CH. 1: Complex Numbers & Complex Plane:

Continued...

E.g.: If $z_1 = 3 + 4i$ & $z_2 = x + y + iy$, find x & y if $z_1 = z_2$.

Sol.:

$$z_1 = z_2 \Rightarrow Re(z_1) = Re(z_2)$$

$$\& \quad Im(z_1) = Im(z_2)$$

$$\Rightarrow \quad 3 = x + y \quad \& \quad y = 4$$

$$\Rightarrow \quad 3 = x + 4 \Rightarrow x = -1$$

E.g.: If $z_1 = 3 + 4i$ & $z_2 = x + y + (x - y)i$, find x & y if $z_1 = z_2$.

Sol.:

$$3 = x + y \text{ --- } (1) \quad \& \quad 4 = x - y \text{ --- } (2)$$

Adding (1) & (2):

$$7 = 2x \Rightarrow x = \frac{7}{2} = 3.5$$

$$\Rightarrow \quad 3 = 3.5 + y \Rightarrow y = -0.5$$

Commutative Laws:

Sum $\rightarrow$ $z_1 + z_2 = z_2 + z_1$

Product $\rightarrow$ $z_1 z_2 = z_2 z_1$

Associative Laws:

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ $\leftarrow$ Sum

$z_1 (z_2 z_3) = (z_1 z_2) z_3$ $\leftarrow$ Product

Distributive Laws:

$\leftarrow$ Sum

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ $\leftarrow$ Product

Ex(1): $z_1 = 3 + 4i$ & $z_2 = -2 - 5i$, find $\dfrac{z_1}{z_2}$.

Sol.:

$$\frac{z_1}{z_2} = \frac{3+4i}{-2-5i} \times \frac{-2+5i}{-2+5i}$$

$$= \frac{-6 + 15i - 8i + \overset{-20}{20i^2}}{(-2)^2 + (-5)^2}$$

$$= \frac{-26 + 7i}{4 + 25} = \frac{-26 + 7i}{29} = \frac{-26}{29} + \frac{7}{29} i$$

Conjugate Properties:

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(2) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$

$\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

(3) $z + \bar{z} = (a + \cancel{ib}) + (a - \cancel{ib}) = 2a$

(4) $z \bar{z} = \text{- - - - - - - - -} = a^2 + b^2$ (as before)

(5) $z - \bar{z} = (2b)i = 2ib$ ← Better formula

(6) $\text{Re}(z) = \dfrac{z + \bar{z}}{2}$

& $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$

Ex (3): Find the reciprocal of $z = 2 - 3i$.

Sol.: $z^{-1} = \dfrac{1}{z} = \dfrac{1}{2-3i} = \dfrac{1}{2-3i} \times \dfrac{2+3i}{2+3i}$

$$= \frac{2+3i}{4+9} = \frac{2+3i}{13} = \underset{\substack{\uparrow \\ a}}{\frac{2}{13}} + \underset{\substack{\uparrow \\ b}}{\frac{3}{13}}\,i$$

The most simplified

Since $z^{-1} = \dfrac{1}{z}$

$\Rightarrow z\,z^{-1} = 1$ can be used to verify the ~~answer~~ answer

$$z\,z^{-1} = (2-3i)\left(\frac{2}{13} + \frac{3}{13}i\right)$$

$$= \frac{(2-3i)(2+3i)}{13} = \frac{4 + \cancel{6i} - \cancel{6i} - 9i^2}{13}$$ ← +9

$$= \frac{13}{13} = 1 \checkmark$$

Exc(1.1): 28 For $z = x+iy$, express $Re(z^2)$ in terms of x & y.

Sol.:

$$z^2 = (x+iy)^2 = (x+iy)(x+iy) = x^2 + ixy + ixy + i^2y^2$$

OR $$x^2 + (2xy)i + \underbrace{i^2y^2}_{-y^2} = \underbrace{x^2 - y^2}_{Re(z^2)} + \underbrace{(2xy)}_{Im(z^2)}i$$

$$\Rightarrow \quad Re(z^2) = x^2 - y^2$$

Exc(1.1): 38 Solve the equation for z:

$$z - 2\bar{z} + 7 - 6i = 0$$

Sol.: Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$$\Rightarrow \quad (x+iy) - 2(x-iy) + 7 - 6i = 0$$

$$x + iy - 2x + 2iy + 7 - 6i = 0$$

$$\underbrace{x - 2x + 7}_{\text{Real}} + \underbrace{(y + 2y - 6)}_{\text{Imaginary}} i = 0$$

$$\Rightarrow \quad x - 2x + 7 = 0 \quad \text{—— (1)}$$

$$-x + 7 = 0 \Rightarrow \boxed{x = 7}$$

&

$$y + 2y - 6 = 0 \quad \text{—— (2)}$$

$$3y - 6 = 0 \Rightarrow 3y = 6 \Rightarrow \boxed{y = 2}$$

$$\Rightarrow \quad \boxed{z = 7 + 2i}$$

Please do this:

$$\frac{z}{1+\bar{z}} = 3+4i \quad \leftarrow \text{solve the equation for } z.$$

Answer is: $z = \underset{\substack{\uparrow \\ x}}{-\frac{7}{6}} - \underset{\substack{\uparrow \\ y}}{\frac{1}{6}}\, i$

1.2 Complex Plane: $\leftarrow$ 2D

A complex number can ~~be~~ easily be determined by an ordered pair (x, y) of $z = x+iy$

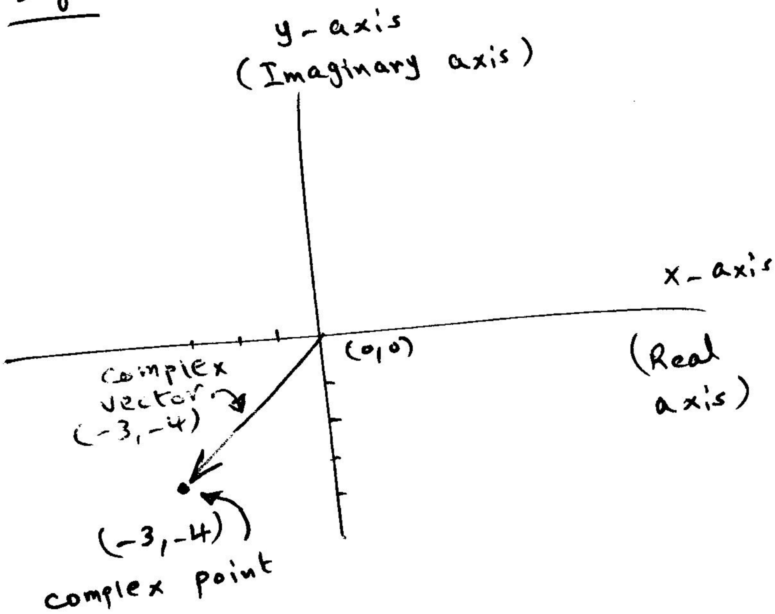
If $z = -3-4i \Rightarrow$ ordered pair is $(-3,-4)$

