The Continuous-Time Fourier Transform:

Introduction:

When the signal is aperiodic, the period $\rightarrow \infty$. This simply means that the fundamental frequency w_0 becomes very small $(\rightarrow 0)$ and hence $2w_0$, $3w_0$, ---, Nw_0 become very close to each other. This results in a continuum range of frequencies. Therefore, the Fourier series sum becomes an integral.

The resulting spectrum of coefficients is called the "Fourier transform". The synthesis integral itself, which uses the coefficients to represent the signal as a linear combination of complex exponential signals, is called the "inverse Fourier transform".

Representation of an aperiodic signals:

The continuous-time Fourier transform:

The Fourier transform pair:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw \quad \text{-----} (I)$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \quad \text{------} (II)$$

Eq.(I) is called :

- The synthesis equation
- The inverse Fourier transform

Eq.(II) is called :

- The Fourier transform of $x(t)$
- The Fourier integral of $x(t)$
- The spectrum of $x(t)$

$X(j\omega)$ provides information needed to describe $x(t)$ as a linear combination of complex exponentials.

Convergence of Fourier transform:

Same as with Fourier series, there are three conditions for an aperiodic signal to have a Fourier transform representation.

Dirichlet conditions:

(1) $x(t)$ must be absolutely integrable :

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

(2) $x(t)$ must have a finite number of maxima & minima within any finite interval.

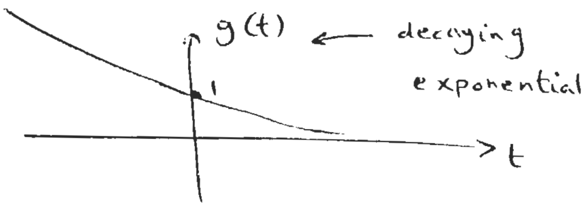
(3) $x(t)$ must have a finite number of discontinuities within any finite interval.

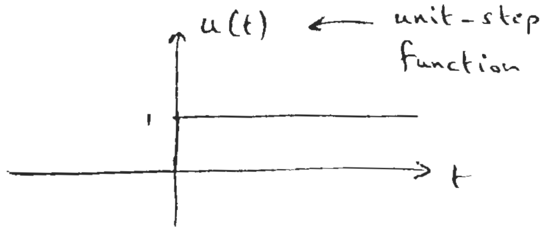
Determination of the continuous-time Fourier transform:

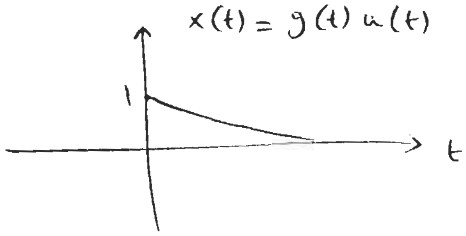
Example: 4.1:

$$x(t) = e^{-at}\, u(t) \qquad a > 0$$

Assume $g(t) = e^{-at}$ $\qquad a > 0$





$\Rightarrow$ 

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty}$$

$$= -\frac{1}{a+j\omega}(0-1) = \frac{1}{a+j\omega}$$

$$\Rightarrow \quad e^{-at}\, u(t) \xleftrightarrow{F} \frac{1}{a+j\omega} \qquad a > 0$$

$X(j\omega) = \dfrac{1}{a+j\omega}$ is a complex number. So it has a magnitude & phase angle.

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

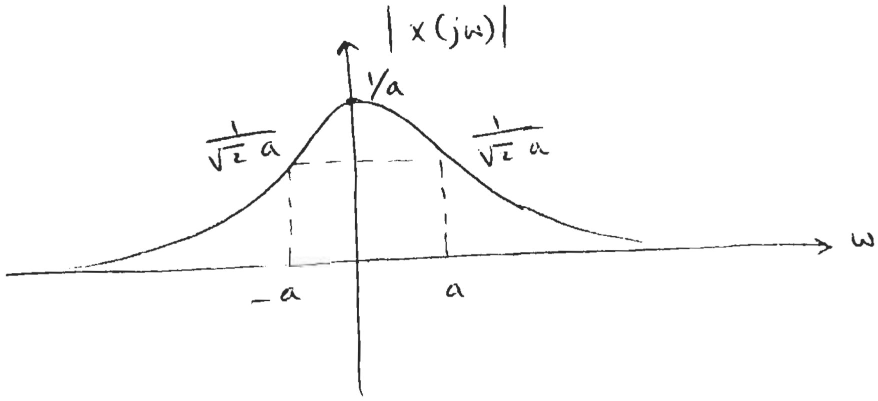
Because: when $z = x + jy \Rightarrow |z| = \sqrt{x^2+y^2}$

but when $z = \frac{1}{x+jy} = \frac{1}{x+jy}\left(\frac{x-jy}{x-jy}\right)$

$$= \frac{x-jy}{x^2+y^2}$$

$$\Rightarrow |z| = \frac{1}{x^2+y^2}\sqrt{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}$$

