Computer Science Department
College of Computer at Al-Lith

جامعة أم القرى
UMM AL-QURA UNIVERSITY

Data Structures

Chapter 7: Graph

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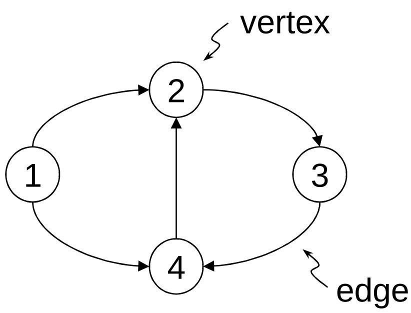
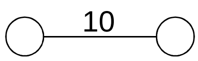
What's a Graph?

- A bunch of vertices connected by edges.



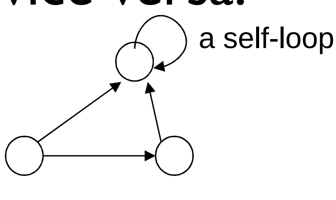
Basic Concepts

- A *graph* is an ordered pair (V, E).
- V is the set of vertices. (You can think of them as integers 1, 2, …, n.)
- E is the set of edges. An edge is a pair of vertices: (u, v).
- Note: since E is a set, there is at most one edge between two vertices. (*Hypergraphs* permit multiple edges.)
- Edges can be labeled with a *weight*:

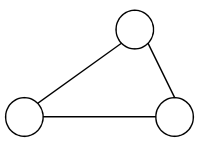


Concepts: Directedness

- In a *directed* graph, the edges are "one-way." So an edge (u, v) means you can go from u to v, but not vice versa.

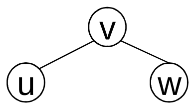


- In an *undirected* graph, there is no direction on the edges: you can go either way. (Also, no self-loops.)

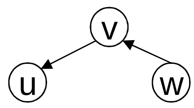


Concepts: Adjacency

- Two vertices are *adjacent* if there is an edge between them.
- For a directed graph, *u* is adjacent to *v* iff there is an edge (*v*, *u*).



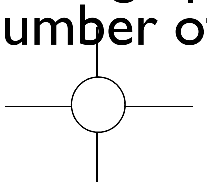
u is adjacent to v.
v is adjacent to u and w.
w is adjacent to v.



u is adjacent to v.
v is adjacent to w.

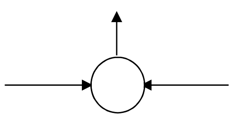
Concepts: Degree

- Undirected graph: The *degree* of a vertex is the number of edges touching it.



degree 4

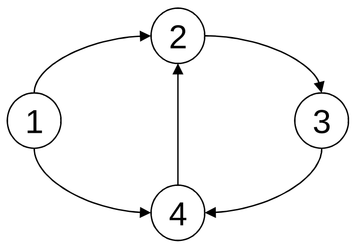
- For a directed graph, the *in-degree* is the number of edges entering the vertex, and the *out-degree* is the number leaving it. The *degree* is the *in-degree* + the *out-degree*.



in-degree 2, out-degree 1

Concepts: Path

- A *path* is a sequence of adjacent vertices. The *length* of a path is the number of edges it contains, i.e. one less than the number of vertices.



Is there a path from 1 to 4?

What is its length?

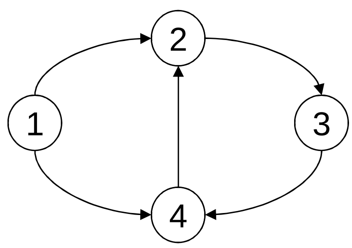
What about from 4 to 1?

How many paths are there from 2 to 3? From 2 to 2? From 1 to 1?

- We write u $\Rightarrow$ v if there is path from u to v. We say v is *reachable* from u.

Concepts: Cycle

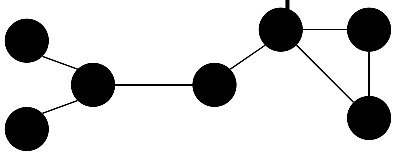
- A *cycle* is a path of length at least 1 from a vertex to itself.
- A graph with no cycles is *acyclic*.
- A path with no cycles is a *simple* path.



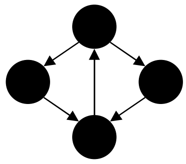
- The path <2, 3, 4, 2> is a cycle.

Connectivity

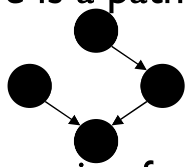
- Undirected graphs are *connected* if there is a path between any two vertices



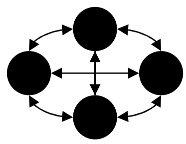
- Directed graphs are *strongly connected* if there is a path from any one vertex to any other



- Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*

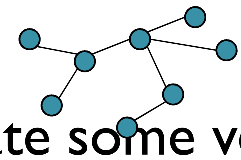


- A *complete* graph has an edge between every pair of vertices



Concepts: Trees

- A *free tree* is a connected, acyclic, undirected graph.



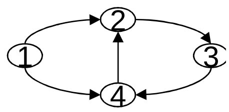
- To get a *rooted tree*, designate some vertex as the root.
- If the graph is disconnected, it's a *forest.*
- Facts about free trees:

- |E| = |V| -1
- Any two vertices are connected by exactly one path.
- Removing an edge disconnects the graph.
- Adding an edge results in a cycle.

Graph Size

- We describe the time and space complexity of graph algorithms in terms of the number of vertices, |V|, and the number of edges, |E|.
- |E| can range from 0 (a totally disconnected graph) to $|V|^2$ (a directed graph with every possible edge, including self-loops).
- we write $\Theta(V + E)$ instead of $\Theta(|V| + |E|)$.

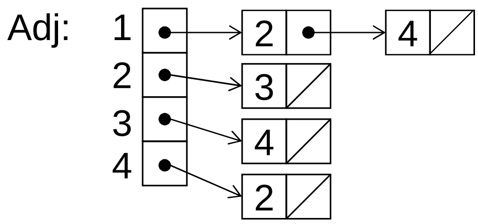
Representing Graphs



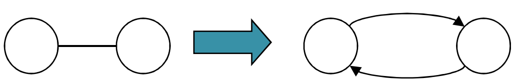
- Adjacency matrix: if there is an edge from vertex i to j, a_{ij} = 1; else, a_{ij} = 0.
- Space: $\Theta(V^2)$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
$$

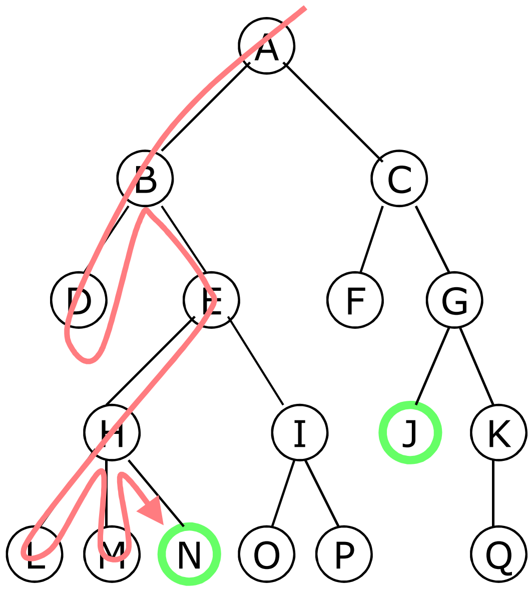
- Adjacency list: Adj[v] lists the vertices adjacent to v.
- Space: $\Theta(V+E)$



Represent an undirected graph by a directed one:



Depth-first searching



- A depth-first search (DFS) explores a path all the way to a leaf before backtracking and exploring another path
- For example, after searching A, then B, then D, the search backtracks and tries another path from B
- Node are explored in the order A B D E H L M N I O P C F G J K Q
- N will be found before J

Fixing Bad-DFS

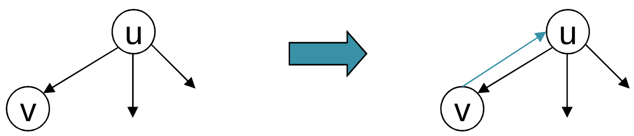
- We've got to indicate when a node has been visited.
- we'll use a color:
- WHITE never seen
- GRAY discovered but not finished (still exploring its descendants)
- BLACK finished

A Better DFS

- ▷ initially, all vertices are WHITE
- Better-DFS(u)
- color[u] ← GRAY
- number u with a "discovery time"
- for each v in Adj[u] do
- if color[v] = WHITE then ▷ avoid looping!
- Better-DFS(v)
- color[u] ← BLACK
- number u with a "finishing time"

Depth-First Spanning Tree

- As we'll see, DFS creates a tree as it explores the graph. Let's keep track of the tree as follows (actually it creates a forest not a tree):
- When v is explored directly from u, we will make u the parent of v, by setting the predecessor, aka, parent (π) field of v to u:



$\pi[v] \leftarrow u$

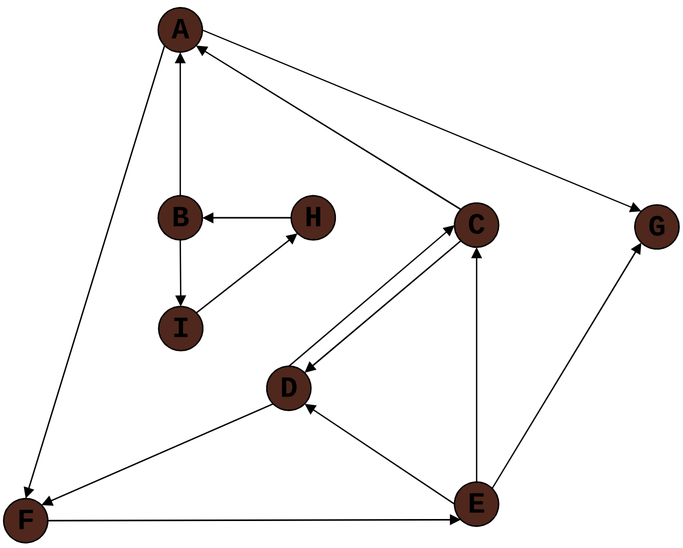
DFS(G)

```
for each vertex u ∈ V[G]
      do color[u] ← WHITE
         π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
      do if color[u] = WHITE
            then DFS-VISIT(u)
```

DFS-VISIT(u)

```
color[u] ← GRAY               ▷ White vertex u has just been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]           ▷ Explore edge (u, v).
      do if color[v] = WHITE
            then π[v] ← u
                 DFS-VISIT(v)
color[u] ← BLACK              ▷ Blacken u; it is finished.
f[u] ← time ← time + 1
```

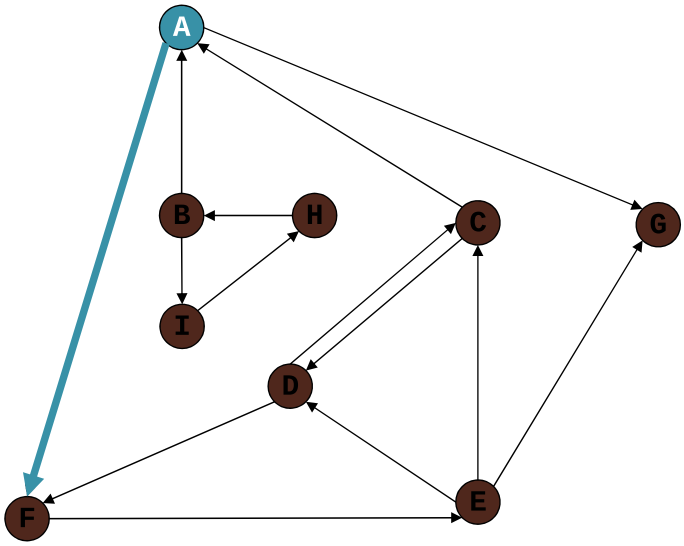
Directed Depth First Search



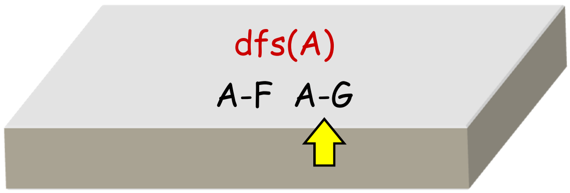
Adjacency Lists

A:	F G
B:	A I
C:	A D
D:	C F
E:	C D G
F:	E
G:	
H:	B
I:	H

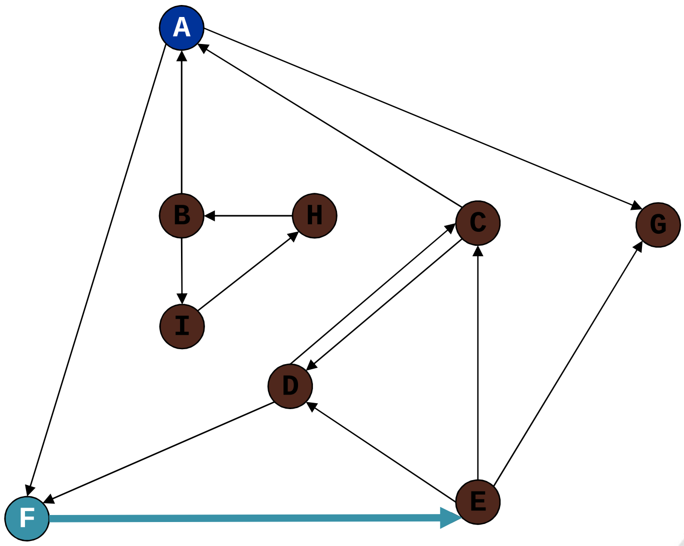
Directed Depth First Search



Function call stack:



Directed Depth First Search



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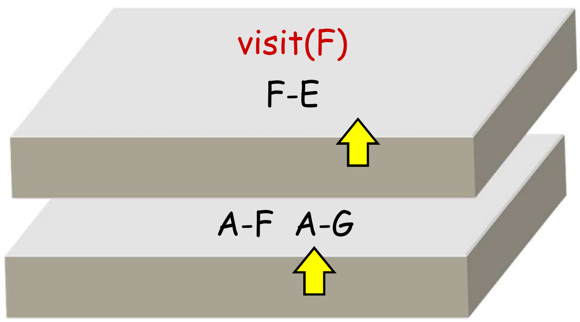
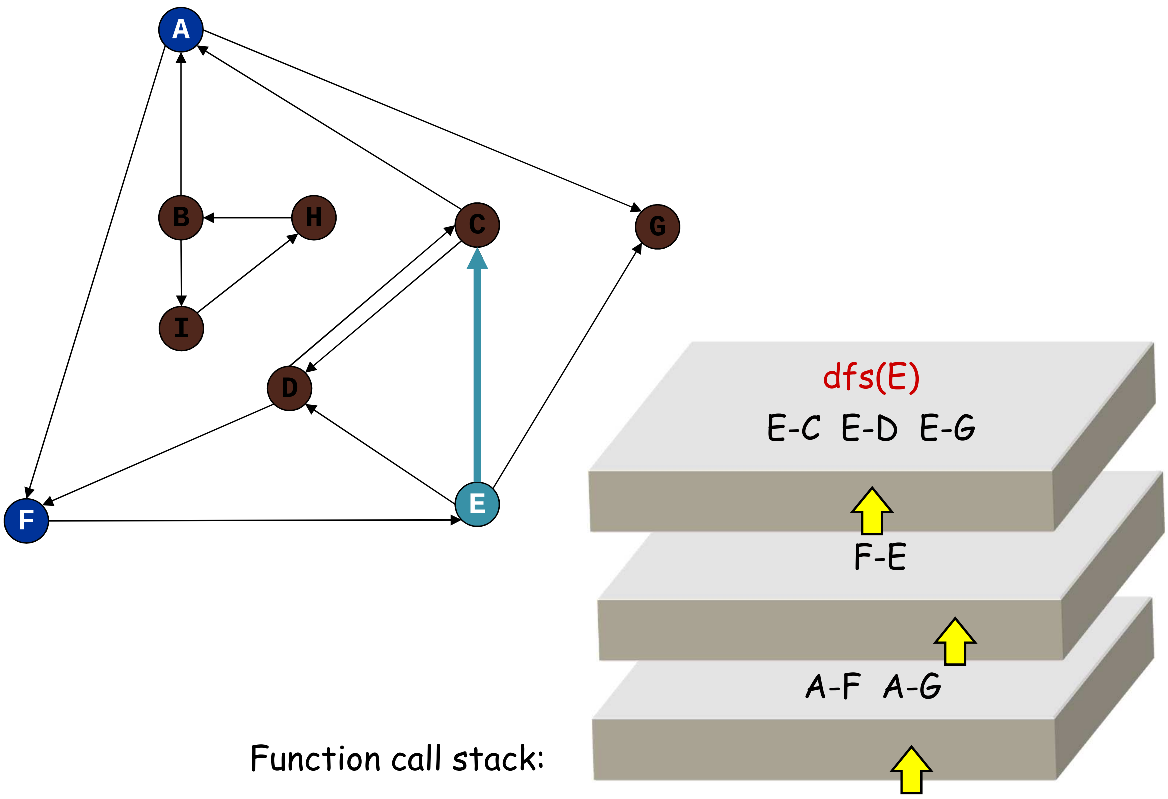


