Fourier Series Representation of Periodic Signals

Introduction:

Fourier series is one way to represent signals and Linear Time-Invariant (LTI) systems.

Fourier series is used to represent periodic signals as linear combinations of a set of basic signals. These basic signals are complex exponentials (or sinusoidal).

Recall:

Euler's rule: $e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$ ------ (1)

$$\Rightarrow \quad \cos \omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \text{ ------ (2)}$$

and

$$\sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] \text{ ------ (3)}$$

These relations are very important to understand how complex exponential signals relate to sinusoidal signals.

Fourier Series Representation of Continuous-time Periodic Signals:

Harmonically Related Complex Exponentials:

From Note 7, a signal is periodic if:

$$x(t) = x(t+T) \quad \text{for all } t$$

The fundamental period of $x(t)$ is the minimum positive nonzero value of T.

The fundamental frequency of $x(t)$ is $\omega_0 \Rightarrow T = \frac{2\pi}{\omega_0}$

Examples of two basic periodic signals are:

$$x_1(t) = \cos w_0 t \quad \text{(sinusoidal signal)}$$

and

$$x_2(t) = e^{jw_0 t} \quad \text{(Complex exponential signal)}$$

Both signals have the fundamental frequency of w_0.

For the complex exponential signal $x_2(t)$, there is a set of harmonically related complex exponentials:

$$\phi_k(t) = e^{jkw_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

For example:

$$\phi_0(t) = 1 \quad \text{(constant with } w = 0\text{)}$$

$$\phi_1(t) = e^{jw_0 t} \quad (w = w_0 \text{, the fundamental frequency})$$

$$\phi_2(t) = e^{j2w_0 t} \quad (w = 2w_0)$$

$$\vdots$$

$$\phi_N(t) = e^{jNw_0 t} \quad (w = Nw_0)$$

For any periodic signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \text{------ (4)}$$

Eq. (4) means that any periodic signal can be represented as a linear combination of a set of harmonically related complex exponentials.

This representation is known as <u>Fourier series</u>

Remember that:

harmonically related complex exponentials are periodic exponentials with fundamental frequencies that are all multiple of a single positive frequency w_0.

Note that harmonically related complex exponentials are orthogonal signals over any interval of duration $T = \frac{2\pi}{w_0}$ (over one period).

Let $\phi_n(t) = e^{jnw_0t} = e^{jn(2\pi/T)t}$

$\phi_m(t) = e^{jmw_0t} = e^{jm(2\pi/T)t}$, $n \neq m$

For orthogonality: $\int_0^T \phi_n(t)\,\phi_m(t)\,dt = 0$

$$\int_0^T e^{jn(2\pi/T)t} \cdot e^{jm(2\pi/T)t}\,dt = \int_0^T e^{j(n+m)(2\pi/T)t}\,dt$$

$$= \frac{1}{j(n+m)(2\pi/T)}\left[e^{j(n+m)(2\pi/T)t}\right]_0^T$$

$$= \frac{T}{j(n+m)(2\pi)}\left(e^{j2\pi(n+m)} - 1\right)$$

$$e^{j2\pi(n+m)} = \cos 2\pi(n+m) + j\sin 2\pi(n+m)$$

For any integer values of n and m,

$$\cos 2\pi(n+m) = \cos 2\pi = 1$$

and $\sin 2\pi(n+m) = \sin 2\pi = 0$

$$\Rightarrow e^{j2\pi(n+m)} = 1 \Rightarrow \int_0^T \phi_n(t)\,\phi_m(t)\,dt = 0$$

$\Rightarrow$ for any values of n and m, $\phi_n(t)$ and $\phi_m(t)$ are orthogonal

Back to Eq. (4):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

- The terms for $k = 1$, $k = -1$ have the same fundamental frequency ω_0. These terms are called the "fundamental components" or the "first harmonic components"

- The terms for $k = 2$, $k = -2$ have the same fundamental frequency $2\omega_0$ (half the period). These terms are called the "second harmonic components"

- More generally, the terms for $k = N$, $k = -N$ are called the "Nth harmonic components"

Example 3.2:

Consider a periodic signal $x(t)$, with fundamental frequency 2π, that is expressed in Fourier series representation as:

$$x(t) = \sum_{k=-3}^{3} a_k \, e^{jk2\pi t}$$

where $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$

(a) What is the fundamental period of $x(t)$

(b) Rewrite $x(t)$ as a combination of harmonic components.