We sketch $|X(j\omega)|$ as:



for $\omega = 0 \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2}} = \frac{1}{a}$

for $\omega = \infty \Rightarrow |X(j\omega)| = \frac{1}{\infty} = 0$

for $\omega = -\infty \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2+(-\infty)^2}} = \frac{1}{\infty} = 0$

for $\omega = a \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2+a^2}} = \frac{1}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}\,a}$

for $\omega = -a \Rightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2+(-a)^2}} = \frac{1}{\sqrt{2a^2}} = \frac{1}{\sqrt{2}\,a}$

$$\measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

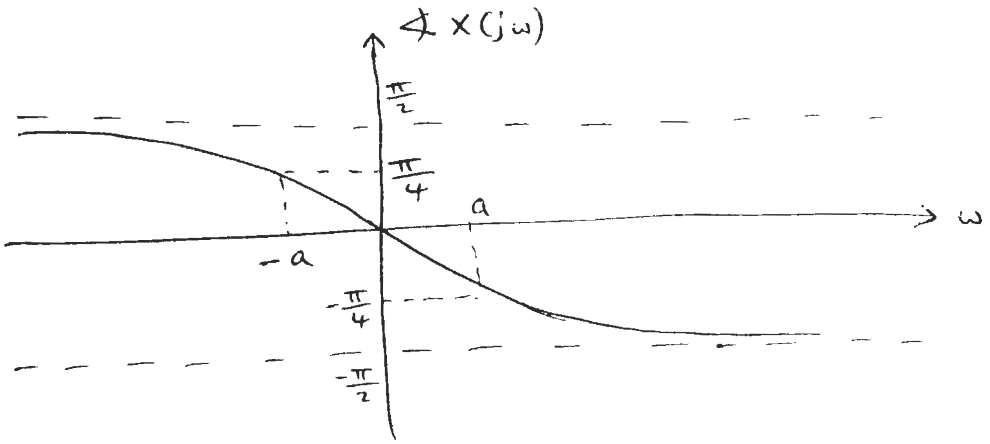
Because: when $z = x + jy \Rightarrow \measuredangle z = \tan^{-1}\left(\frac{y}{x}\right)$

but when $z = \dfrac{1}{x+jy}$

$$\Rightarrow \measuredangle z = \tan^{-1}\left(\frac{-\frac{y}{x^2+y^2}}{\frac{x}{x^2+y^2}}\right) = \tan^{-1}\left(\frac{-y}{x}\right)$$

$$= -\tan^{-1}\left(\frac{y}{x}\right)$$

We scketch $\measuredangle X(j\omega)$ as:



for $\omega = 0 \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{0}{a}\right) = 0^\circ$

for $\omega = a \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{a}{a}\right) = -\tan^{-1}(1) = -\frac{\pi}{4}$

for $\omega = -a \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(-\frac{a}{a}\right) = -\tan^{-1}(-1)$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

for $\omega = \infty \Rightarrow \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\infty}{a}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$

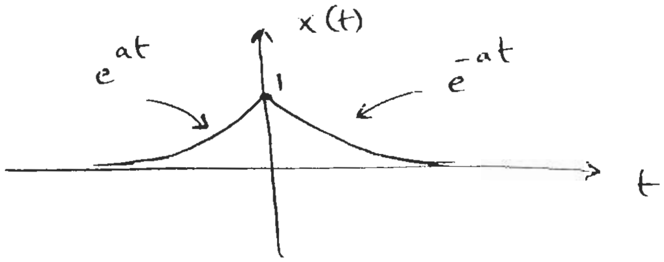
remember that: $\tan\left(-\frac{\pi}{2}\right) = \frac{-1}{0} = -\infty$

for $\omega = -\infty \Rightarrow \measuredangle X(j\omega) = -\tan\left(\frac{-\infty}{a}\right) = -\tan^{-1}(-\infty) = \tan^{-1}(\infty) = \frac{\pi}{2}$