Directed Depth First Search



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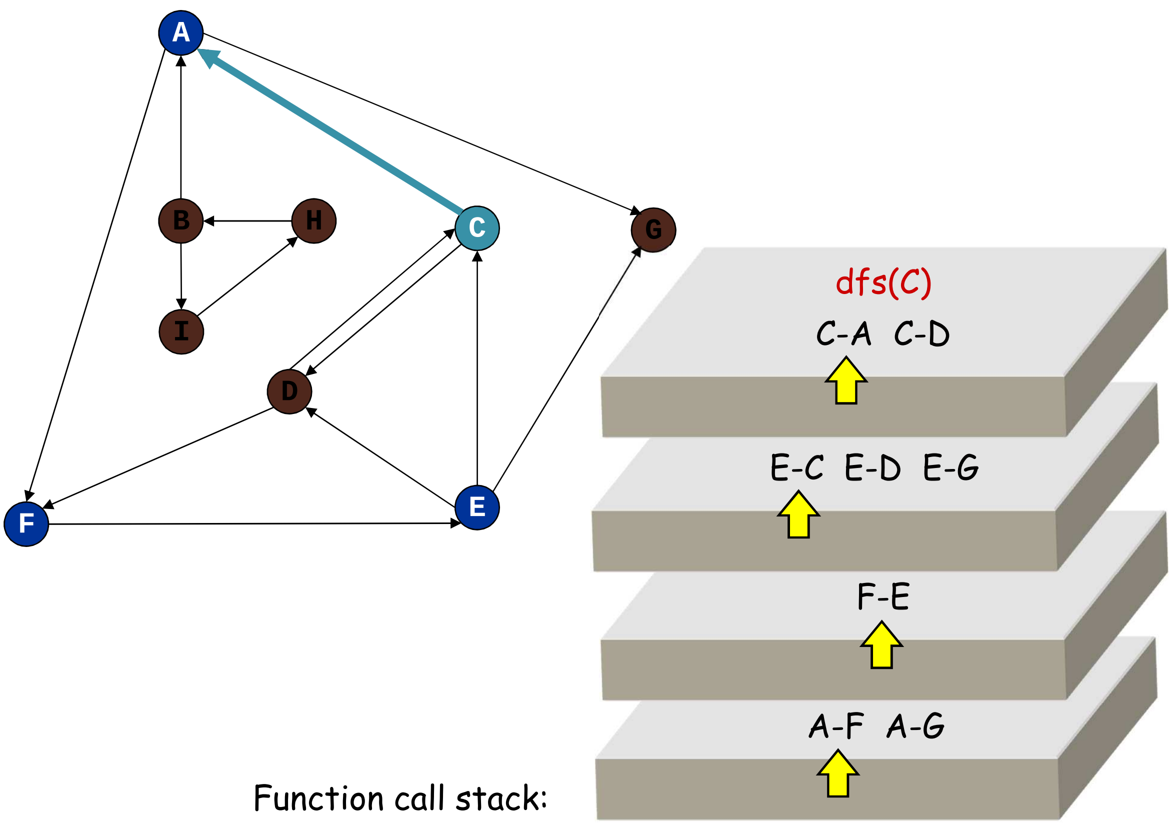

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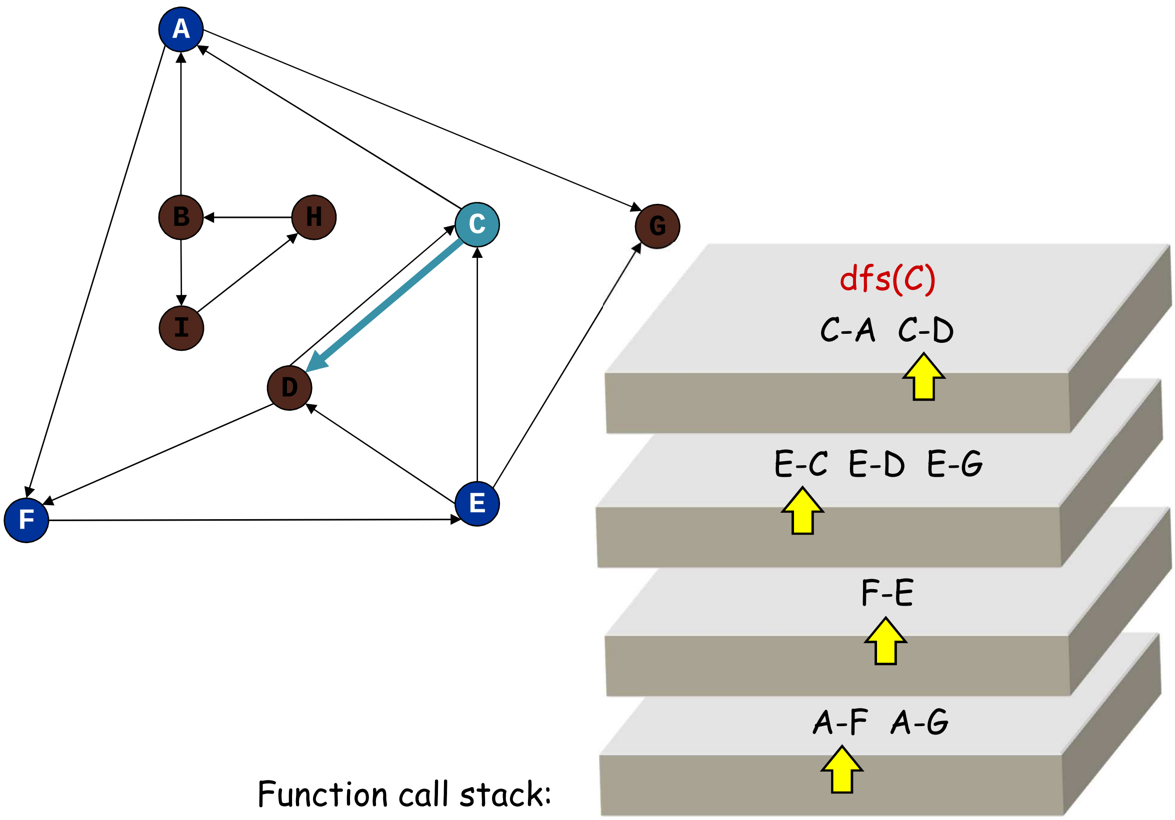

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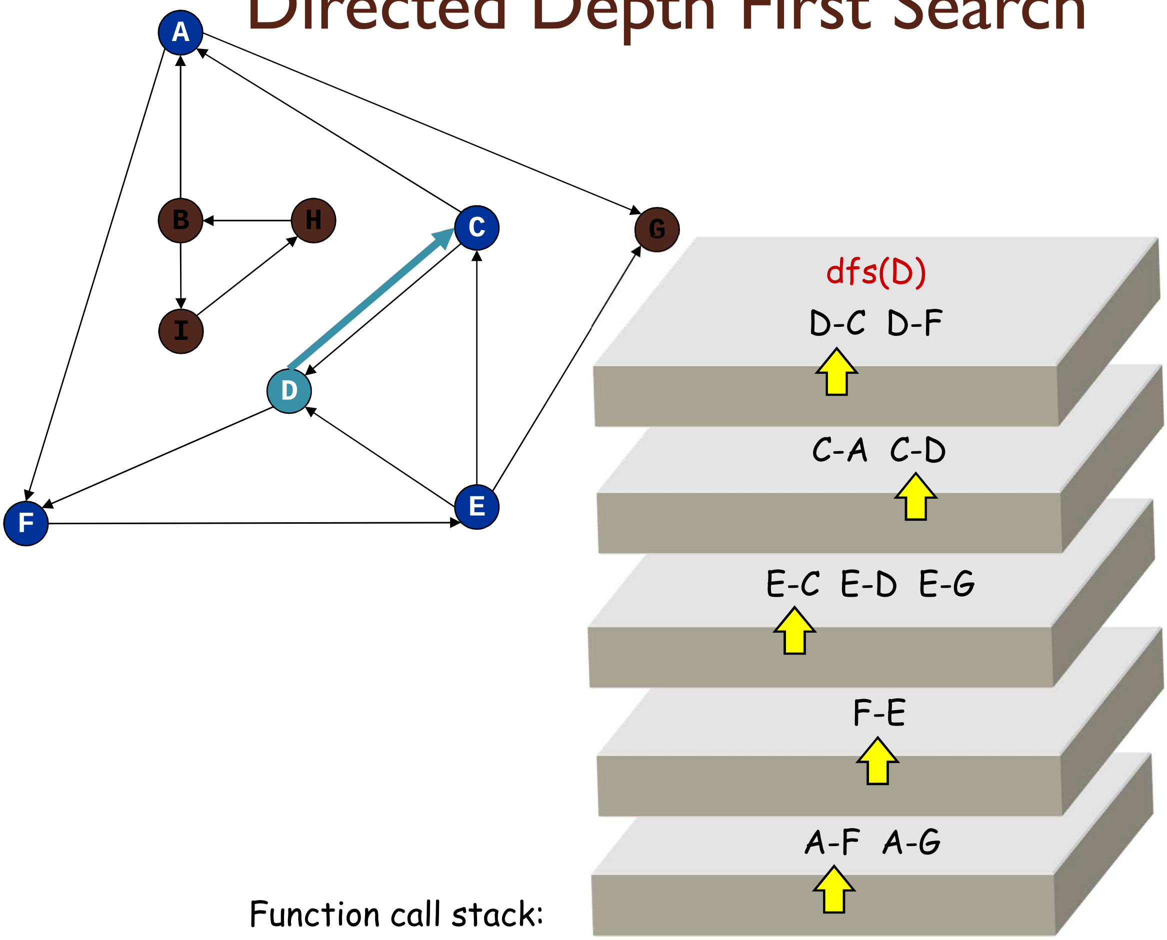

