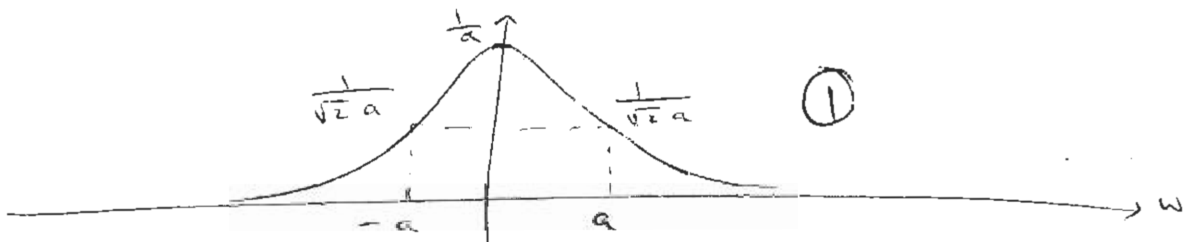
a) i) $x(t) = e^{-at}\,u(t) \quad , a > 0$

$$X(j\omega) = \int_0^{\infty} e^{-at}\, e^{-j\omega t}\, dt \quad 1 \quad = \int_0^{\infty} e^{-(a+j\omega)t}\, dt \quad 1$$

$$= -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = -\frac{1}{a+j\omega}(0-1) = \boxed{\frac{1}{a+j\omega}}$$

Complex

Magnitude: $|X(j\omega)| = \dfrac{1}{\sqrt{a^2+\omega^2}}$ $\frac{1}{2}$



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$\omega = 0 \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2}} = \frac{1}{a}$

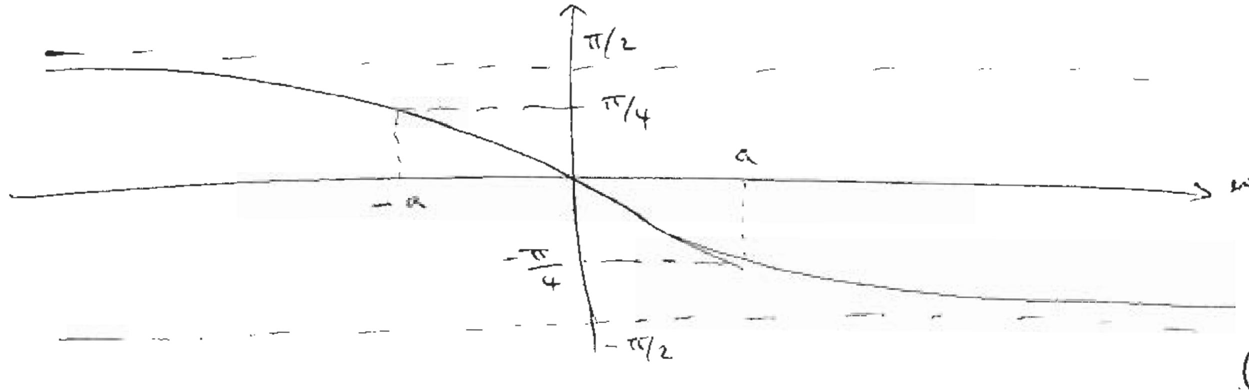
$\omega = \infty \Rightarrow |X(j\omega)| = \frac{1}{\infty} = 0$

$\omega = -\infty \Rightarrow |X(j\omega)| = \frac{1}{\infty} = 0$

$\omega = a \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}\,a}$

$\omega = -a \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{2}\,a}$

Phase: $\measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$ $\frac{1}{2}$



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$$\omega = 0 \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{0}{a}\right) = 0$$
$$\omega = a \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{a}{a}\right) = -\tan^{-1}(1) = -\pi/4$$
$$\omega = -a \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{-a}{a}\right) = -\tan^{-1}(-1) = \tan^{-1}(1) = \frac{\pi}{4}$$
$$\omega = \infty \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\infty}{a}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$$
$$\omega = -\infty \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{-\infty}{a}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

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2) $x(t) = \delta(t)$ $\frac{1}{2}$

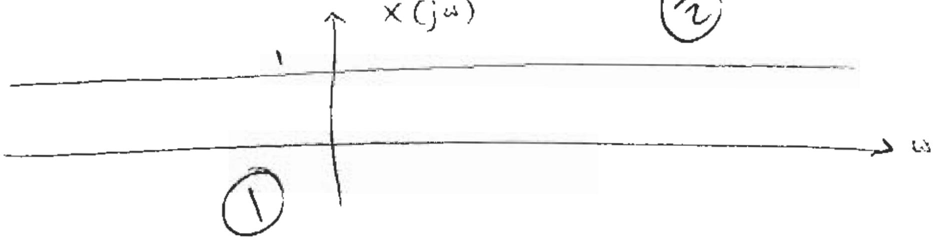
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{0}\, dt$$

since $\delta(t)\, g(t) = g(0)\, \delta(t)$ 1

If $g(t) = e^{-j\omega t} \Rightarrow \delta(t)\, g(t) = \delta(t)\, g(0) = \delta(t)\cdot 1 = \delta(t)$

$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, dt = \boxed{1} \leftarrow (\text{real})$$