(c) Simplify the expression for $x(t)$ — obtained in (b) — by collecting each harmonic components with the same fundamental frequency.

Solution:

(a) The fundamental period of $x(t)$ equals to $\frac{2\pi}{\omega_0}$. This equals $\frac{2\pi}{2\pi} = 1$ (sec).

(b)
$$x(t) = a_{-3} e^{-j6\pi t} + a_{-2} e^{-j4\pi t} + a_{-1} e^{-j2\pi t} + a_0 + a_1 e^{j2\pi t} + a_2 e^{j4\pi t} + a_3 e^{j6\pi t}$$

$$= \frac{1}{3} e^{-j6\pi t} + \frac{1}{2} e^{-j4\pi t} + \frac{1}{4} e^{-j2\pi t} + 1 + \frac{1}{4} e^{j2\pi t} + \frac{1}{2} e^{j4\pi t} + \frac{1}{3} e^{j6\pi t}$$

(c)
$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

Using Eq. (2) in Page 1:

$$\Rightarrow \quad x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t \quad \text{.....}(5)$$

Figure 3.4 in the book (Page 188) shows how the signal $x(t)$ is built up from its harmonic components.

Eq. (5) is an example of an alternative form for the Fourier series of real periodic signals. This representation is known as "trigonometric Fourier series" since it uses trigonometric functions.

- If a_k's are real (as in Example 3.2), $a_k = a_{-k}$
- If a_k's are complex, $a_k = a^*_{-k}$ or $a^*_k = a_{-k}$

For real signals:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\} \quad \text{-----------} \quad (6)$$

if a_k is real, then:

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos k\omega_0 t \quad \text{-----------} \quad (7)$$

if a_k is complex, $a_k = A_k e^{j\theta_k}$, then:

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \quad \text{-----------} \quad (8)$$

Also, for complex a_k, if $a_k = B_k + jC_k$, then:

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t] \quad \text{-----------} \quad (9)$$

Determination of the Fourier Series Representation of a Continuous-time Periodic Signal:

From Eq. (4):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

How can we determine the coefficients a_k ?

By multiplying Eq. (4) by $e^{-jn\omega_0 t}$

$$\Rightarrow \quad x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t}$$

By integrating both sides from 0 to $T = \frac{2\pi}{\omega_0}$

$$\Rightarrow \quad \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t}\, dt$$

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}\, dt \right] \quad \text{------ (10)}$$

Now,

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j \int_0^T \sin(k-n)\omega_0 t\, dt$$

For $k = n \Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T dt = T$

For $k \neq n \Rightarrow$ $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic with fundamental frequency $(k-n)\omega_0$ and period $\frac{T}{|k-n|}$.

The area under $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ over one period equals zero.

$$\Rightarrow \int_0^T e^{j(k-n)w_0 t}\, dt = \begin{cases} T & , \; k=n \\ 0 & , \; k \neq n \end{cases}$$

Back to Eq.(10):

$$\int_0^T x(t)\, e^{-jnw_0 t}\, dt = a_n T \qquad (\text{since } k=n)$$

$$\Rightarrow \quad a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jnw_0 t}\, dt$$

if we integrate over any interval of length T, we will have the same results, so:

$$a_n = \frac{1}{T} \int_T x(t)\, e^{-jnw_0 t}\, dt \quad \text{-----------} (11)$$

As a summary:

Fourier series of a periodic continuous-time signal is defined as:

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad \text{-------} (\mathrm{I}) \\ a_k &= \frac{1}{T} \int_T x(t)\, e^{-jkw_0 t}\, dt = \frac{1}{T} \int_T x(t)\, e^{-jk(2\pi/T)t}\, dt \quad \text{-------} (\mathrm{II}) \end{aligned}}$$

Eq.(I) is called "synthesis equation"

Eq.(II) is called "analysis equation"

The set $\{a_k\}$ are called "Fourier series coefficients" of $x(t)$.