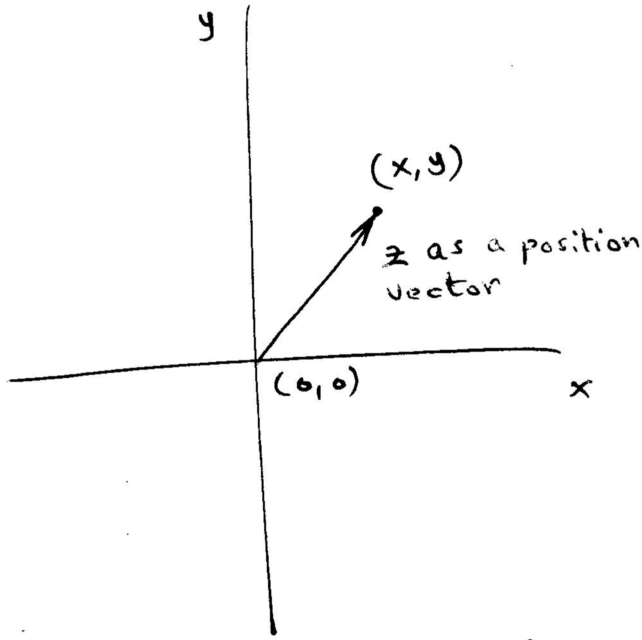
" $z = 2i \Rightarrow$ " " " $(0, 2)$

complex no. $\rightarrow$ complex point $\rightarrow$ complex vector

$x+iy$ $\qquad$ (x, y) $\qquad$ (x, y)

E.g.: For $z = -3-4i$:





General Model for z

This is called z-plane or complex plane.

⑥

Position vector: is a vector whose initial point is the origin $(0,0)$ and whose terminal point is (x,y) in our case; where $z = x + iy$.

$\Rightarrow$ Any complex no. z can be viewed as a position vector

Modulus of complex no.:

If $z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$ called modulus or absolute value.

$|z|$ is called magnitude in vectors.

$|z|$ is always positive & real.

So any complex no. has a modulus which represents the length of the position vector representing it.

Ex (1): Find the modulus of z.

(a) $z = 2 - 3i$

$$\Rightarrow |z| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

(b) $z = -9i$

$$\Rightarrow |z| = \sqrt{0^2 + (-9)^2} = \sqrt{9^2} = \sqrt{81} = 9$$

= imaginary part absolute value

$\Rightarrow$ If $z = -9 \Rightarrow |z| = 9$

= real part absolute value

(7) Properties of Complex No. Modulus:

(1) Recall: $z\bar{z} = x^2 + y^2 \Rightarrow |z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2 + y^2$

$\Rightarrow |z|^2 = z\bar{z}$ or $|z| = \sqrt{z\bar{z}}$

(2) $|z_1 z_2| = |z_1|\,|z_2|$ & $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

If $z_1 = z_2 = z$

$\Rightarrow |z z| = |z|\,|z|$

$\Rightarrow |z^2| = |z|^2$

<u>Note:</u> $|z_1 + z_2| \neq |z_1| + |z_2|$

<u>Ex:</u> $z_1 = 1 + i$ & $z_2 = 3 - 2i$, find:

(a) $|z_1|$ (b) $|z_2|$ (c) $z_1 z_2$

(d) Prove that $|z_1 z_2| = |z_1|\,|z_2|$

<u>Sol.:</u>

(a) $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$

(b) $|z_2| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$

(c) $z_1 z_2 = (1+i)(3-2i) = 3 - 2i + 3i \underbrace{- 2i^2}_{+2}$

$= 5 + i$

(d) $|z_1 z_2| = |5 + i| = \sqrt{5^2 + 1^2} = \sqrt{25+1} = \sqrt{26}$ —(1)

$|z_1|\,|z_2| = \sqrt{2}\sqrt{13} = \sqrt{2 \times 13} = \sqrt{26}$ —— (2)

From (1) & (2):

$$\Rightarrow \quad |z_1 z_2| = |z_1|\,|z_2|$$

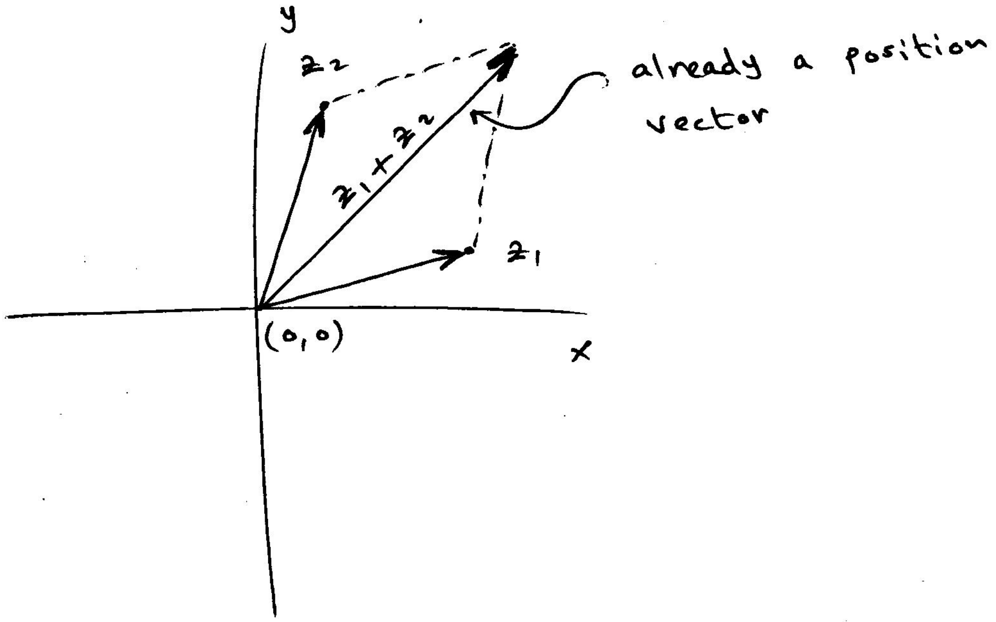
Back to addition:

If $z_1 = x_1 + i y_1$ & $z_2 = x_2 + i y_2$

$$\Rightarrow \quad z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

Using ordered-pairs:

$z_1 + z_2 = (x_1 + x_2\ ,\ y_1 + y_2)$ ← Same as vector addition



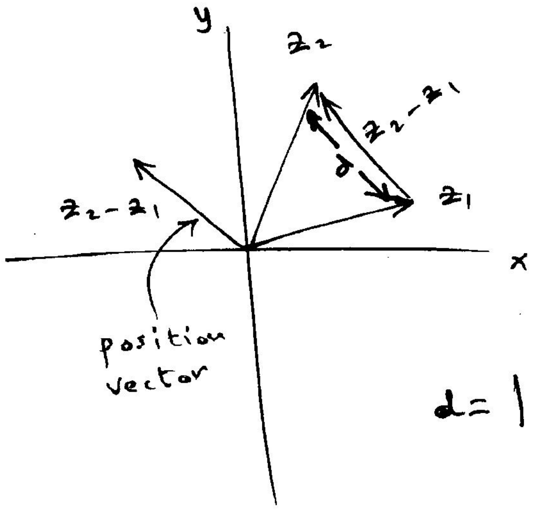
The distance between two points in the complex plane is:

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The order is not important here
(distance, not a vector)

<u>Note:</u> $|z_2 - z_1| = |z_1 - z_2|$

(9)



$d = |z_2 - z_1| = |z_1 - z_2|$

Distance is always <u>real</u> & <u>positive</u>.