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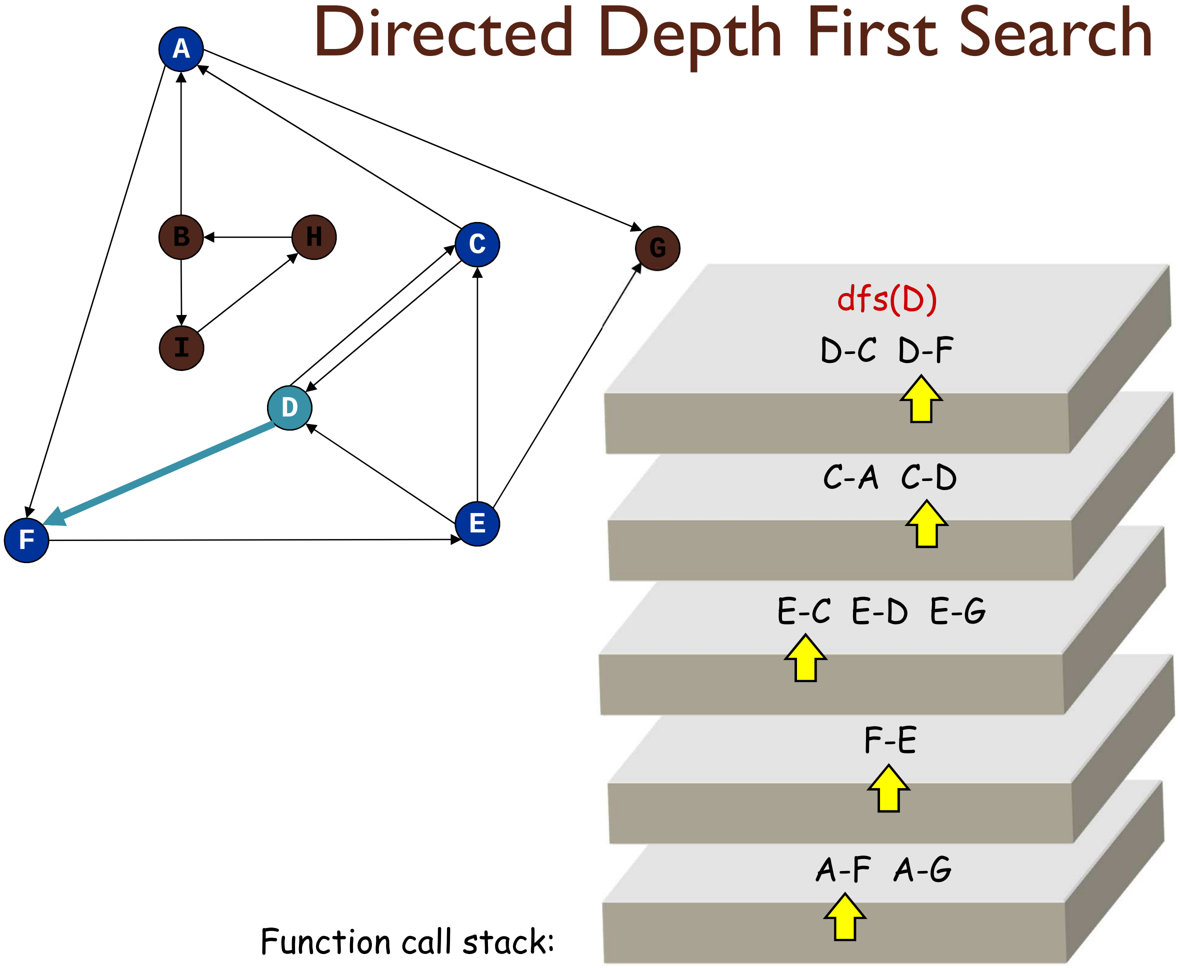

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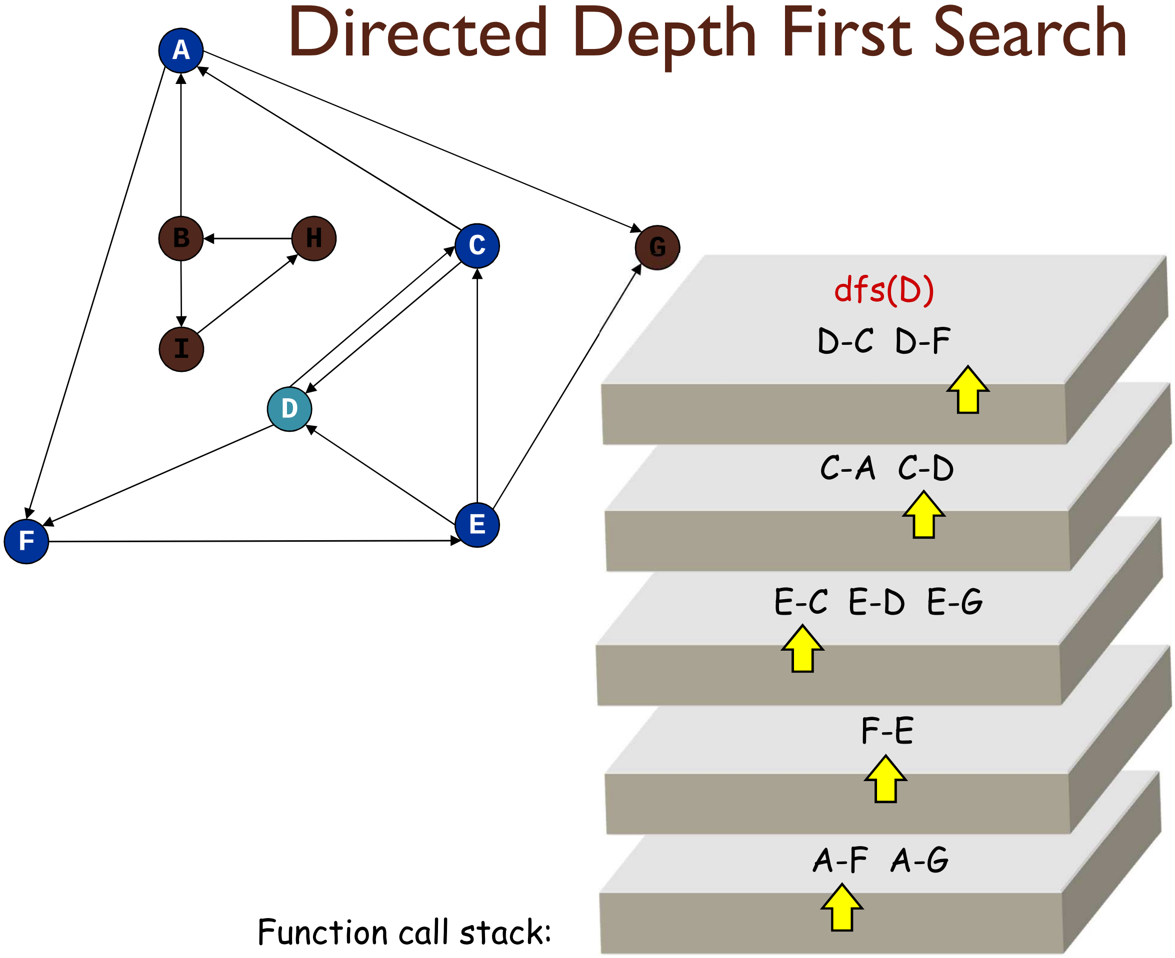

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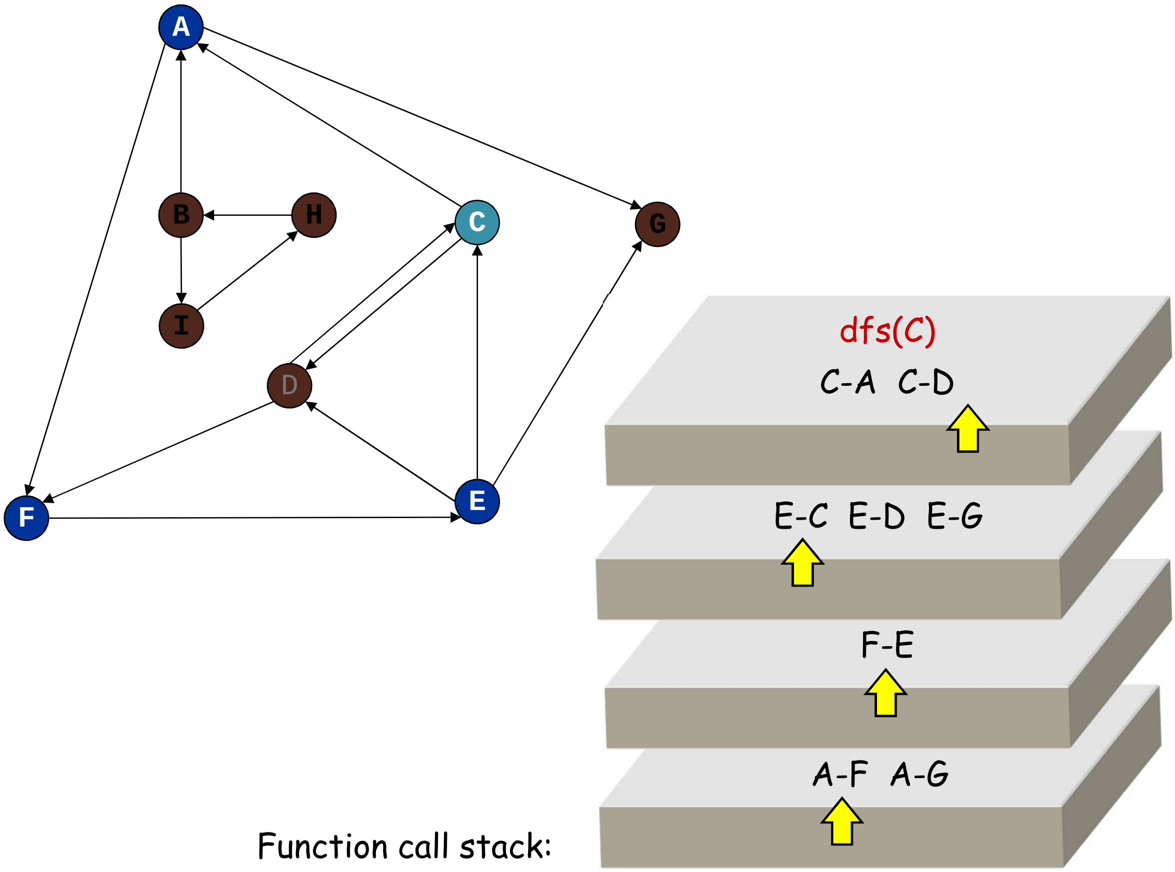