Example : 4.2 :

$$x(t) = e^{-a|t|} \qquad a > 0$$

$$\Rightarrow \quad x(t) = \begin{cases} e^{at} & t < 0 \\ e^{-at} & t \geq 0 \end{cases}$$



remember that at $t = 0 \Rightarrow x(t) = e^{-0} = 1$

at $t = \infty \Rightarrow x(t) = e^{-\infty} = 0$

at $t = -\infty \Rightarrow x(t) = e^{-\infty} = 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

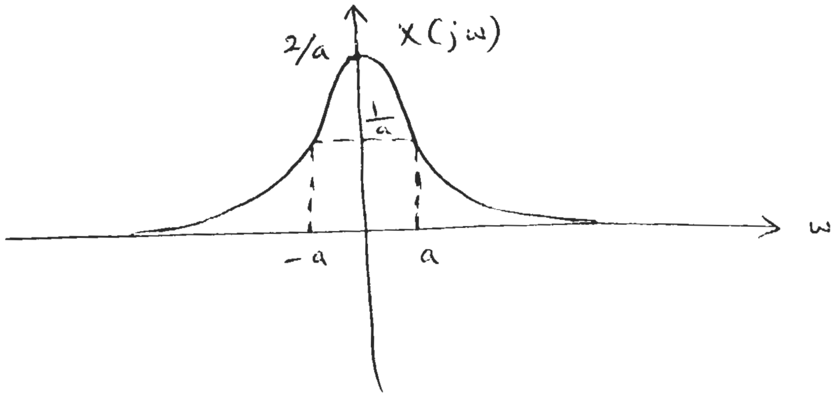
$$= \frac{1}{a-j\omega}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} - \frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_{0}^{\infty}$$

$$= \frac{1}{a-j\omega}\left(1 - 0\right) - \frac{1}{a+j\omega}\left(0 - 1\right)$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a + \cancel{j\omega} + a - \cancel{j\omega}}{a^2 + \omega^2} = \frac{2a}{a^2 + \omega^2}$$

Note that $X(j\omega)$ is real, so it has no phase angle.

we sketch $X(j\omega)$ as:



for $\omega = 0 \Rightarrow X(j\omega) = \frac{2a}{a^2} = \frac{2}{a}$

for $\omega = \infty \Rightarrow X(j\omega) = \frac{2a}{\infty} = 0$

for $\omega = -\infty \Rightarrow X(j\omega) = \frac{2a}{\infty} = 0$

for $\omega = a \Rightarrow X(j\omega) = \frac{2a}{a^2 + a^2} = \frac{2a}{2a^2} = \frac{1}{a}$

for $\omega = -a \Rightarrow X(j\omega) = \frac{2a}{a^2 + (-a)^2} = \frac{1}{a}$

Example: 4.3:

$$x(t) = \delta(t)$$

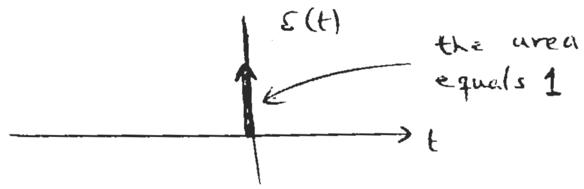
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

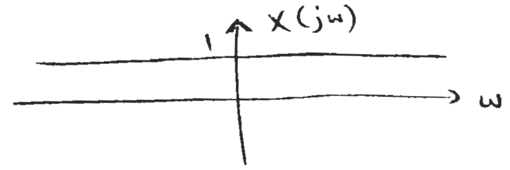
remember that: $\delta(t)\, g(t) = g(0)\, \delta(t)$. If $g(t) = e^{-j\omega t}$

$$\Rightarrow \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t) \cdot e^{0}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

remember that for the unit impulse function $\delta(t)$:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$





Example: 4.4:

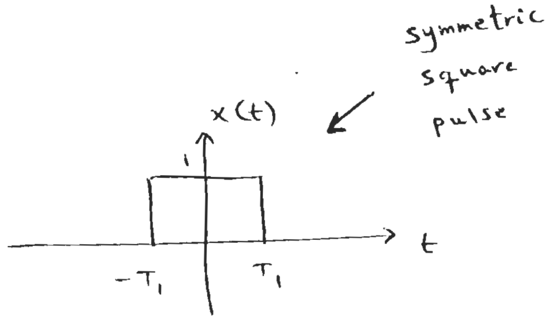
Consider the rectangular pulse:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

Find $X(j\omega)$ and sketch it?