<u>Note:</u> $|z_2 - z_1| \neq |z_2| - |z_1|$

This is the modulus of the new complex no. $z_2 - z_1$

<u>Exc(1.2):</u> (4) Interpret z_1 & z_2 as vectors, graph z_1 & z_2 and the indicated sum and difference as vectors:

$z_1 = 4 - 3i$ & $z_2 = -2 + 3i$

Sum: $2z_1 + 4z_2$

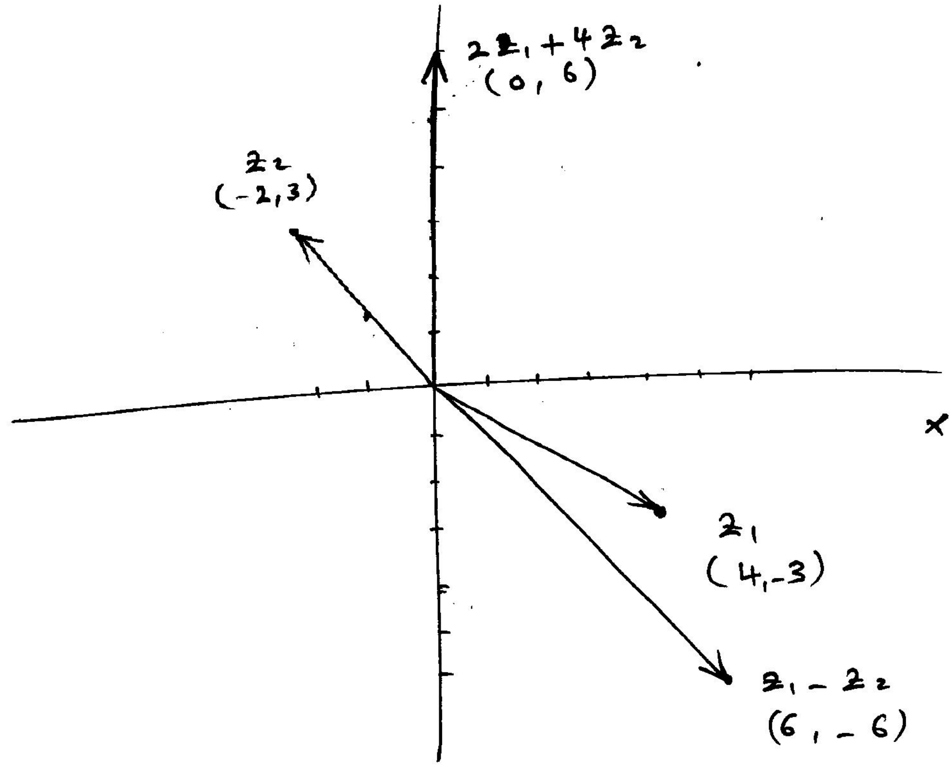
Diff.: $z_1 - z_2$

<u>Sol.:</u> $z_1 = (4, -3)$ & $z_2 = (-2, 3)$

Sum: $2z_1 + 4z_2 = 2(4, -3) + 4(-2, 3)$
$= (8, -6) + (-8, 12)$
$= (0, 6) = 6i$

Diff.: $z_1 - z_2 = (4, -3) - (-2, 3i)$
$= (6, -6) = 6 - 6i$

(10) $\Rightarrow$ The graph:



Exc (1.2): (12) Find the modulus of $z = \dfrac{1-2i}{1+i} + \dfrac{2-i}{1-i}$

Sol.: Simplify z.

$$z = \frac{(1-2i)(1-i) + (2-i)(1+i)}{(1+i)(1-i)}$$

$$= \frac{1 - i - 2i + \overbrace{2i^2}^{-2} + 2 + 2i - i - \overbrace{i^2}^{+1}}{1 - \cancel{i} + \cancel{i} - i^2}$$

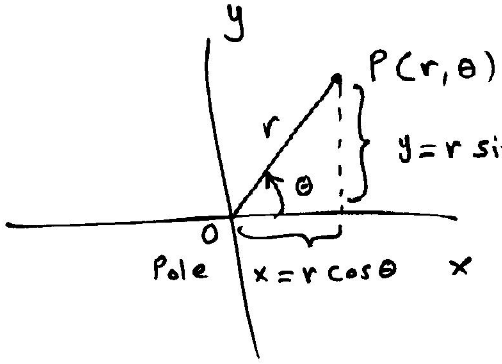
$a^2 + b^2 = 1^2 + 1^2 = 1 + 1$

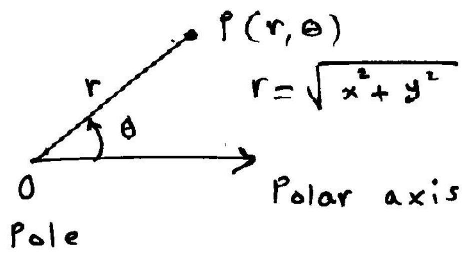
$$= \frac{1 - 3i - \cancel{2} + \cancel{2} + i + 1}{1+1}$$

$$= \frac{2-2i}{2} = 1 - i$$

$$\Rightarrow |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

1.3 Polar Forms of Complex Numbers:





$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Any point in the plane with rectangular coordinates (x, y) can be described in polar coordinates (r, θ).

r: is the distance from the pole O to the point P.

θ: is the angle of inclination (in radians) from the polar axis to the line OP.

$r = |z| \leftarrow$ the modulus of z.

$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \quad , \quad n = 0, \pm 1, \pm 2, \cdots$

Ex (1): Express $-\sqrt{3} - i$ in polar form.

Sol.: $x = -\sqrt{3}$ & $y = -1 \Rightarrow r = |z| = \sqrt{(-\sqrt{3})^2 + (-1)^2}$

$\Rightarrow r = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) + 2n\pi$

$= \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right)$ for $n = 0$

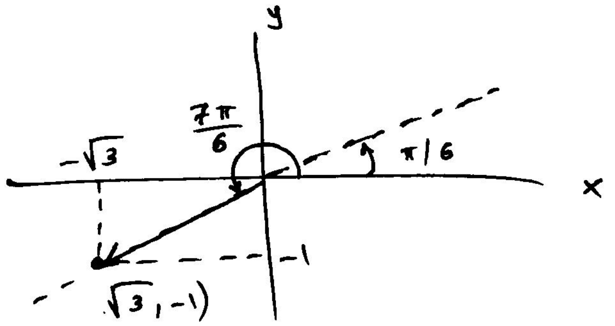
Using calculator:

$\theta = \frac{\pi}{6}$ (30° , in the 1^{st} quadrant)

But $z = (-\sqrt{3}, -1)$ is located in the 3^{rd} quadrant

x & y are both negative

$$\Rightarrow \quad \theta = \frac{\pi}{6} + \pi = \frac{7\pi}{6} \qquad (210^\circ \text{ in degrees})$$



In polar form, z is represented as:

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

$$\Rightarrow \quad z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$$

<u>Principal</u> <u>Argument</u> : is a unique value which lies between $-\pi$ and π radians.

capital letter

$$\boxed{-\pi < \operatorname{Arg}(z) \leq \pi} \qquad (\text{since } \pi = -\pi)$$

<u>Principal</u> <u>Argument</u>

(or <u>principal</u> <u>value</u> <u>of</u> <u>arg(z)</u>)

Note:

$Arg(z)$: is the principal argument.

$arg(z)$: is any argument.

Recall: $\quad arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \quad arg(z) = Arg(z) + 2n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad Arg(z) = arg(z) \quad \text{for} \quad n = 0$$

E.g.: What is the argument of i ? $\leftarrow arg(i)$

What is $Arg(i)$?