$\frac{1}{2}$



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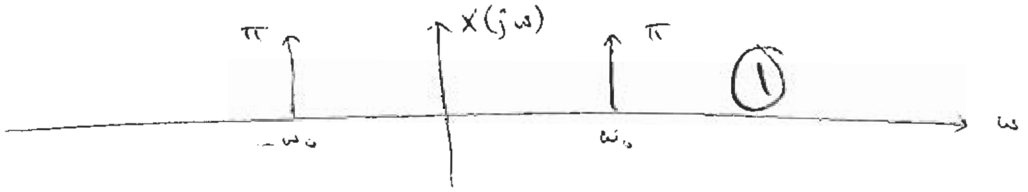
3) $x(t) = \cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$ $\frac{1}{2}$

$$F\left\{e^{j\omega_0 t}\right\} = 2\pi\,\delta(\omega - k\omega_0) \quad 1$$

$$\Rightarrow X(j\omega) = \frac{1}{2}\cdot 2\pi\,\delta(\omega - \omega_0) + \frac{1}{2}\cdot 2\pi\,\delta(\omega + \omega_0)$$

$$= \pi\,\delta(\omega - \omega_0) + \pi\,\delta(\omega + \omega_0) \quad \frac{1}{2}$$



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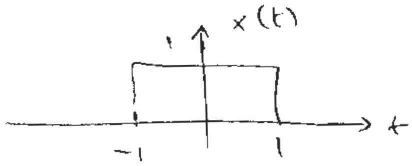
Q 6. a) 4) $x(t) = \begin{cases} 1 & |t|<1 \\ 0 & |t|>1 \end{cases}$ ← Square pulse



$\frac{1}{2}$

$$X(j\omega) = \int_{-1}^{1} 1 \cdot e^{-j\omega t}\,dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-1}^{1} = -\frac{1}{j\omega}\left(e^{-j\omega} - e^{j\omega}\right) \quad \frac{1}{2}$$

$$= \frac{1}{j\omega}\left(e^{j\omega} - e^{-j\omega}\right) = \boxed{\frac{2}{\omega}\sin\omega} \quad 1$$

↑ real

$$\boxed{\sin\theta = \frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right)}$$

$\omega = 0 \Rightarrow X(j\omega) = \frac{0}{0}$, L'Hopital rule $\Rightarrow \lim_{\omega \to 0} \frac{2\cos\omega}{1} = 2$ $\frac{1}{2}$

$$\frac{2}{\omega}\sin\omega = 2\,\text{sinc}\left(\frac{\omega}{\pi}\right)$$

$$\boxed{\text{sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}}$$

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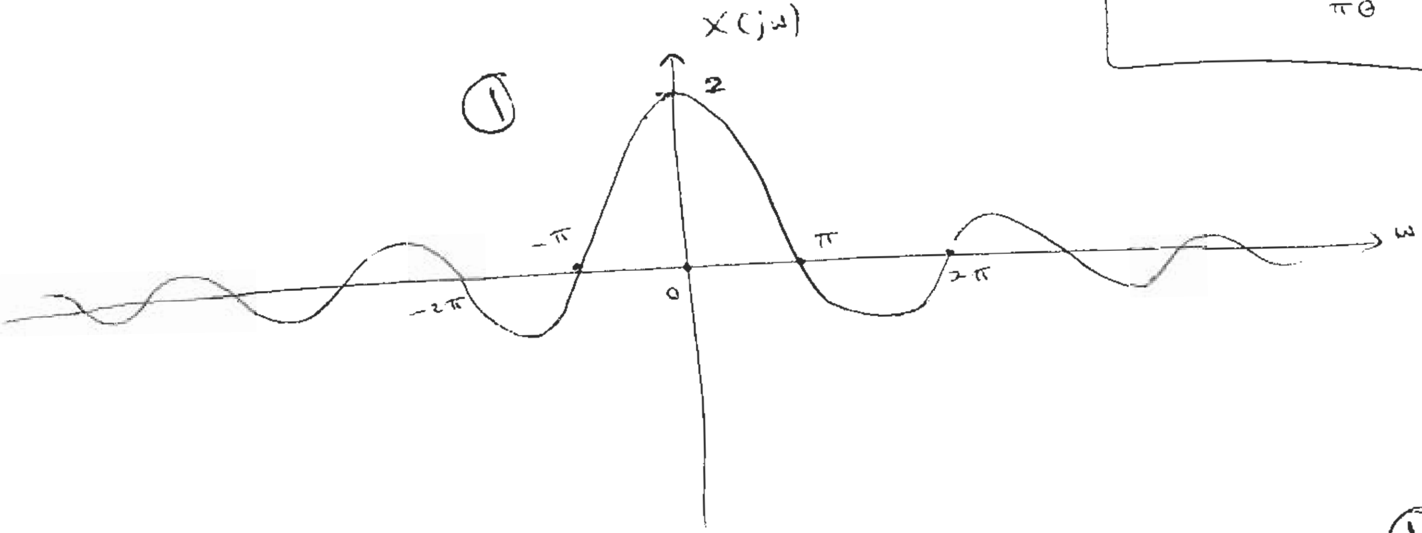

$\frac{2\sin\omega}{\omega} = 0 \Rightarrow 2\sin\omega = 0 \Rightarrow \omega = \pi$, $\omega = -\pi$ $\frac{1}{2}$

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b) $g(t) = 2\,x(t-2)$ 1 where $x(t) = \begin{cases} 1 & |t|<1 \\ 0 & |t|>1 \end{cases}$ from a)

Using shifting property

$x(t) \xleftrightarrow{F} X(j\omega)$ 1

$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$ $t_0 = 2$ in this example

$$\Rightarrow G(j\omega) = 2\,e^{-j2\omega} \cdot \frac{2}{\omega}\sin\omega = \boxed{\frac{4\sin\omega}{\omega}e^{-j2\omega}} \quad 1$$