Solution:

$$x(t) = \begin{cases} 1 & -T_1 < t < T_1 \\ 0 & \text{elsewhere} \end{cases}$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right) = \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right)$$

$$\Rightarrow X(j\omega) = \frac{2}{\omega}\sin(\omega T_1) = \frac{2\sin(\omega T_1)}{\omega} \quad \text{-------- (III)}$$

remember that: $\sin\theta = \frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right)$

$$\Rightarrow \quad x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \quad \xleftrightarrow{F} \quad X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

To sketch $X(j\omega)$: (note that $X(j\omega)$ is <u>real</u>)

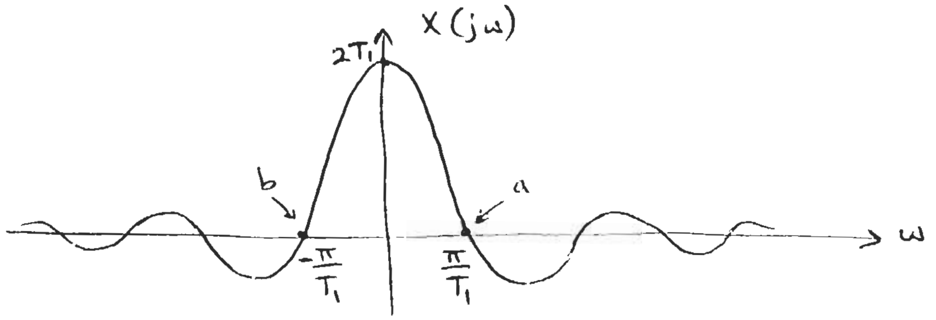
for $\omega = 0 \Rightarrow X(j\omega) = \frac{2\sin 0}{0} = \frac{0}{0}$

so we use Hopital's rule:

$$\boxed{\text{If } \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty \quad \Rightarrow \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}}$$

$$\Rightarrow X(j\omega) = \lim_{\omega \to 0} \frac{2T_1 \cos(\omega T_1)}{1} = 2T_1$$

since $X(j\omega) = \frac{2}{\omega} \cdot \sin(\omega T_1) \Rightarrow$ it can be sketched as:



for $\omega = 0 \Rightarrow X(j\omega) = 2T_1$. To find a & b:

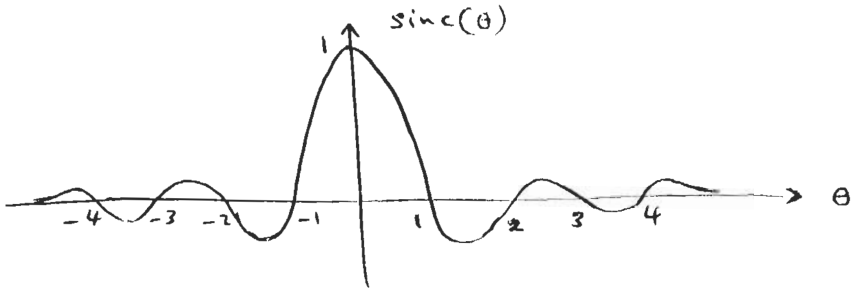
$\frac{2\sin(\omega T_1)}{\omega} = 0 \Rightarrow 2\sin(\omega T_1) = 0$ for $\omega \neq 0$

$\Rightarrow \omega T_1 = \pi \Rightarrow \omega = \frac{\pi}{T_1}$ (this is a)

also $\omega T_1 = -\pi \Rightarrow \omega = -\frac{\pi}{T_1}$ (this is b)

In the study of Fourier and LTI systems, an important function that appears frequently is the sinc function. The sinc function can be defined as:

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta} \qquad \text{-------(IV)}$$



The signal in Eq.(III) can be expressed in terms of the sinc function as:

$$\frac{2\sin(\omega T_1)}{\omega} = \frac{2\sin\left(\pi \frac{\omega T_1}{\pi}\right)}{\frac{\pi}{T_1} \; \frac{\omega T_1}{\pi}} = 2T_1 \,\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

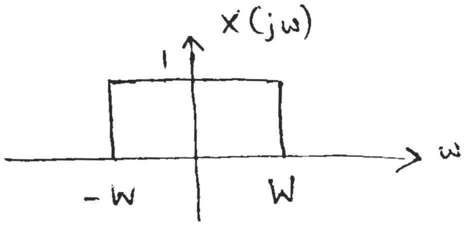
Example: 4.5:

Consider the signal $x(t)$ whose Fourier transform is:

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

Find $x(t)$?