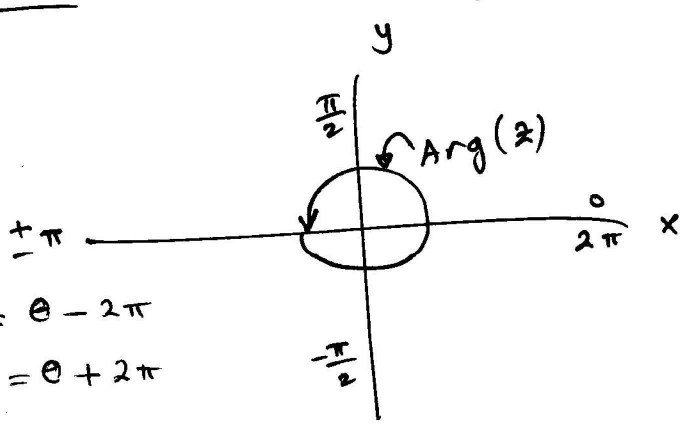
Sol.: $\quad z = 0 + i \quad \Rightarrow \quad Re(z) = 0 \;\&\; Im(z) = 1$

$$\begin{aligned} arg(i) &= Arg(i) + 2n\pi \\ &= \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi \\ &= \tan^{-1}\left(\frac{1}{0}\right) + 2n\pi \\ &= \frac{\pi}{2} + 2n\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Since $\cos\frac{\pi}{2} = 0 \;\&\; \sin\frac{\pi}{2} = 1$

For $n = 0 \Rightarrow arg(i) = \frac{\pi}{2} = Arg(i)$

Note: $\quad -\pi < \frac{\pi}{2} \leq \pi$



$$\boxed{-\pi < Arg(z) \leq \pi}$$

If $\theta > \pi \Rightarrow Arg(z) = \theta - 2\pi$

If $\theta \leq -\pi \Rightarrow Arg(z) = \theta + 2\pi$

Multiplication and Division in Polar Form:

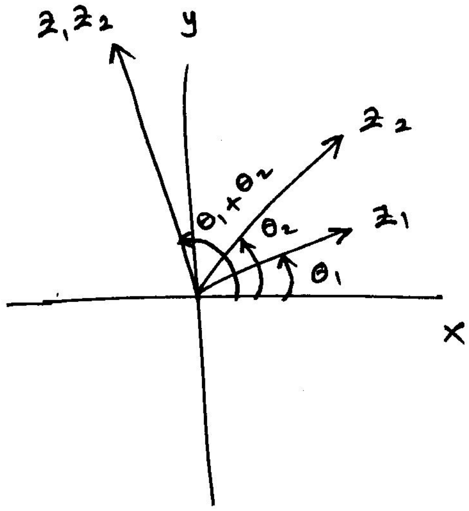
$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right]$$

& $$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right]$$

Also:

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

& $$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$



Ex (2): $z_1 = i$, $z_2 = -\sqrt{3} - i$, find the principal argument of the product and the quotient.

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Sol.:

$$z_1 z_2 = i(-\sqrt{3} - i) = -\sqrt{3}\, i - i^2 = -\sqrt{3}\, i + 1 = 1 - \sqrt{3}\, i$$

$$\frac{z_1}{z_2} = \frac{i}{-\sqrt{3} - i} = \frac{i}{-\sqrt{3} - i} \times \frac{-\sqrt{3} + i}{-\sqrt{3} + i}$$

$$= \frac{-\sqrt{3}\, i + i^2}{3 + 1} = \frac{-1 - \sqrt{3}\, i}{4} = -\frac{1}{4} - \frac{\sqrt{3}}{4} i$$

$$\cancel{\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{2} + \frac{7\pi}{6} = \frac{10\pi}{6} = \frac{\pi}{3}}$$

$$\cancel{\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{2} - \frac{7\pi}{6} =}$$

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$
$$= \mathrm{Arg}(i) + \mathrm{Arg}(-\sqrt{3} - i)$$
$$= \tan^{-1}\left(\frac{1}{0}\right) + \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right)$$
$$= \frac{\pi}{2} + \left(\frac{\pi}{6} + \pi\right)$$

from Ex(1)

$$= \frac{\pi}{2} + \frac{7\pi}{6}$$

$$\boxed{\mathrm{Arg}(z_1) = \frac{\pi}{2}} \quad \cancel{= \frac{10\pi}{6} = \frac{5\pi}{3}}$$

(Not in the range between $-\pi$ & π)

$$\Rightarrow \boxed{\mathrm{Arg}(z_2)} = \frac{7\pi}{6} - 2\pi = \boxed{\frac{-5\pi}{6}}$$

$$\cancel{\Rightarrow \mathrm{Arg}(z_1 z_2)} =$$

$$\Rightarrow \boxed{\mathrm{Arg}(z_1 z_2)} = \frac{\pi}{2} - \frac{5\pi}{6} = \boxed{\frac{-\pi}{3}}$$ (in range)

$$\& \quad \mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} - \left(-\frac{5\pi}{6}\right) = \frac{4\pi}{3}$$ (not in range)

$$\Rightarrow \boxed{\mathrm{Arg}\left(\frac{z_1}{z_2}\right)} = \frac{4\pi}{3} - 2\pi = \boxed{\frac{-2\pi}{3}}$$ (in range)

Another method to find $\mathrm{Arg}(z)$:

- Determine the quadrant.
- Calculate $\tan^{-1}\left(\frac{y}{x}\right)$ without signs (that is θ).
- Using the following formulas:

2nd quadrant: $\mathrm{Arg}(z) = \pi - \theta$	1st quadrant: $\mathrm{Arg}(z) = \theta$
3rd quadrant: $\mathrm{Arg}(z) = \theta - \pi$	4th quadrant: $\mathrm{Arg}(z) = -\theta$

Integer Power of z:

Recall: $z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$

$$\Rightarrow \quad z\,z = r^2 (\cos 2\theta + i \sin 2\theta)$$

Similarly:

$$z^3 = r^3 (\cos 3\theta + i \sin 3\theta)$$

Also:

even $\swarrow$ odd $\swarrow$

$$\frac{1}{z^2} = z^{-2} = r^{-2} \left[\cos(-2\theta) + i \sin(-2\theta) \right]$$

$$= r^{-2} (\cos 2\theta - i \sin 2\theta)$$

Generally:

The n^{th} power of z is:

$$z^n = r^n (\cos n\theta + i \sin n\theta) \qquad (n \text{ is integer})$$

Ex(3): Compute z^3 for $z = -\sqrt{3} - i$.

Sol.: From Ex(1), $r = 2$ & $\theta = \frac{7\pi}{6}$

$$\Rightarrow \quad z^3 = 2^3 \left[\cos 3\left(\frac{7\pi}{6}\right) + i \sin 3\left(\frac{7\pi}{6}\right) \right]$$

$$= 8 \left[\cos\left(\frac{7\pi}{2}\right) + i \sin\left(\frac{7\pi}{2}\right) \right]$$

$$= 8 (0 - i) = -8i \quad \leftarrow \text{in Cartesian}$$

de Moivre's Formula:

If $z = \cos\theta + i\sin\theta \Rightarrow r = |z| = 1$

Then: $\boxed{z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Ex (4): Given z with $|z| = 1$, and $\theta = \frac{\pi}{6}$, find z^3.

Sol.: Since $|z| = 1 \Rightarrow$ use de Moivre's formula.

$$z^3 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 = \cos\left(3.\frac{\pi}{6}\right) + i\sin\left(3.\frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = i$$

Exc (1.3):

(28) Find $(-\sqrt{2} + \sqrt{6}\,i)^4$.

