Convergence of the Fourier series:

Is Fourier series valid for all periodic signals?

The answer is: No.

However, Fourier series can be used to represent an extremely large class of periodic signal (including the square wave)

Fourier series approximates the periodic signal by a linear combination of a number of harmonically related complex exponential signals. This number can be finite or infinite.

Let us assume that the number of the harmonically related complex exponentials is finite. Thus:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

If $e_N(t)$ is the error signal: $e_N(t) = x(t) - x_N(t)$

$$\Rightarrow e_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

The average energy of the error signal (over one period) is:

$$E_N = \int_T |e_N(t)|^2 \, dt$$

$$= \int_T \left| x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \right|^2 dt$$

To find the optimum value of a_k that minimizes the error E_N:

$$\frac{dE_N}{da_k} = 0 \Rightarrow a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \, dt$$ (identical to Eq. II Note 10)

- As N increases, E_N decreases.
- If $x(t)$ has a Fourier series representation, E_N approaches zero as N goes to infinity.

Dirichlet conditions for a signal to have a Fourier series representation:

(1) over any period, $x(t)$ must be absolutely integrable:

$$\int_T |x(t)| < \infty$$

For square signal:

$$|a_k| \leq \frac{1}{T} \int_T |x(t)|\, dt$$

If $\int_T |x(t)|\, dt$ is finite $\Rightarrow$ a_k will be finite

Example of a signal that is not absolutely integrable:

$$x(t) = \frac{1}{t} \qquad 0 < t \leq 1 \qquad (\text{Fig. } 3.8\ (a))$$

(2) In any finite interval of time, $x(t)$ must have a finite number of maxima (maximum values) and minima (minimum values).

Example of a signal that have infinite number of maxima and minima is:

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \qquad 0 < t \leq 1 \qquad (\text{Fig. } 3.8\ (b))$$

(3) In any finite interval of time, $x(t)$ must have a finite number of discontinuities.

Example of a signal that does not satisfy this condition is shown in Fig. 3.8 (c).

- Fig. 3.9 illustrates the convergence of the Fourier series representation of a square wave. (Please have a look)

Properties of continuous-time Fourier series:

We will use the notation:

$$x(t) \xleftrightarrow{FS} a_k$$

to relate a periodic signal $x(t)$ to its Fourier series coefficients.

(1) Linearity:

$$\left.\begin{array}{ll} \text{Let} & x(t) \xleftrightarrow{FS} a_k \\ \text{and} & y(t) \xleftrightarrow{FS} b_k \end{array}\right\} \text{both with period } T$$

$$\Rightarrow z(t) = A\,x(t) + B\,y(t) \xleftrightarrow{FS} c_k = A\,a_k + B\,b_k$$

Meaning: the Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$ are given by the same linear combination of the Fourier series coefficients for $x(t)$ and $y(t)$.

(2) Time-shifting:

Let $x(t) \overset{FS}{\longleftrightarrow} a_k$

$$\Rightarrow \quad x(t-t_0) \overset{FS}{\longleftrightarrow} b_k = e^{-jk\omega_0 t_0} a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

Meaning: when a periodic signal $x(t)$ is shifted in time, the magnitudes of its Fourier series coefficients are not affected:

$$|b_k| = |a_k|$$

Only the phase of a_k will be affected (shifted by $-\omega_0 t_0$).

(3) Time-reversal:

Let $x(t) \overset{FS}{\longleftrightarrow} a_k$

$$\Rightarrow \quad x(-t) \overset{FS}{\longleftrightarrow} b_k = a_{-k}$$

If $x(t)$ is even; $x(-t) = x(t) \Rightarrow a_{-k} = a_k$

If $x(t)$ is odd; $x(-t) = -x(t) \Rightarrow a_{-k} = -a_k$

(4) Time-scaling:

This affects the period (and frequency) of the signal.

If $x(t)$ has the period T and fundamental frequency ω_0.

$\Rightarrow \quad x(\alpha t)$ is periodic with period $\frac{T}{\alpha}$ and frequency $\alpha\omega_0$

$\uparrow$

Positive real number

$$\Rightarrow \quad x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

(5) Multiplication:

Let $x(t) \xleftrightarrow{FS} a_k$

and $y(t) \xleftrightarrow{FS} b_k$

$$\Rightarrow \quad x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell \, b_{k-\ell}$$

(6) Conjugation & conjugate symmetry:

Let $x(t) \xleftrightarrow{FS} a_k$

$$\Rightarrow \quad x^*(t) \xleftrightarrow{FS} a^*_{-k}$$

If $x(t)$ is real; $x(t) = x^*(t)$:

$$\Rightarrow \left.\begin{aligned} a_k &= a^*_{-k} \\ Re\{a_k\} &= Re\{a_{-k}\} \\ Im\{a_k\} &= -Im\{a_{-k}\} \\ |a_k| &= |a_{-k}| \\ \measuredangle a_k &= -\measuredangle a_k \end{aligned}\right\}$$

the Fourier series coefficients are conjugate symmetric

(7) Differentiation:

Let $x(t) \xleftrightarrow{FS} a_k$

$$\Rightarrow \quad \frac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k = jk\left(\frac{2\pi}{T}\right) a_k$$

(8) Integration:

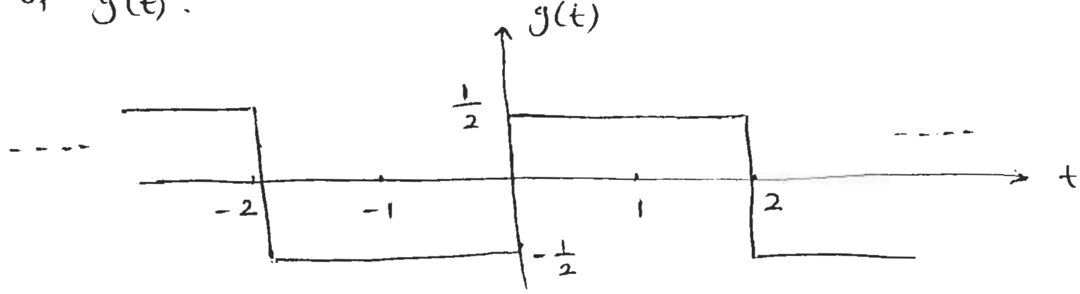
Let $x(t) \xleftrightarrow{FS} a_k$

$$\Rightarrow \quad \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{FS} \left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$$

For the complete list of Fourier series properties, see Table 3.1 (Page 206)

Example 3.6:

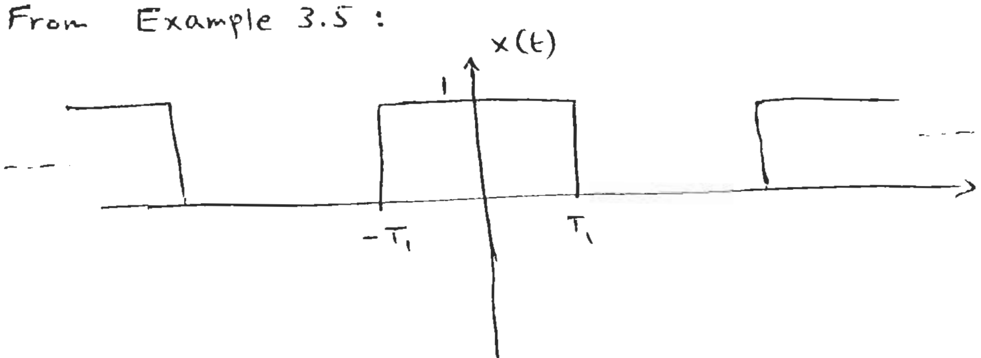
Consider the signal $g(t)$ with a fundamental period of 4, shown in the figure. Use the Fourier series properties to determine the coefficients of the Fourier series representation of $g(t)$.



Solution:

$T = 4$ sec. $\Rightarrow w_o = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$ rad/sec.

From Example 3.5 :



$$\boxed{a_o = \frac{2T_1}{T}} \quad , \quad \boxed{a_k = \frac{2\sin(kw_oT_1)}{kw_oT} = \frac{\sin(kw_oT_1)}{k\pi} \ , \ k \neq 0}$$

In our example, $T_1 = 1$, and $g(t)$ is shifted to the right by 1, and shifted down by $\frac{1}{2}$.

$\Rightarrow g(t) = x(t-1) - \frac{1}{2}$ $\qquad g(t) \xleftrightarrow{FS} d_k$

Using the time-shifting property:

$$x(t-1) \xleftrightarrow{FS} e^{-jk\omega_0} a_k = e^{-jk\frac{\pi}{2}} a_k \qquad \text{for } k \neq 0$$

$$\underbrace{g(t)}_{\downarrow FS \; d_k} = \underbrace{x(t-1)}_{\downarrow FS \; b_k} - \underbrace{\tfrac{1}{2}}_{\downarrow FS \; c_k}$$

$$b_k = \begin{cases} e^{-jk\frac{\pi}{2}} a_k, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases} = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\frac{\pi}{2}}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0 \\ -\frac{1}{2}, & \text{for } k = 0 \end{cases}$$