$\{a_k\}$ are also called "Spectral coefficients" of $x(t)$.

These coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component.

a_0 is the dc or constant component of $x(t)$.

From Eq. (II) with $k=0$:

$$\boxed{a_0 = \frac{1}{T}\int_T x(t)\,dt \qquad \text{-----------} \quad (\text{III})}$$

Eq. (III) represents the average value of $x(t)$ over one period.

Example: (3.3)

Consider the signal: $x(t) = \sin w_0 t$ whose fundamental frequency is w_0. Determine the Fourier series coefficients for $x(t)$?

Solution:

We can use Eq. (II). But for this simple case (where $x(t)$ is sinusoidal signal), we can use Euler's rule:

$$\sin w_0 t = \frac{1}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right) \qquad \text{from Eq. (3), page 1}$$

$$= \frac{1}{2j} e^{jw_0 t} - \frac{1}{2j} e^{-jw_0 t}$$

$$\Rightarrow \quad a_0 = 0 \;,\quad a_1 = \frac{1}{2j} \;,\quad a_{-1} = -\frac{1}{2j}$$

$$a_2 = a_{-2} = 0 \;,\quad a_3 = a_{-3} = 0 \;,\quad \text{------}$$

OR: $a_k = 0$ for $k \neq 1$ or -1

Example: (3.4)

$$x(t) = 1 + \sin w_o t + 2\cos w_o t + \cos\left(2w_o t + \frac{\pi}{4}\right)$$

$x(t)$ has fundamental frequency w_o. Determine the Fourier series coefficients for $x(t)$?

Solution:

Since $x(t)$ is a combination of sinusoidal signals, we can also use Euler's rule:

$$x(t) = 1 + \frac{1}{2j}\left(e^{jw_o t} - e^{-jw_o t}\right) + \left(e^{jw_o t} + e^{-jw_o t}\right) + \frac{1}{2}\left(e^{j(2w_o t + \pi/4)} + e^{-j(2w_o t + \pi/4)}\right)$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{jw_o t} + \left(1 - \frac{1}{2j}\right)e^{-jw_o t} + \left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)e^{j2w_o t} + \left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)e^{-j2w_o t}$$

$$e^{j\pi/4} = \cos\pi/4 + j\sin\pi/4 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+j)$$

$$e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1-j)$$

$\Rightarrow$ $a_0 = 1$ $\leftarrow$ dc component

$a_1 = 1 + \frac{1}{2j} = 1 - j\frac{1}{2}$ $\leftarrow$ frequency is w_o

$a_{-1} = 1 - \frac{1}{2j} = 1 + j\frac{1}{2}$ $\leftarrow$ frequency is w_o

$a_2 = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}(1+j) = \frac{1}{2\sqrt{2}}(1+j) = \frac{\sqrt{2}}{4}(1+j)$ $\leftarrow$ freq. is $2w_o$

$a_{-2} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}(1-j) = \frac{1}{2\sqrt{2}}(1-j) = \frac{\sqrt{2}}{4}(1-j)$ $\leftarrow$ freq. is $2w_o$

Note that: a_1 and a_{-1} are the coefficients for the 1st harmonic components.
a_2 and a_{-2} " " " " " 2nd " "

Also note that :

$$a_1 = a_{-1}^*$$

$$a_2 = a_{-2}^*$$

This is because the coefficients $\{a_k\}$ here are complex.