Sol.: $r = |z| = \sqrt{2+6} = \sqrt{8}$

$$\theta = \tan^{-1}\left(\frac{\sqrt{6}}{-\sqrt{2}}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$\uparrow$ 2nd quadrant $\qquad$ $\uparrow$ 4th quadrant

$$\Rightarrow \theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3} \quad (120^\circ)$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta) = \sqrt{8}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\Rightarrow z^4 = (\sqrt{8})^4\left[\cos\left(4.\frac{2\pi}{3}\right) + i\sin\left(4.\frac{2\pi}{3}\right)\right]$$

$$= 64\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$

$$= 64\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -32 + 32\sqrt{3}\,i$$

$$= -32 + 55.43i$$

Exc (1.3):

(32) Find
$$\frac{\left[8\left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right)\right]^{10}}$$

Sol.:

$$\frac{8^3\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)}{2^{10}\left(\cos\frac{10\pi}{16}+i\sin\frac{10\pi}{16}\right)} =$$

$$= \frac{(2^3)^3}{2^{10}}\left[\cos\left(\frac{9\pi}{8}-\frac{10\pi}{16}\right)+i\sin\left(\frac{9\pi}{8}-\frac{10\pi}{16}\right)\right]$$

In division: the arguments are subtracted from each other.

Same as $\dfrac{8^3\, e^{j\,9\pi/8}}{2^{10}\, e^{j\,10\pi/16}} = \dfrac{(2^3)^3}{2^{10}}\, e^{j\left(\frac{9\pi}{8}-\frac{10\pi}{16}\right)}$

$$= \frac{2^9}{2^{10}}\left[\cos\left(\frac{18\pi-10\pi}{16}\right)+i\sin\left(\frac{18\pi-10\pi}{16}\right)\right]$$

$$= \frac{1}{2}\left(\cos\frac{8\pi}{16}+i\sin\frac{8\pi}{16}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$= \frac{1}{2}(0+i) = \frac{i}{2} \quad \left(\text{or } \frac{1}{2}i\right)$$

1.4 Power and Roots :

Roots:

Let $z = r(\cos\theta + i\sin\theta)$ & $\omega = \rho(\cos\phi + i\sin\phi)$

If $\omega^n = z \Rightarrow \rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$

$\Rightarrow \rho^n = r$

& $\cos n\phi + i\sin n\phi = \cos\theta + i\sin\theta$

$\Rightarrow n\phi = \theta + 2k\pi$ where k is integer

$\Rightarrow \boxed{\phi = \dfrac{\theta + 2k\pi}{n}}$

So, the n n^{th} roots of a nonzero complex no. $z = r(\cos\theta + i\sin\theta)$ is :

$$\boxed{\omega_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]}$$

where $k = 0, 1, 2, 3, \ldots$

Ex (1): Cube Roots of a Complex No. :

Find the three cube roots of $z = i$

Sol.: Recall : for $z = i$, $r = 1$ & $\theta = \frac{\pi}{2}$

$$\left.\begin{matrix}\boxed{n = 3} \\ \& \ \boxed{k = \underbrace{0, 1, 2}_{3 \text{ roots}}}\end{matrix}\right\} \Rightarrow \boxed{\omega_k = \sqrt[3]{1}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right]}$$

Cont. →

(20) The three cube roots are:

$$k=0 \Rightarrow w_0 = \sqrt[3]{1}\left[\cos\frac{(\pi/2)}{3} + i\sin\left(\frac{\pi/2}{3}\right)\right]$$

$$= (1)\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k=1 \Rightarrow w_1 = \sqrt[3]{1}\left[\cos\left(\frac{\frac{\pi}{2}+2\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi}{3}\right)\right]$$

$$= (1)\left[\cos\left(\frac{\pi+4\pi}{2\times 3}\right) + i\sin\left(\frac{\pi+4\pi}{2\times 3}\right)\right]$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k=2 \Rightarrow w_2 = \sqrt[3]{1}\left[\cos\left(\frac{\frac{\pi}{2}+4\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+4\pi}{3}\right)\right]$$

$$= (1)\left[\cos\left(\frac{\pi+8\pi}{2\times 3}\right) + i\sin\left(\frac{\pi+8\pi}{2\times 3}\right)\right]$$

$$= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$$

$$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

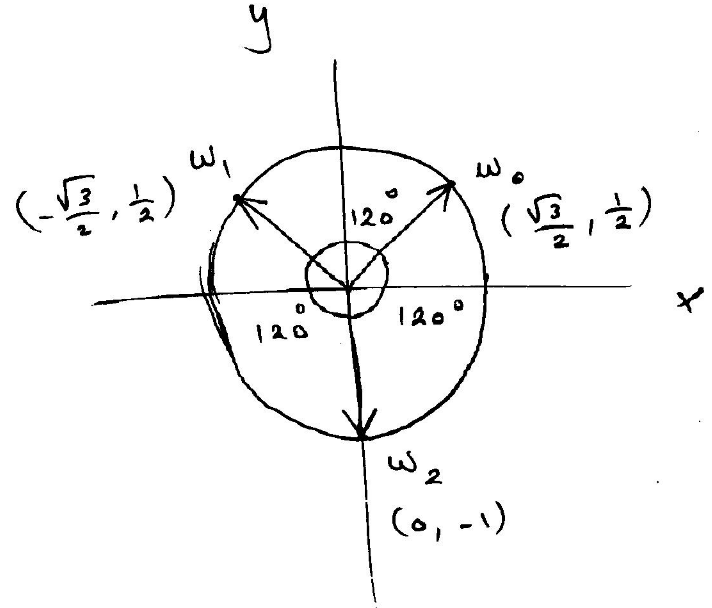
$$= 0 - i = -i$$

$\therefore$ For $z = i$, $n = 3$, $k = 0, 1, 2$, and the 3-cube roots of z are:

$$w_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_2 = -i$$



Note that: $\left.\begin{array}{l} |w_0| = 1 \\ |w_1| = 1 \\ |w_2| = 1 \end{array}\right\}$ check it please

Also: Angle between each two roots is fixed (120° in our example).

Also: $\left.\begin{array}{c} w_0 \cdot w_1 \cdot w_2 = z \\ \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)(-i) = i \end{array}\right\}$ check it please

Recall: $\sqrt[3]{27} = 3 \Rightarrow 3 \times 3 \times 3 = 27$

& $\sqrt{4} = \sqrt[2]{4} = \pm 2$ ← two roots in each case ← From Real Analysis studied before

$\Rightarrow (+2)(+2) = 4$ ← $(+2)^2$

& $(-2)(-2) = 4$ ← $(-2)^2$

Principal n^{th} Root :

The root with $k=0$ is called the principal n^{th} root.

E.g. $z = 1+i$, find the 4 4^{th} roots & specify the principal 4^{th} root.