Solution:



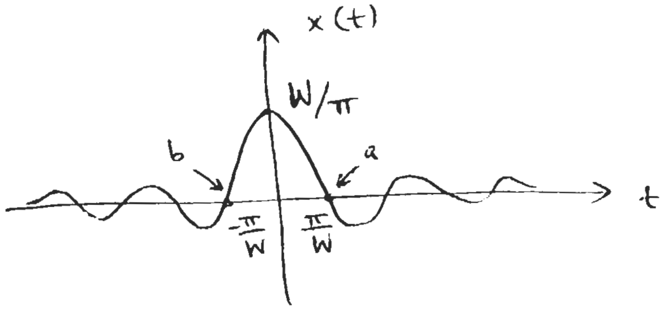
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\cdot\frac{1}{jt}\left[e^{j\omega t}\right]_{-W}^{W}$$

$$= \frac{1}{j2\pi t}\left(e^{jWt} - e^{-jWt}\right)$$

$$\Rightarrow \quad x(t) = \frac{1}{\pi t}\sin Wt = \frac{\sin Wt}{\pi t} \quad \text{------- (V)}$$

$x(t)$ can be sketched as:



for $w = 0 \Rightarrow x(t) = \frac{\sin 0}{0} = \frac{0}{0}$

using Hopital's rule: $w(t) = \lim_{t\to 0} \frac{W\cos Wt}{\pi} = \frac{W}{\pi}$

To find a and b:

$\frac{\sin Wt}{\pi t} = 0 \Rightarrow \sin Wt = 0 \quad$ for $t \neq 0$

$\Rightarrow Wt = \pi \Rightarrow t = \frac{\pi}{W}$ (this is a)

also $Wt = -\pi \Rightarrow t = -\frac{\pi}{W}$ (this is b)

The signal in Eq. (V) can be expressed in terms of the sinc function as:

$$\frac{\sin Wt}{\pi t} = \frac{\sin\left(\pi \frac{Wt}{\pi}\right)}{\pi t \frac{W}{\pi}\ \frac{\pi}{W}} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

From Examples 4.4 & 4.5, we can conclude that if $x(t)$ is square pulse, its Fourier transform is sinc function. Also, if $X(j\omega)$ is square pulse, its inverse Fourier transform is sinc function. This special relationship is a result of the duality property for Fourier transforms. (See Table 4.2, Page 329)

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \quad \xleftrightarrow{F} \quad \frac{2\sin\omega T_1}{\omega}$$

$$\frac{\sin Wt}{\pi t} \quad \xleftrightarrow{F} \quad X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

For the square pulse $X(j\omega)$ in Example 4.5:

As W increases $\Rightarrow$ $\frac{W}{\pi}$ increases but $\frac{\pi}{W}$ decreases.

This means that the main peak of $x(t)$ at $t=0$ becomes higher and the width of the first lobe (the part for $|t| < \pi/W$) becomes narrower. (See Fig. 4.11, Page 296)

A result of this is that when $W \to \infty$ (dc function), $x(t)$ becomes a unit impulse signal. Because of duality property:

$$x(t) = 1 \quad \xleftrightarrow{F} \quad 2\pi\,\delta(\omega)$$

$$\delta(t) \quad \xleftrightarrow{F} \quad 1$$

The Fourier transform for periodic signals:

Remember that:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

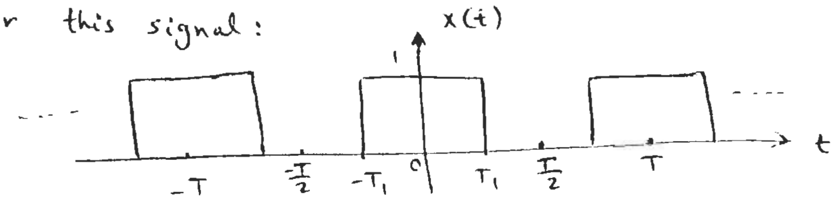
The Fourier transform of the periodic signal $x(t)$ is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0) \quad \text{--------- (VI)}$$

Periodic signal $x(t)$	$\xrightarrow[\text{Fourier series}]{FS}$	$x(t) = \sum$	$\xrightarrow[\text{Fourier transform}]{F}$	$X(j\omega) = \sum$
Aperiodic signal $x(t)$	$\xrightarrow[\text{Fourier transform}]{F}$	$X(j\omega)$		