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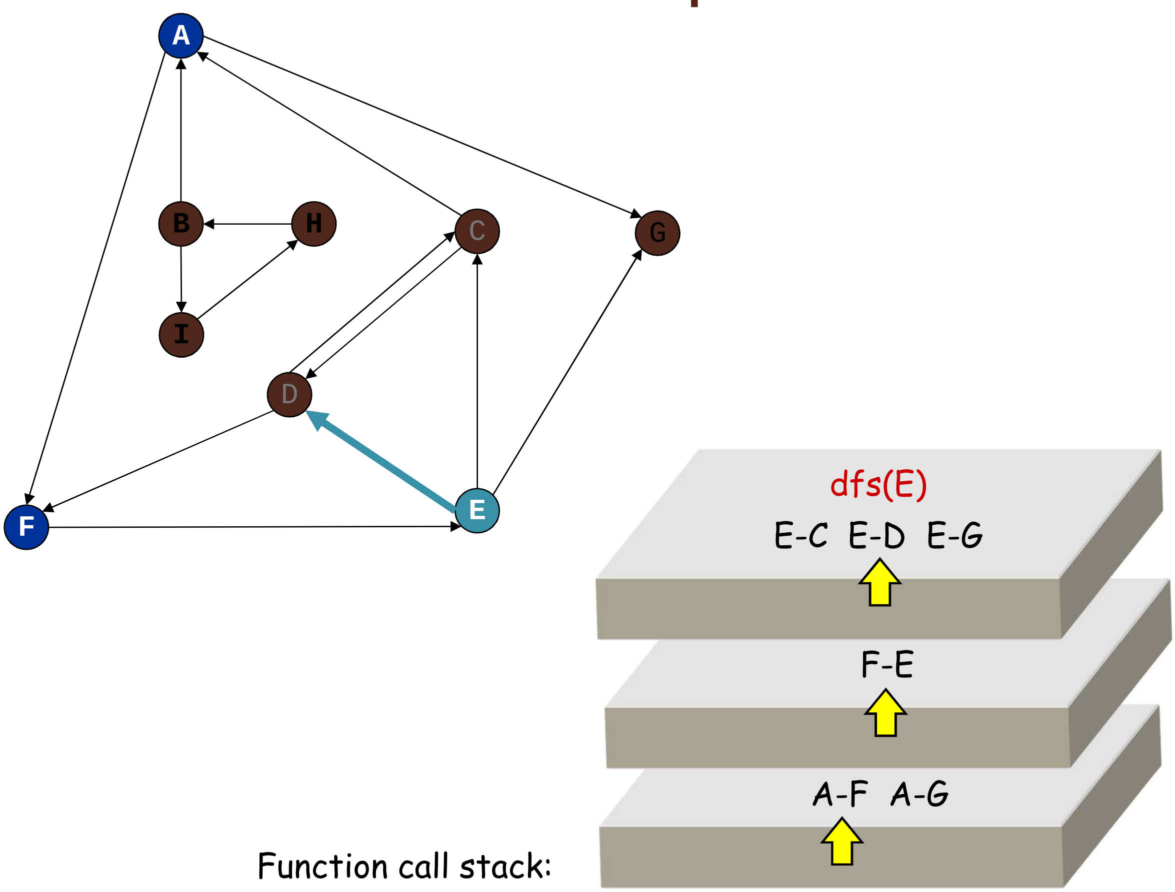

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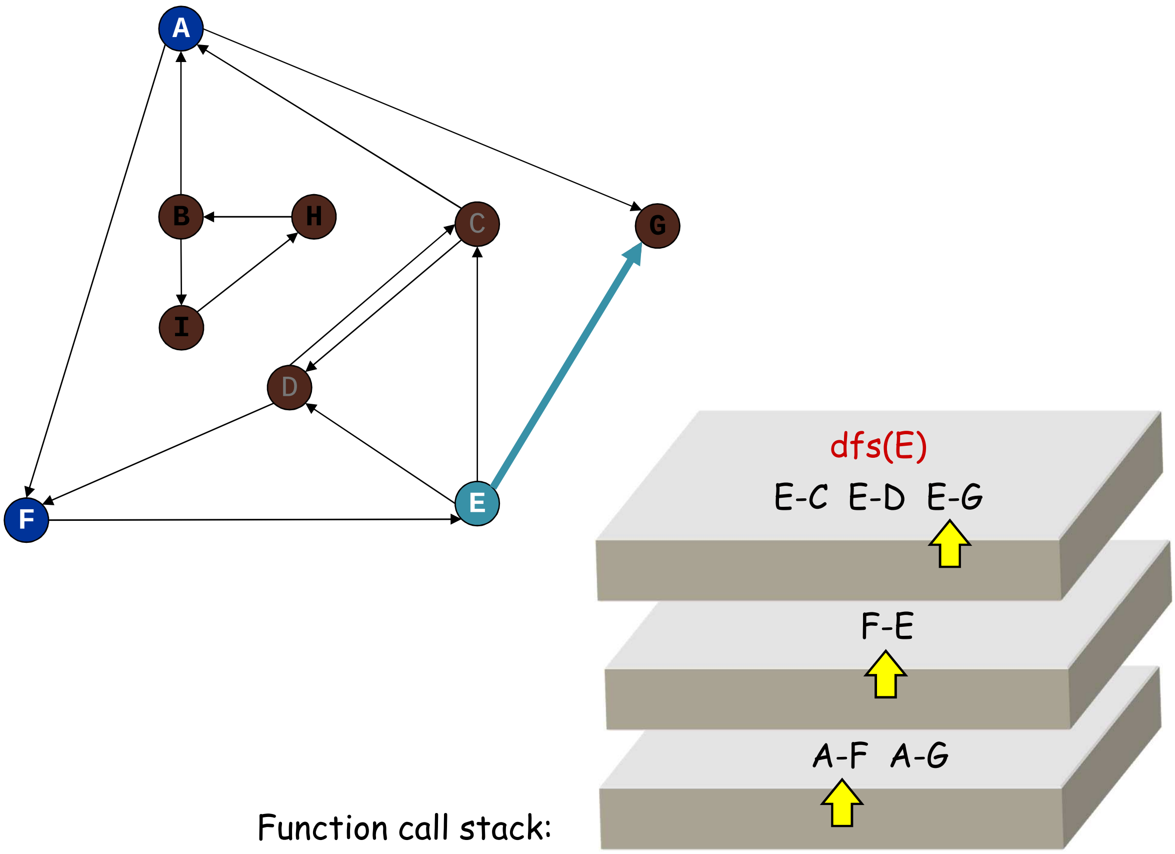

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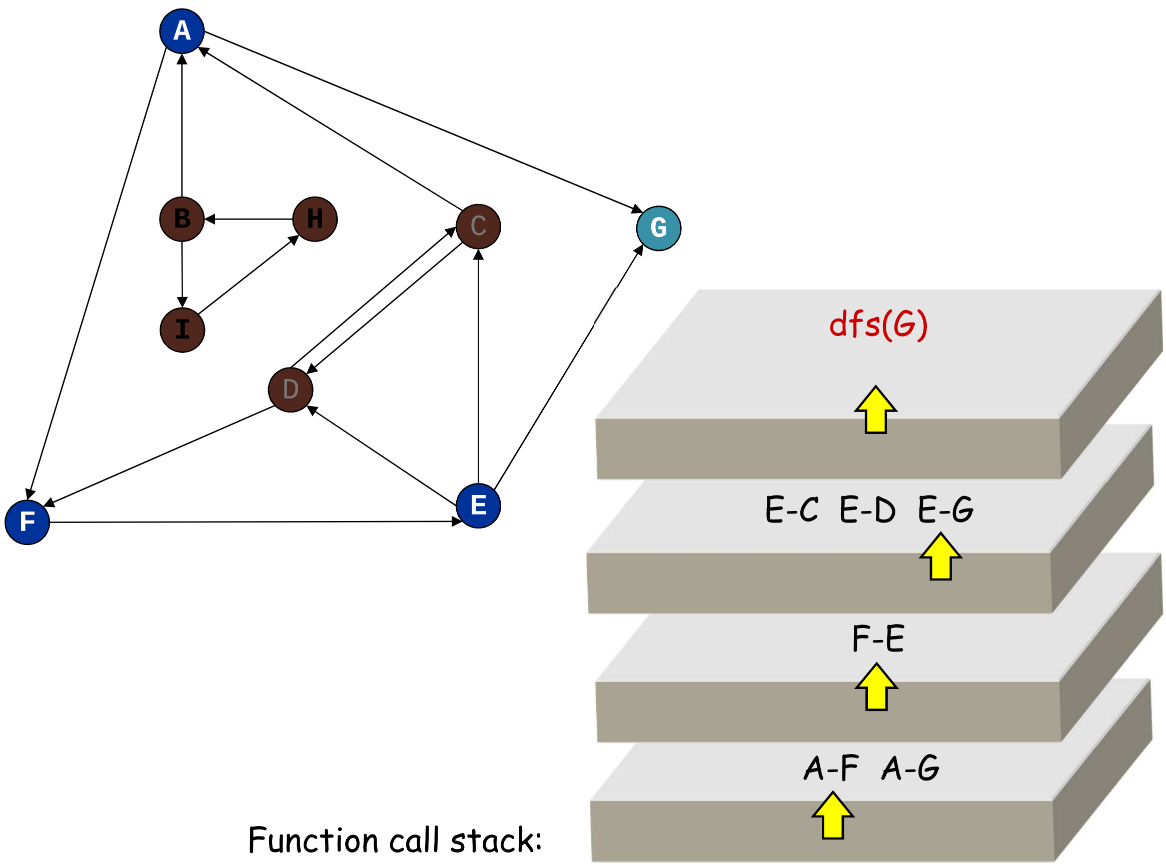