Sol.:

$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

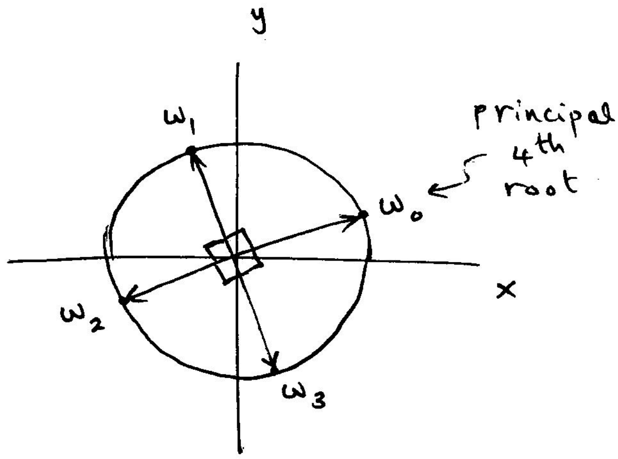
$$n = 4 \quad , \quad k = 0, 1, 2, 3$$

$$w_k = \sqrt[4]{\sqrt{2}}\left[\cos\left(\frac{\frac{\pi}{4}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{4}+2k\pi}{4}\right)\right]$$

$$\Rightarrow w_0 = \left[(2)^{\frac{1}{2}}\right]^{\frac{1}{4}}\left[\cos\left(\frac{\pi+0}{16}\right) + i\sin\left(\frac{\pi+0}{16}\right)\right]$$

$$= (2)^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$$

$$\Rightarrow w_0 = 1.07 + 0.21i$$



Similarly:

$$w_1 = -0.21 + 1.07i$$

$$w_2 = -1.07 - 0.21i$$

$$w_3 = 0.21 - 1.07i$$

check: $w_0^4 = 1+i$ $\qquad$ $w_1^4 = 1+i$

$w_2^4 = 1+i$ $\qquad$ $w_3^4 = 1+i$

$w_0 \cdot w_1 \cdot w_2 \cdot w_3 \neq 1+i$ (answer is $-1-i$)

Also:

$$w_0 = 1.07 + 0.21i = (2)^{1/8}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$$

$$w_1 = (2)^{1/8}\left[\cos\left(\frac{\pi}{16}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{16}+\frac{\pi}{2}\right)\right]$$

because the angle between each two successive roots is $\pi/2$ radian.

$$= (2)^{1/8}\left[\cos\left(\frac{\pi+8\pi}{16}\right) + i\sin\left(\frac{\pi+8\pi}{16}\right)\right]$$

$$= (2)^{1/8}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$$

$$= -0.21 + 1.07i$$

Third angle will be $\frac{9\pi}{16} + \frac{\pi}{2} = \frac{17\pi}{16}$ ← angle of w_2

Fourth " " " $\frac{17\pi}{16} + \frac{\pi}{2} = \frac{25\pi}{16}$ ← " " w_3

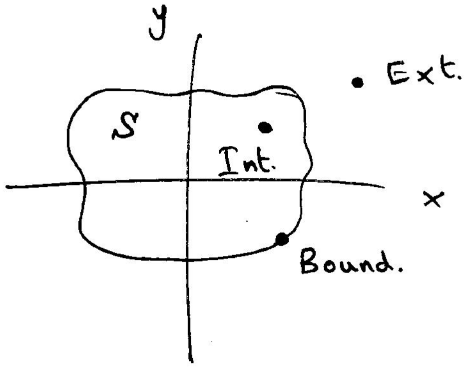
The principal 4^{th} root is $\underset{k=0}{w_0} = 1.07 + 0.21i$

1.5: Sets of Points in a Complex Plane:

Any set of points S is called a region.

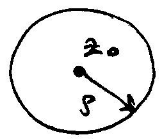
There are three types of points:

1) Interior point
2) Exterior point
3) Boundary point



Circles: The equation of a circle of center z_0 and radius ρ is:

$$|z - z_0| = \rho$$



Ex (1):

(a) Write the equation of a unit circle centered at the origin.

(b) Describe the circle of the equation $|z - 1 + 3i| = 5$

Sol:

(a) Unit circle $\Rightarrow \rho = 1$ at origin $\Rightarrow z_0 = (0,0)$ or $0 + 0i$

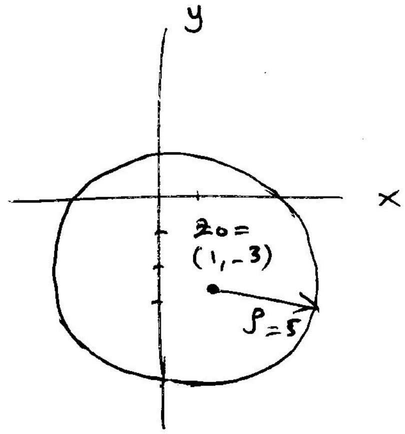
$\Rightarrow$ The equation is: $|z| = 1$

25

(b) By rewriting the equation:

$$|z - \underbrace{(1-3i)}_{z_0}| = \underset{\uparrow \atop \rho}{5}$$

$\Rightarrow$ It is a circle of radius 5, centered at $z_0 = 1-3i$
$(1,-3)$



<u>Circle equation:</u>

$$x^2 + y^2 = r^2$$ if centered at origin

$$|z| = |x + iy| = \sqrt{x^2+y^2}$$

$$|z|^2 = x^2 + y^2 = r^2$$ r is the radius

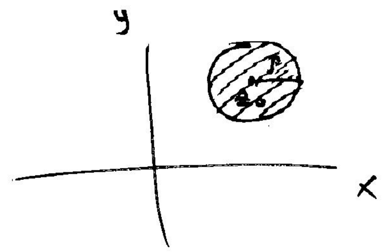
$\Rightarrow$ $\boxed{|z| = \rho}$ same

OR: $$(x-x_0)^2 + (y-y_0)^2 = r^2$$ if centered at (x_0, y_0)

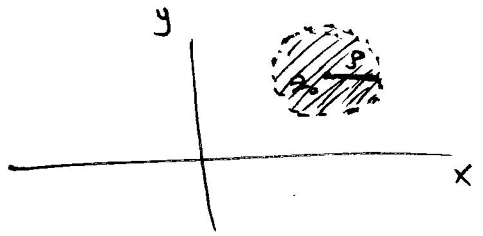
$\Rightarrow$ $\boxed{|\underset{(x,y)}{z} - \underset{(x_0,y_0)}{z_0}| = \rho}$ same

Examples of sets in different circles:

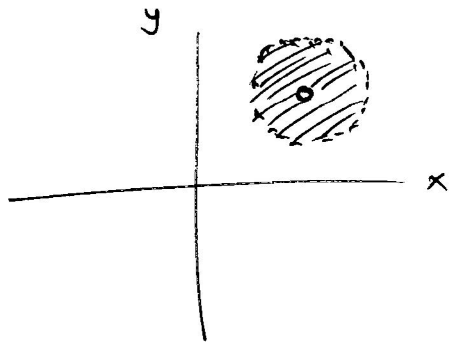
1) Disk: $|z - z_0| \leq \rho$



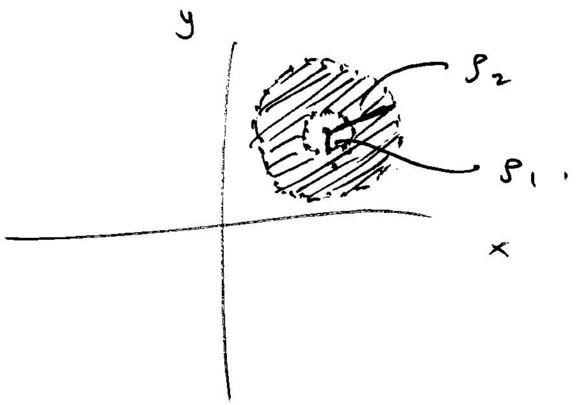
2) Neighborhood: $|z - z_0| < \rho$



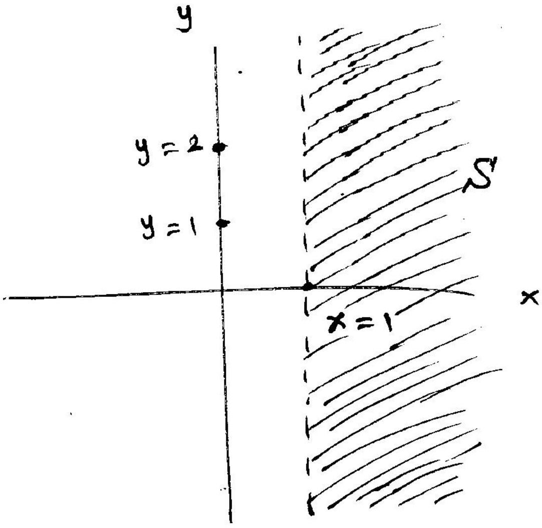
3) Deleted Neighborhood: $0 < |z - z_0| < \rho$