Example: 4.6:

Consider this signal:



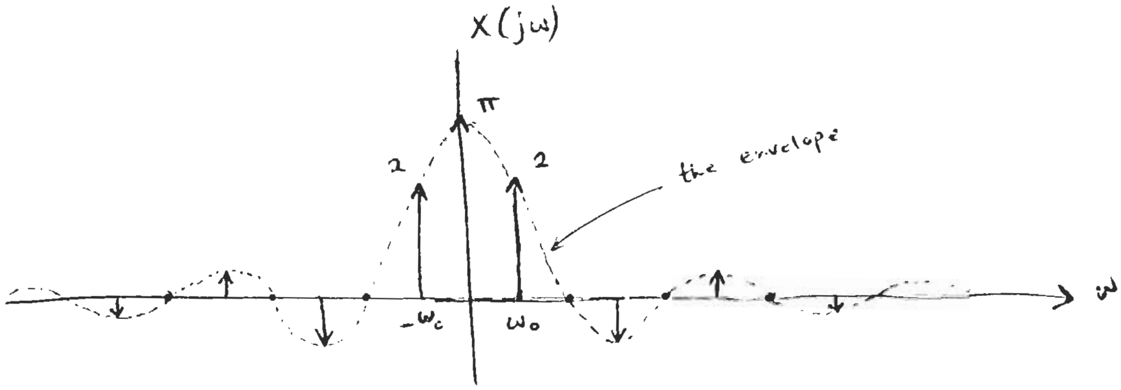
Remember that for this periodic square signal:

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

$$\Rightarrow \quad X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \frac{\sin k\omega_0 T_1}{\pi k} \cdot \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \, \delta(\omega - k\omega_0)$$

For $T = 4T_1$, the signal $X(j\omega)$ can be sketched as:



Note that $X(j\omega)$ is a train of pulses with different areas. The envelop of these pulses is a sinc function.

From Eq. (VI), we can conclude that the Fourier transform of a complex exponential signal $x(t) = e^{j\omega_0 t}$ is an impulse signal. To prove this:

assume that $X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$, Find $x(t)$?

This is an impulse signal with area equals 2π and it is shifted to the right by ω_0

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

remember that $\delta(\omega - \omega_0) = 0$ for $\omega \neq \omega_0$

and $\delta(\omega - \omega_0)\cdot g(\omega) = g(\omega_0)\cdot\delta(\omega - \omega_0)$

and $\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, d\omega = 1$

$$\Rightarrow\ x(t) = e^{j\omega_0 t}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)\, d\omega = e^{j\omega_0 t}$$

$$\Rightarrow \quad \boxed{\begin{array}{lll} e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\,\delta(\omega-\omega_0) & \text{-------} & (\text{VII}) \\ \delta(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} & & (\text{because of the } \underline{\text{duality}}) \end{array}}$$

<u>Example: 4.7:</u>

Find the Fourier transform for the signals:

$x_1(t) = \sin \omega_0 t$

$x_2(t) = \cos \omega_0 t$

<u>Solution:</u>

$$x_1(t) = \sin \omega_0 t = \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

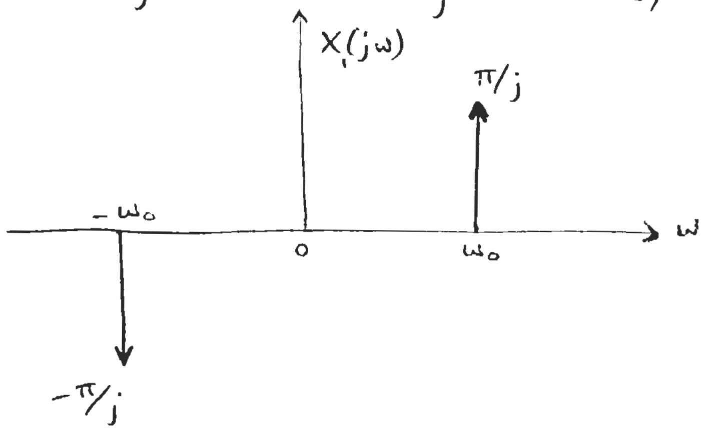
$$= \frac{1}{j2} e^{j\omega_0 t} - \frac{1}{j2} e^{-j\omega_0 t}$$

$$F\left\{\frac{1}{j2} e^{j\omega_0 t}\right\} = \frac{2\pi}{j2}\delta(\omega-\omega_0) = \frac{\pi}{j}\delta(\omega-\omega_0) \qquad \text{From Eq. (VII)}$$