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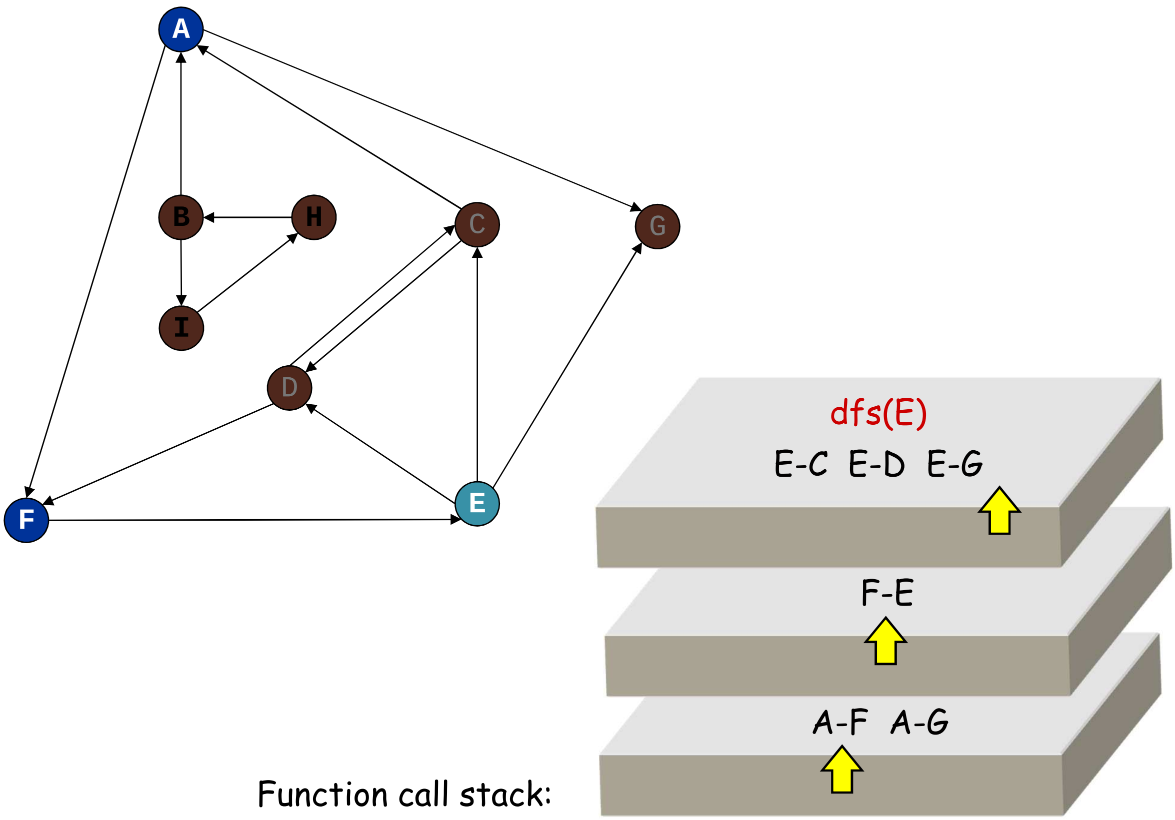

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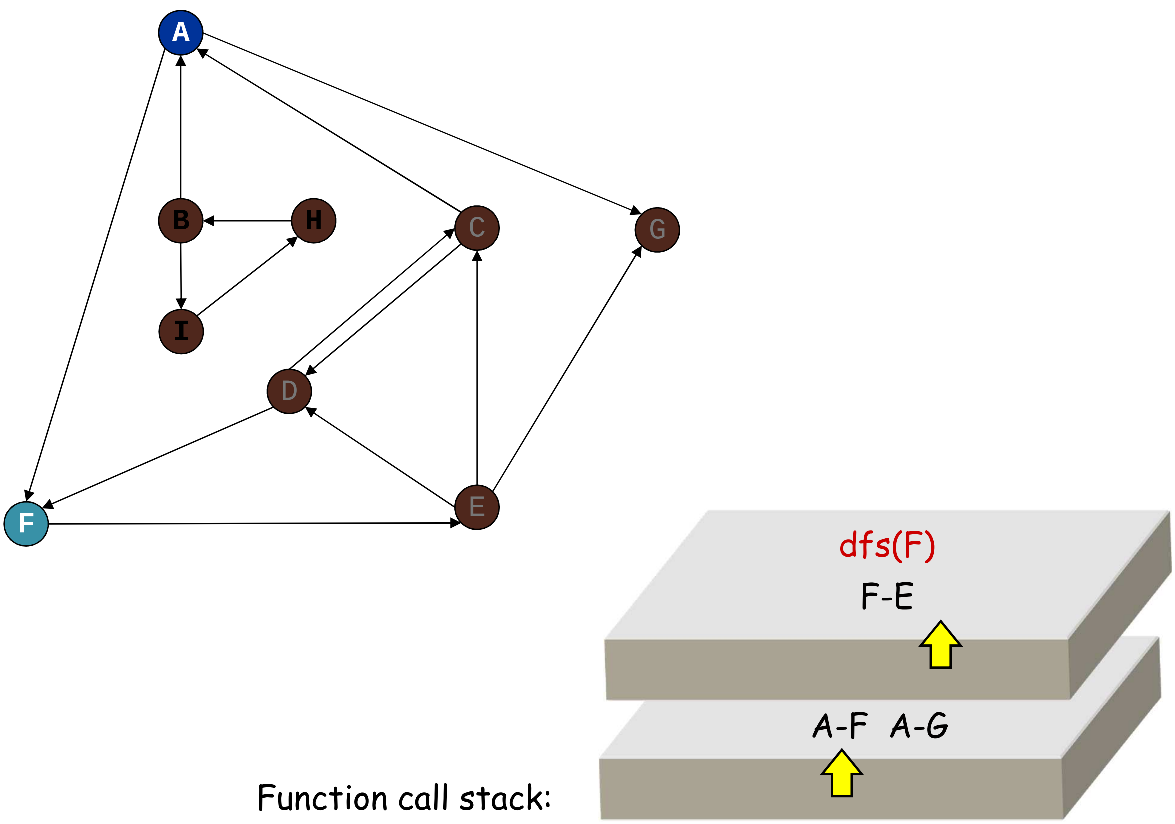

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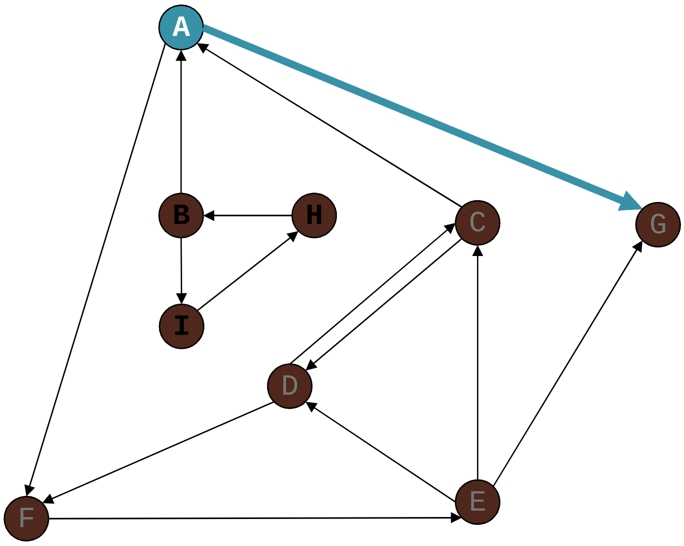
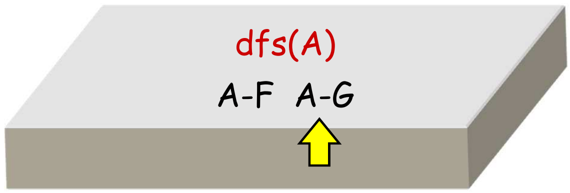


Function call stack:



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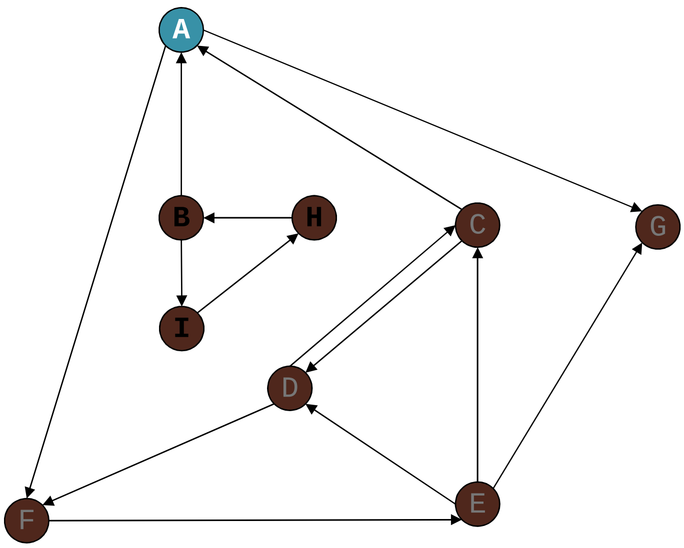