4) Annulus: $\rho_1 < |z - z_0| < \rho_2$



<u>Open set:</u> is a set which only contains interior points (no exterior or boundary points).



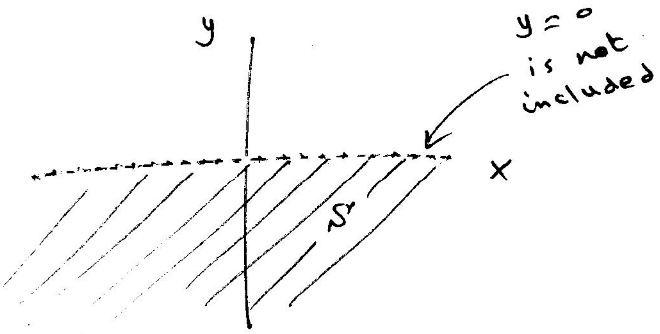
S is an open set

For each point in S, there exists some neighborhood of z_0 that lies entirely within S.

Examples of open sets:

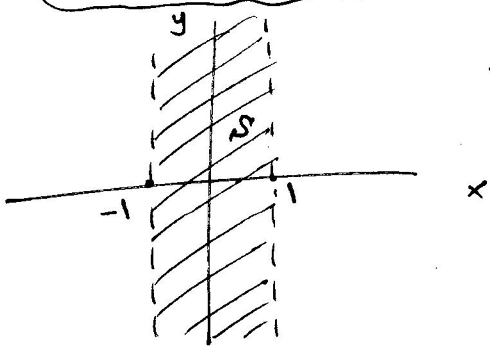
(1) Lower half plane

$$\boxed{Im(z) < 0}$$



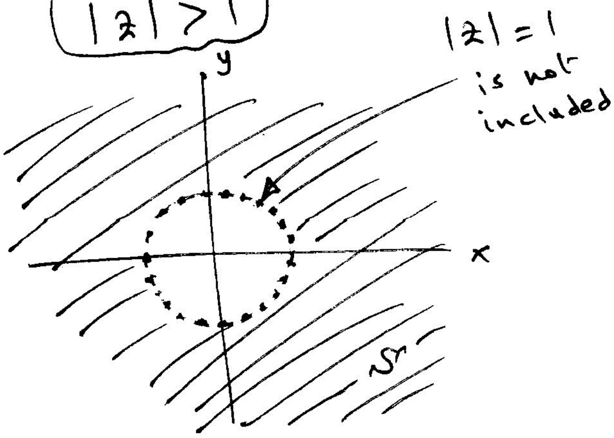
(2) Infinite Vertical Strip

$$\boxed{-1 < Re(z) < 1}$$



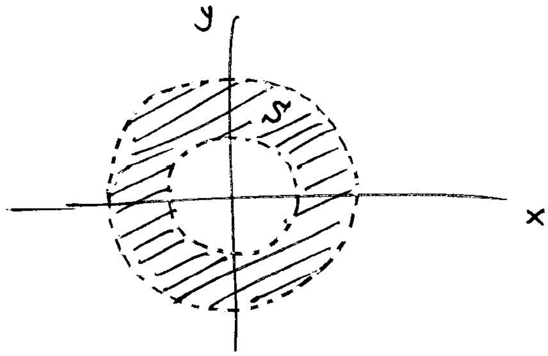
(3) Exterior of unit circle

$$\boxed{|z| > 1}$$

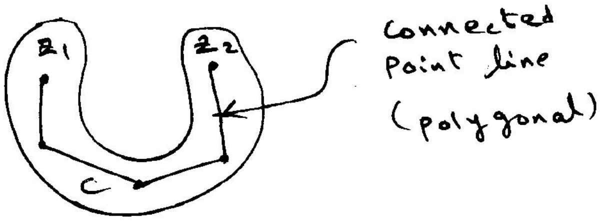


(4) Interior of circular ring

$$\boxed{1 < |z| < 2}$$



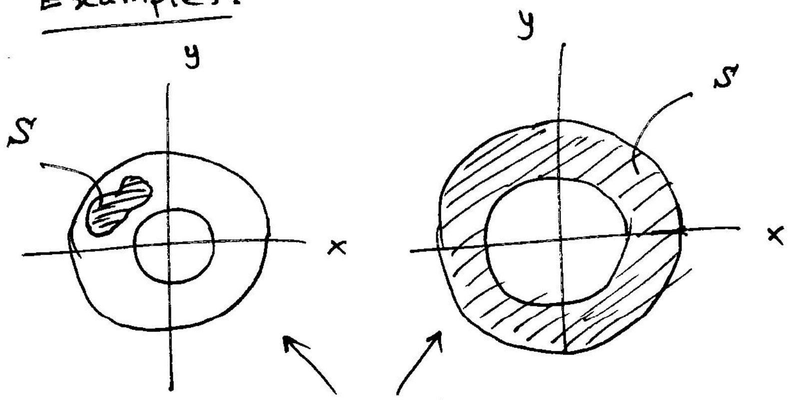
Connected set: is a set which ~~contains~~ consists of connected set of points (any two points can be connected by a polygonal line).



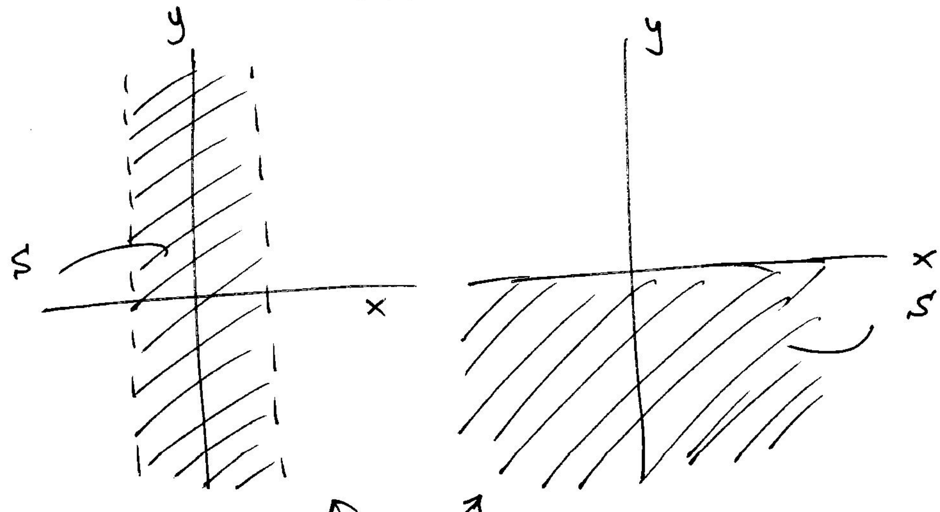
Domain: is an open connected set.