↑ Fourier transform

$$F\left\{-\frac{1}{j2} e^{-j\omega_0 t}\right\} = -\frac{2\pi}{j2}\delta(\omega+\omega_0) = -\frac{\pi}{j}\delta(\omega+\omega_0) \qquad \text{From Eq. (VII)}$$

$$\Rightarrow X_1(j\omega) = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$$

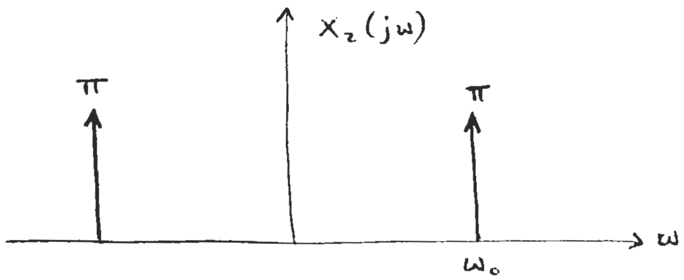


$$x_2(t) = \cos w_o t = \frac{1}{2}\left(e^{jw_o t} + e^{-jw_o t}\right)$$

$$F\left\{\frac{1}{2} e^{jw_o t}\right\} = \frac{2\pi}{2}\,\delta(\omega - \omega_o) = \pi\,\delta(\omega - \omega_o)$$

$$F\left\{\frac{1}{2} e^{-jw_o t}\right\} = \frac{2\pi}{2}\,\delta(\omega + \omega_o) = \pi\,\delta(\omega + \omega_o)$$

$$\Rightarrow \quad X_2(j\omega) = \pi\,\delta(\omega - \omega_o) + \pi\,\delta(\omega + \omega_o)$$



- In the book (page 298), the solution to this example is slightly different, please have a look at it.

Properties of the continuous-time Fourier transform:

Notations:

$$x(t) \xleftrightarrow{F} X(j\omega)$$

$\uparrow$ inverse Fourier transform

$\uparrow$ Fourier transfom

$$X(j\omega) = F\{x(t)\}$$

$$x(t) = F^{-1}\{X(j\omega)\}$$

For example:

$$F\{e^{-a} u(t)\} = \frac{1}{a + j\omega}$$

$$F^{-1}\left\{\frac{1}{a+j\omega}\right\} = e^{-at} u(t)$$

(1) Linearity:

Let $x(t) \xleftrightarrow{F} X(j\omega)$

and $y(t) \xleftrightarrow{F} Y(j\omega)$

$$\Rightarrow \quad a\,x(t) + b\,y(t) \xleftrightarrow{F} a\,X(j\omega) + b\,Y(j\omega)$$

(2) Time-shifting:

Let $x(t) \xleftrightarrow{F} X(j\omega)$

$$\Rightarrow \quad x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$$

No effect on the magnitude of $X(j\omega)$

Only the phase is shifted by $-\omega_0 t$

This means that:

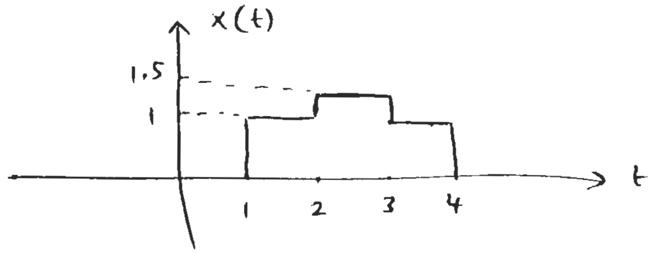
$$F\{x(t)\} = X(j\omega) = |X(j\omega)|\, e^{j \measuredangle X(j\omega)}$$

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)|\, e^{j[\measuredangle X(j\omega) - \omega t_0]}$$

The magnitude is the same in both cases.

Example: 4.9:

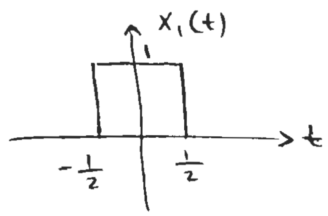
Consider the signal $x(t)$ shown in the Figure

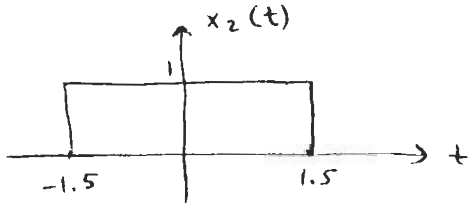


Use the linearity and time-shifting properties to find the Fourier transform of $x(t)$?