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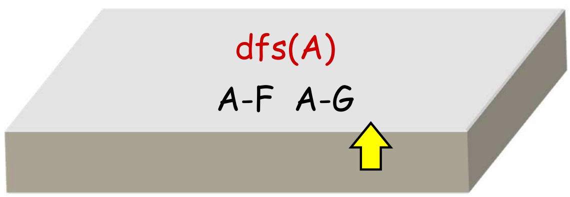
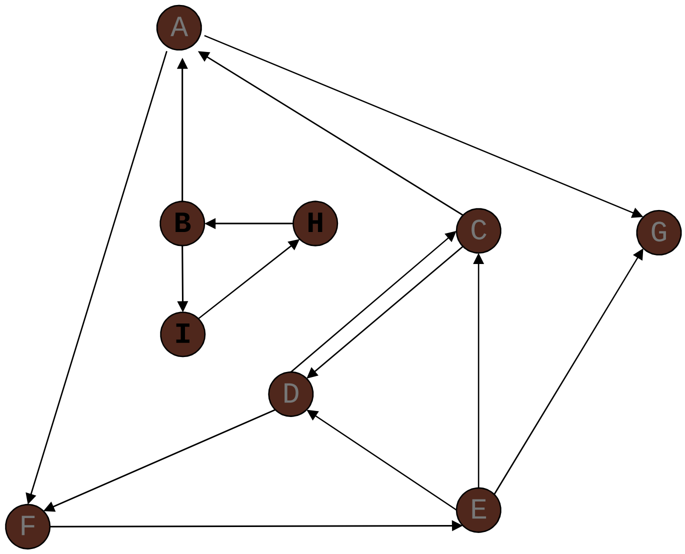
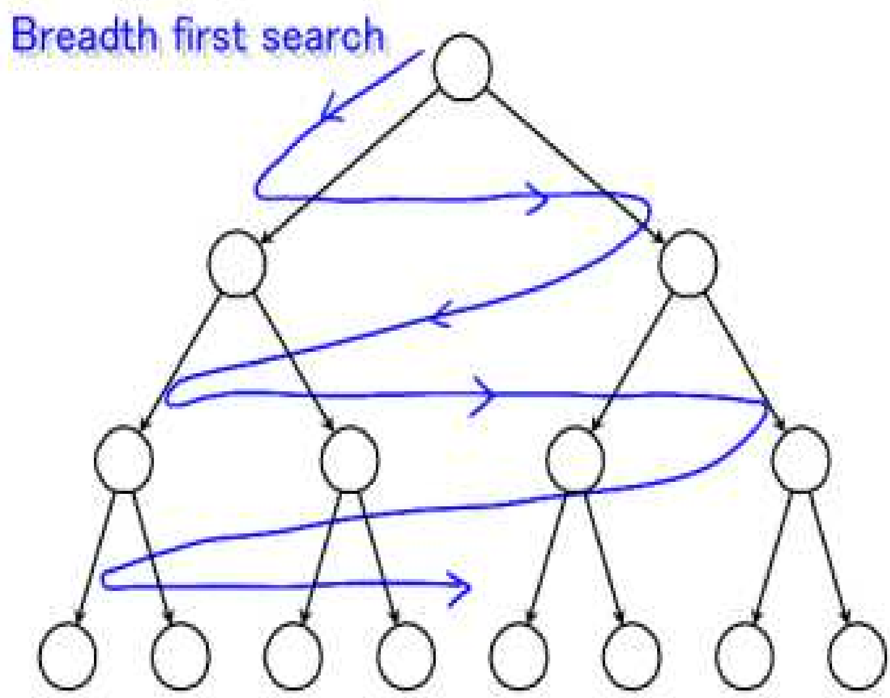


Directed Depth First Search



Nodes reachable from A: A, C, D, E, F, G

Breadth First Search



Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with ***n*** vertices and ***m*** edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