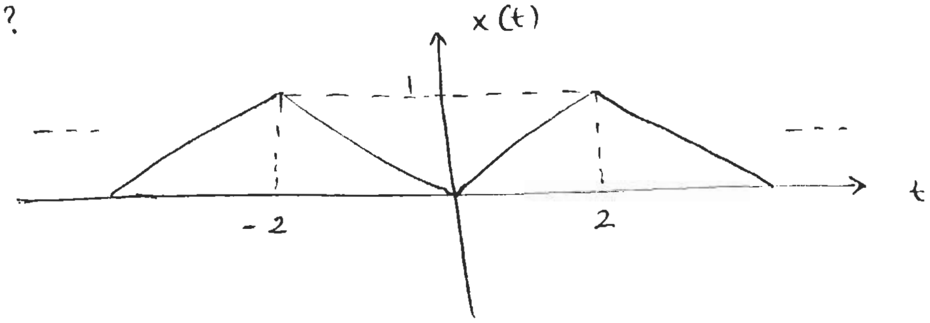
Using linearity property:

$$d_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

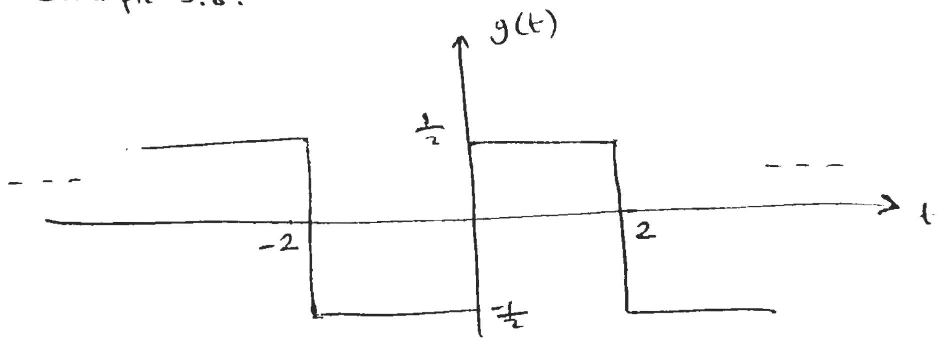
$$\Rightarrow d_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

Example 3.7:

Consider the triangular signal $x(t)$ with period $T = 4$ and fundamental frequency $w_0 = \pi/2$. Determine the Fourier series coefficients of $x(t)$?



From Example 3.6:



Note that:

$$x(t) = \begin{cases} \frac{1}{2}t & 0 \le t \le 2 \\ -\frac{1}{2}t & -2 \le t \le 0 \end{cases}$$

$$g(t) = \begin{cases} \frac{1}{2} & 0 \le t \le 2 \\ -\frac{1}{2} & -2 \le t \le 0 \end{cases}$$

$$\Rightarrow \quad x(t) = \int_T g(t)\,dt \qquad \text{or} \qquad g(t) = \frac{dx(t)}{dt}$$

Using the integration property:

If $\quad g(t) \xleftrightarrow{FS} d_k$

and $\quad x(t) \xleftrightarrow{FS} e_k$

$$\Rightarrow \quad e_k = \left(\frac{1}{jk\omega_o}\right) d_k$$

$$= \left(\frac{1}{jk(\pi/2)}\right) \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2} \quad , \text{ for } k \neq 0$$

$$= \frac{2 \sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2} \quad , \text{ for } k \neq 0$$

To find e_o :

$$e_o = \frac{1}{T}\int_{-2}^{2} x(t)\, dt = \frac{1}{4} \int_{-2}^{0} -\frac{1}{2} t\, dt + \frac{1}{4} \int_{0}^{2} \frac{1}{2} t\, dt$$

$$= -\frac{1}{8}\left[\frac{t^2}{2}\right]_{-2}^{0} + \frac{1}{8}\left[\frac{t^2}{2}\right]_{0}^{2}$$

$$= -\frac{1}{8}(0-2) + \frac{1}{8}(2-0)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \quad e_k = \begin{cases} \dfrac{2 \sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2} & , \text{ for } k \neq 0 \\ \dfrac{1}{2} & , \text{ for } k = 0 \end{cases}$$

Note that you can also use the differentiation property:

$$y(t) = \frac{dx(t)}{dt}$$

$$d_k = jk\omega_o\, e_k = jk(\pi/2)\, e_k \quad \Rightarrow e_k = \frac{2}{jk\pi} d_k$$

$$\Rightarrow e_k = \frac{2 \sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2} \quad , \text{ for } k \neq 0$$