Solution:

Let us decompose the signal $x(t)$ into two simpler signals:





$$x(t) = \frac{1}{2} x_1(t-2.5) + x_2(t-2.5)$$

$$X(j\omega) = \frac{1}{2} e^{-j\frac{5\omega}{2}} X_1(j\omega) + e^{-j\frac{5\omega}{2}} X_2(j\omega)$$

remember that: $X_1(j\omega) = \dfrac{2\sin(\omega/2)}{\omega}$ $\quad (T_1 = \frac{1}{2})$

$X_2(j\omega) = \dfrac{2\sin(3\omega/2)}{\omega}$ $\quad (T_1 = \frac{3}{2})$

$$\Rightarrow X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

(3) Conjugation and conjugate symmetry:

Let $x(t) \xleftrightarrow{F} X(j\omega)$

$$\Rightarrow x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

If $x(t)$ is real, then $X(j\omega)$ has conjugate symmetry

$$\Rightarrow X(-j\omega) = X^*(j\omega)$$

$$Re\{X(j\omega)\} = Re\{X(-j\omega)\}$$

$$Im\{X(j\omega)\} = -Im\{X(j\omega)\}$$

(4) Differentiation and integration:

Let $x(t) \xleftrightarrow{F} X(j\omega)$

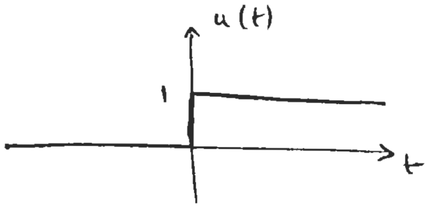
$$\Rightarrow \frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

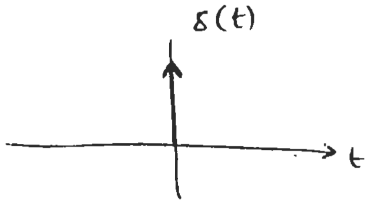
and $$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega)$$

Example: 4.11:

What is the Fourier transform $X(j\omega)$ of the unit-step $x(t) = u(t)$? Use the integration property of the Fourier transform.

Solution:





Recall that:

$$\frac{du}{dt} = \delta(t) \quad \text{and} \quad \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

$$g(t) = \delta(t) \xleftrightarrow{F} G(j\omega) = 1$$

$$x(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \xleftrightarrow{F} \frac{1}{j\omega} + \pi\, G(0)\, \delta(\omega)$$

$$= \frac{1}{j\omega} + \pi\, \delta(\omega)$$

$$= X(j\omega)$$

$$\Rightarrow \quad u(t) \xleftrightarrow{F} \frac{1}{j\omega} + \pi\, \delta(\omega)$$

If you want to find $G(j\omega)$ from $X(j\omega)$, you can use the differentiation property as:

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{F} j\omega\, X(j\omega)$$

$$= j\omega \left[\frac{1}{j\omega} + \pi\, \delta(\omega) \right]$$

$$= 1 + j\pi\omega\, \delta(\omega) = 1$$

remember that $\omega\, \delta(\omega) = 0$ because $\delta(\omega) = 0$ for $\omega \neq 0$

(5) Time and Frequency scaling:

Let $x(t) \xleftrightarrow{F} X(j\omega)$

$$\Rightarrow \quad x(at) \xleftrightarrow{F} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

a is nonzero real number

A special case: when $a = -1$

$$\Rightarrow \quad x(-t) \xleftrightarrow{F} X(-j\omega)$$

Meaning: reversing a signal in time will also reverse its Fourier transform in frequency

(6) Duality: (see Fig. 4.17, Page 310)

There is a dual relationship between any Fourier transform pair.

For example:

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$\Rightarrow \quad x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

Also:

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\Rightarrow \quad -jt\,x(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}$$

Also:

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\,\delta(\omega - \omega_0)$$

$$\Rightarrow \quad \delta(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0}$$

Also: $\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau = \frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega) \Rightarrow -\frac{1}{jt} x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\infty}^{\omega} x(\alpha)\,d\alpha$