We can rewrite $x(t)$ as:

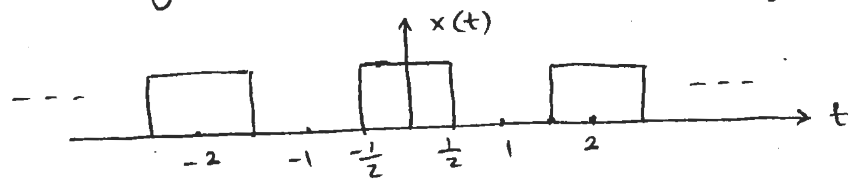
Using Eq.(9) $\Rightarrow$ $x(t) = 1 + 2\cos w_0 t + \sin w_0 t + \frac{\sqrt{2}}{2}\cos 2w_0 t - \frac{\sqrt{2}}{2}\sin 2w_0 t$

Here $B_1 = 1$, $C_1 = -\frac{1}{2}$

$B_2 = \frac{\sqrt{2}}{4}$, $C_2 = \frac{\sqrt{2}}{4}$

Example:

Consider the signal $x(t)$ shown in the figure:



Determine the Fourier series coefficients for $x(t)$?

Solution:

$x(t)$ can be expressed mathimatically as:

$$x(t) = \begin{cases} 1 , & |t| < \frac{1}{2} \\ 0 , & \frac{1}{2} < |t| < 1 \end{cases}$$

$x(t)$ is periodic signal with period $T = 2$ sec.

$\Rightarrow$ the fundamental frequency is $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ rad./sec.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\int_{-1}^{1} x(t)\,dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dt = \frac{1}{2}\left[t\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2}}$$

$$a_1 = \frac{1}{T}\int_T x(t)\,e^{-j\omega_0 t}\,dt = \frac{1}{2}\int_{-1}^{1} x(t)\,e^{-j\omega_0 t}\,dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega_0 t}\,dt$$

$$= -\frac{1}{j2\omega_0}\left[e^{-j\omega_0 t}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{j2\pi}\left[e^{-j\pi t}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{j2\pi}\left(e^{-j\frac{\pi}{2}} - e^{j\frac{\pi}{2}}\right)$$

$$= \frac{1}{\pi}\cdot\frac{1}{j2}\left(e^{j\pi/2} - e^{-j\pi/2}\right) = \frac{1}{\pi}\sin\frac{\pi}{2} = \frac{1}{\pi}$$

since a_1 is real $\Rightarrow \boxed{a_{-1} = a_1 = \frac{1}{\pi}}$

$$a_2 = \frac{1}{T}\int_T x(t)\, e^{-j2w_0 t}\, dt = \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-j2w_0 t}\, dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2w_0 t}\, dt$$

$$= -\frac{1}{j4w_0}\left[e^{-j2w_0 t}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{j4\pi}\left[e^{-j2\pi t}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{j4\pi}\left(e^{-j\pi} - e^{j\pi}\right) = \frac{1}{2\pi}\cdot\frac{1}{j2}\left(e^{j\pi} - e^{-j\pi}\right) = \frac{1}{2\pi}\sin\pi = 0$$

$$\Rightarrow \boxed{a_2 = a_{-2} = 0}$$

We can say that $a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$

$$\Rightarrow a_3 = \frac{1}{3\pi}\sin\left(\frac{3\pi}{2}\right) = -\frac{1}{3\pi}$$

$$\boxed{a_3 = a_{-3} = -\frac{1}{3\pi}}$$

$$a_4 = \frac{1}{4\pi}\sin(2\pi) = 0$$

$$\boxed{a_4 = a_{-4} = 0}$$

$$a_5 = \frac{1}{5\pi}\sin\left(\frac{5\pi}{2}\right) = \frac{1}{5\pi}$$

$$\boxed{a_5 = a_{-5} = \frac{1}{5\pi}}$$

$$\boxed{a_6 = a_{-6} = 0}\ , \quad \boxed{a_7 = a_{-7} = -\frac{1}{7\pi}} \quad ------$$

$$\Rightarrow x(t) = \frac{1}{2} + \frac{1}{\pi}\left(e^{j\pi t} + e^{-j\pi t}\right) - \frac{1}{3\pi}\left(e^{j3\pi t} + e^{-j3\pi t}\right) + \frac{1}{5\pi}\left(e^{j5\pi t} + e^{-j5\pi t}\right) + -----$$

Also

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos\pi t - \frac{2}{3\pi}\cos 3\pi t + \frac{2}{5\pi}\cos 5\pi t + -----$$