Bounded set: is a set whose all points have bounded distance from the origin.

Examples:



Bounded sets



Unbounded sets (points are scattared in the region with any distance from O)

(29)

Exc (1.5):

(8) Sketch the graph of the given equation in the z-plane:

$$Im(z - i) = Re(z + 4 - 3i)$$

Sol.:

Let $z = x + iy \Rightarrow Im(z - i) = Im(x + iy - i)$

$= Im(x + i(y - 1)) = y - 1$ —— (1)

$Re(z + 4 - 3i) = Re(x + iy + 4 - 3i)$

$= Re(x + 4 + i(y - 3)) = x + 4$ —— (2)

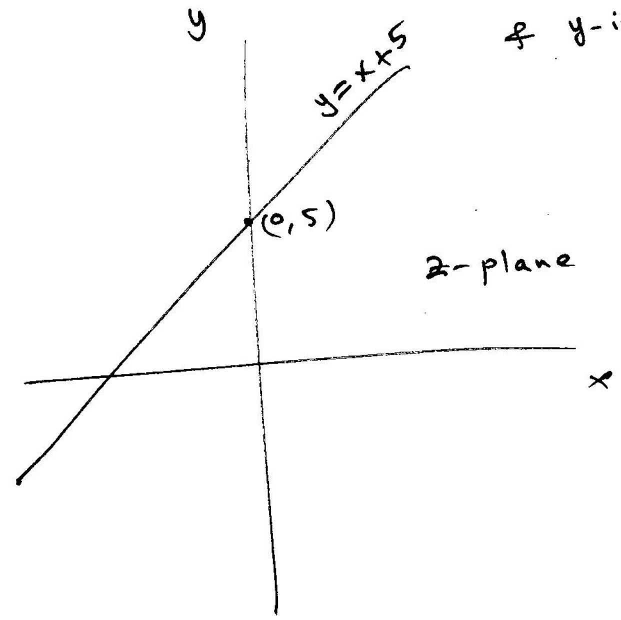
$\Rightarrow$ The given equation becomes:

$y - 1 = x + 4 \Rightarrow \boxed{y = x + 5}$

linear function

with slope = 1

& y-intersection = 5



Exc (1.5):

(12) Sketch the graph of the given equation in the complex plane:

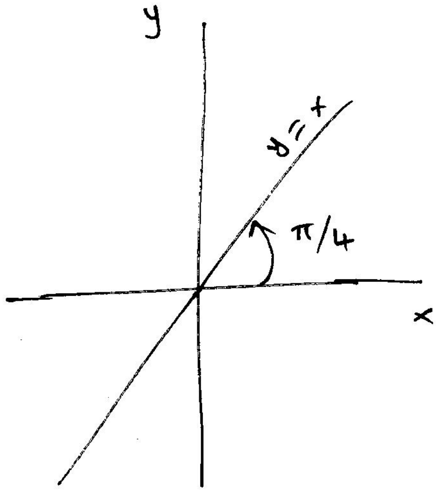
$$\arg(z) = \frac{\pi}{4}$$

Sol.:

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$$

Since $\tan\left(\frac{\pi}{4}\right) = 1$

$$\Rightarrow \frac{y}{x} = 1 \Rightarrow \boxed{y = x}$$

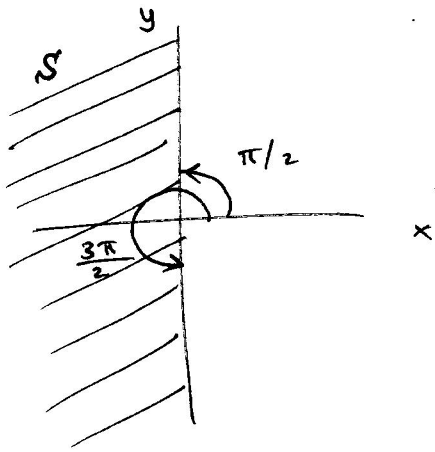


Exc (1.5):

(30) Describe the shaded area using $\arg(z)$.

Sol.:

$$\frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{2}$$



- Not open
- Unbounded
- Connected
- Not a domain ← (connected but not open)

Exc (1.5):

14 Sketch $|Re(z)| > 2$, then determine whether the set is:

inequality

(a) Open. (b) Closed. (c) A domain.

(d) Bounded. (e) Connected.

Sol.:

$$|Re(z)| > 2 \Rightarrow \boxed{Re(z) > 2}$$

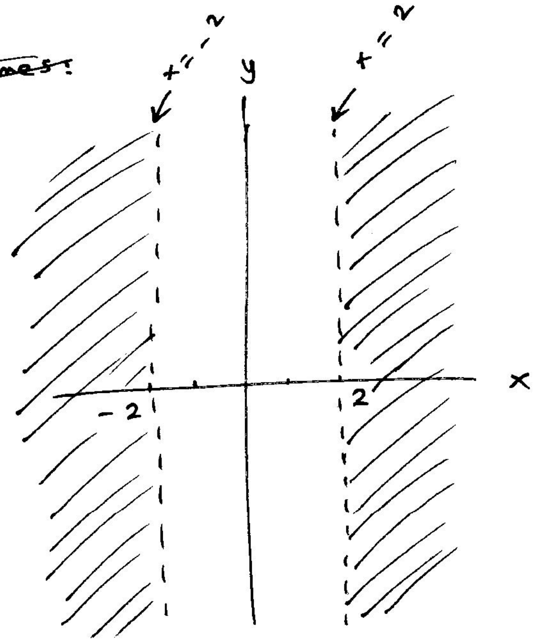
$$-Re(z) > 2 \Rightarrow \boxed{Re(z) < -2}$$

absolute value

$\Downarrow$

~~⇒ The inequality becomes:~~

$\cancel{-2 < Re(z) < 2}$



(a) Open

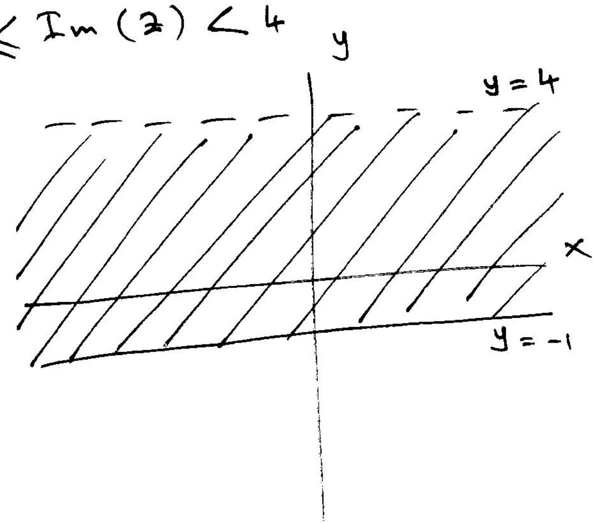
(b) Not closed

(c) Not a domain ← (open but not connected)

(d) Not bounded

(e) Not connected

15 Repeat for $-1 \leq Im(z) < 4$



(a) Not open

(b) Not closed

(c) Not a domain ← (connected but not open)

(d) Not bounded

(e) Connected

closed set: Contains all of its boundary points.