BFS Algorithm

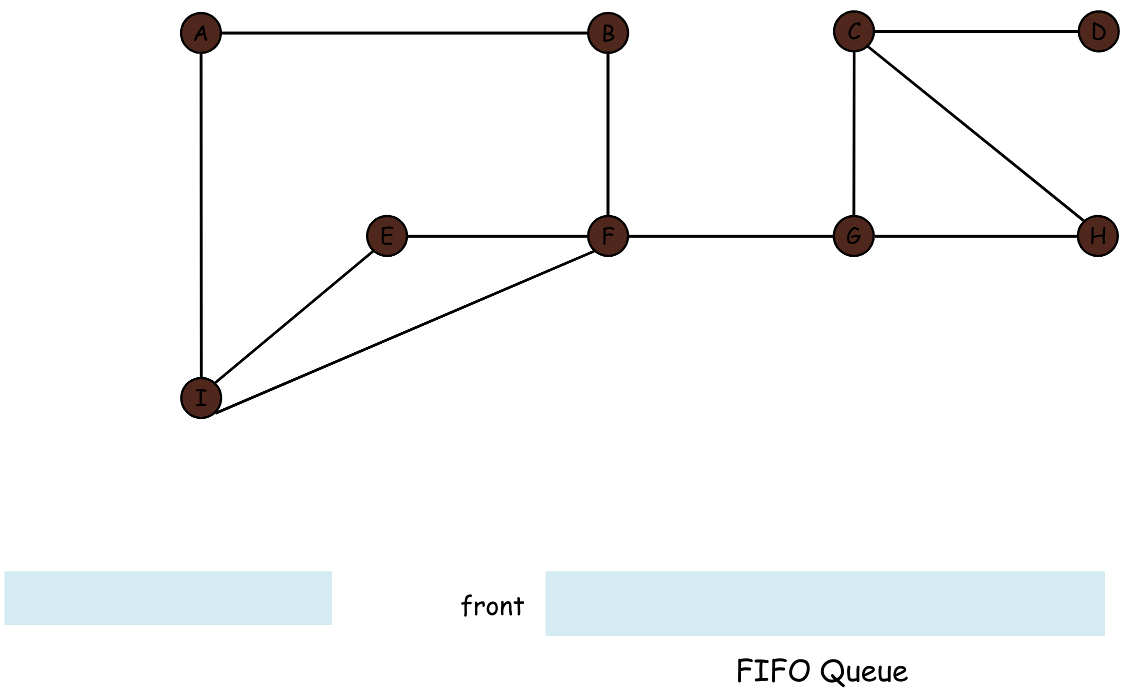
- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
  for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
  for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
  for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        BFS(G, v)
```

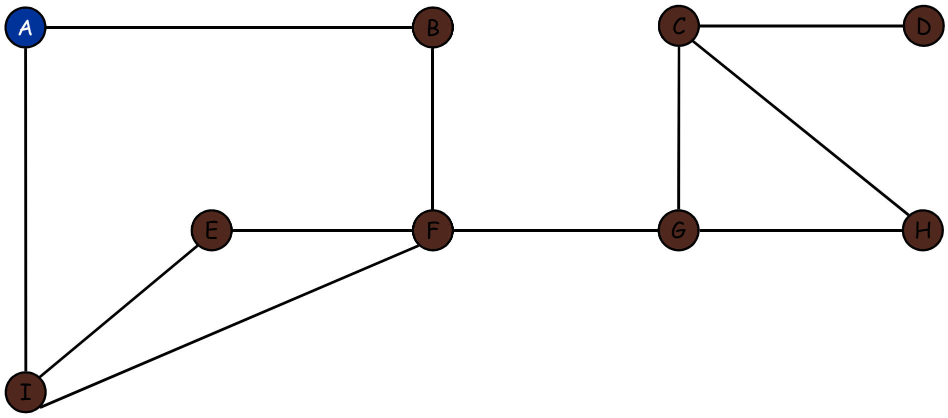
```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.insertLast(s)
  setLabel(s, VISITED)
  i ← 0
  while ¬Li.isEmpty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            setLabel(w, VISITED)
            Li+1.insertLast(w)
          else
            setLabel(e, CROSS)
    i ← i +1
```

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - A vertex is "discovered" the first time it is encountered during the search.
  - A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - White – Undiscovered.
  - Gray – Discovered but not finished.
  - Black – Finished.
    - Colors are required only to reason about the algorithm. Can be implemented without colors.

Breadth First Search



Breadth First Search

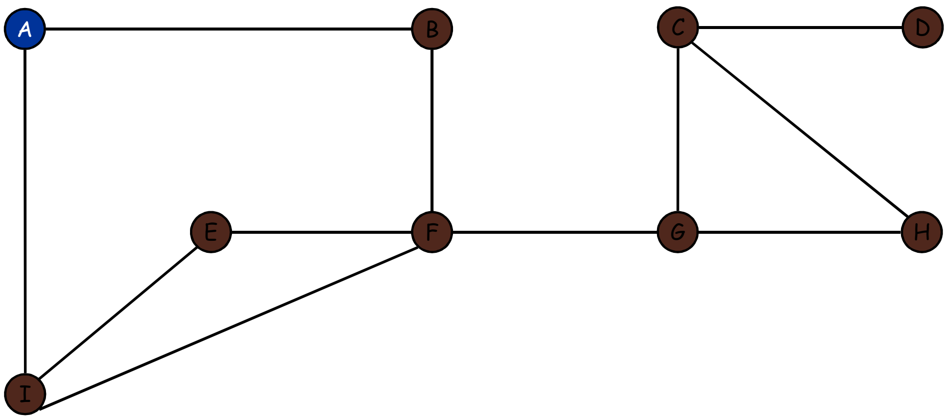


enqueue source node

front **A**

FIFO Queue

Breadth First Search

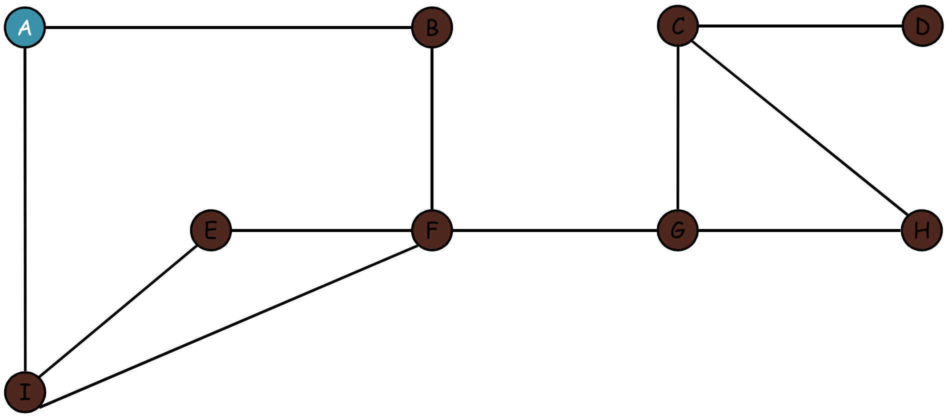


dequeue next vertex

front **A**

FIFO Queue

Breadth First Search

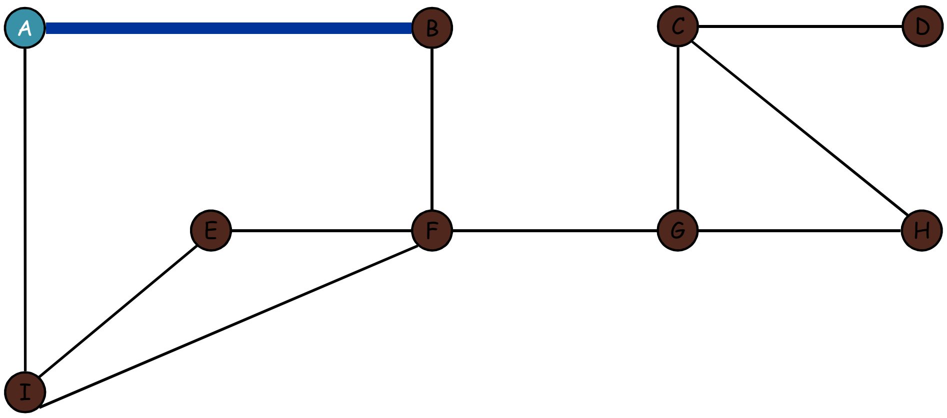


visit neighbors of A

front

FIFO Queue

Breadth First Search

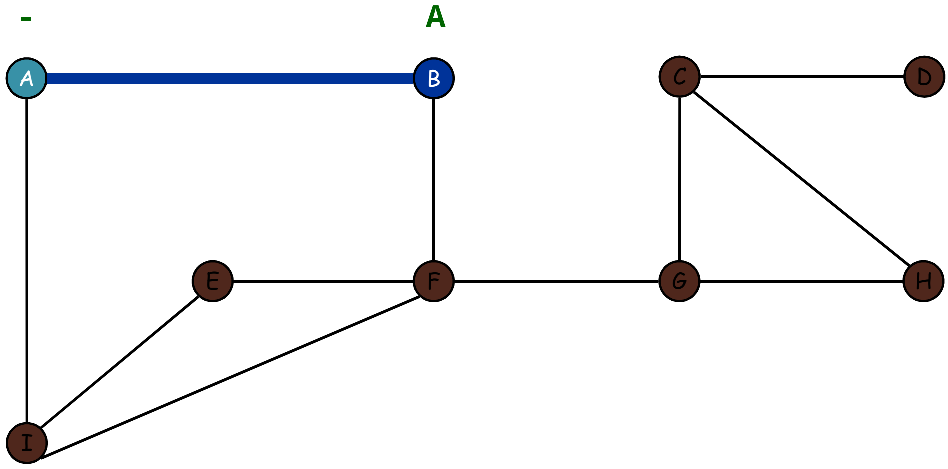


visit neighbors of A

front

FIFO Queue

Breadth First Search

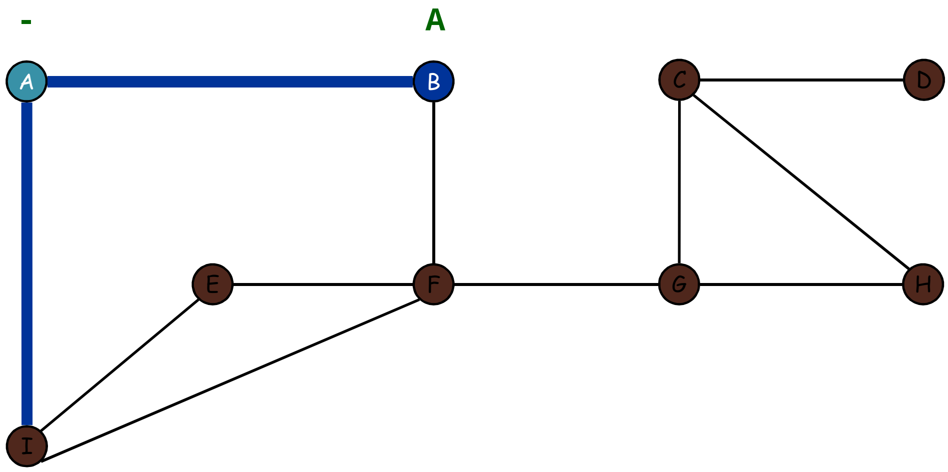


B discovered

front **B**

FIFO Queue

Breadth First Search

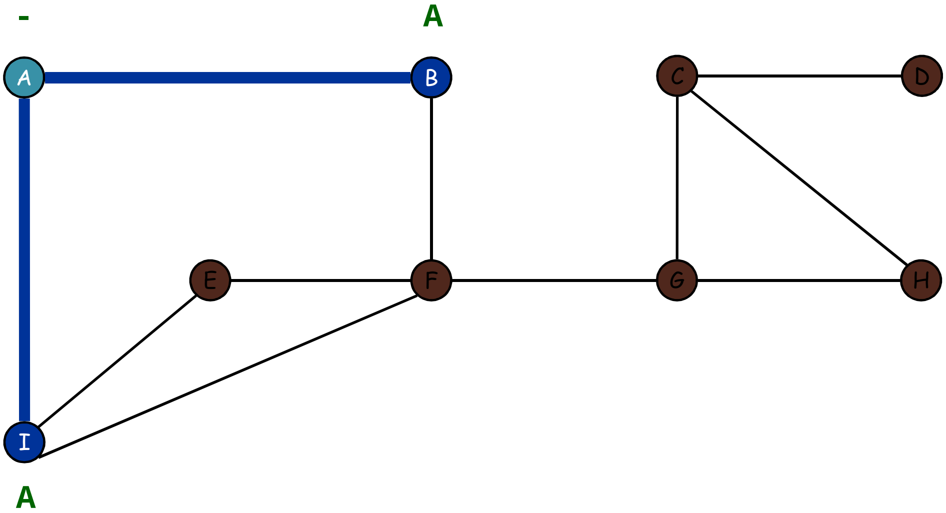


visit neighbors of A

front **B**

FIFO Queue

Breadth First Search

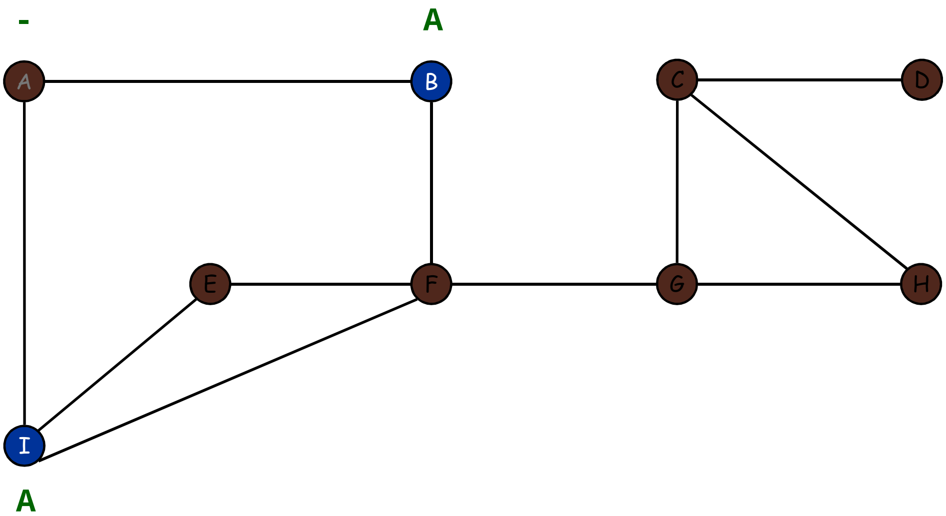


I discovered

front **B I**

FIFO Queue

Breadth First Search

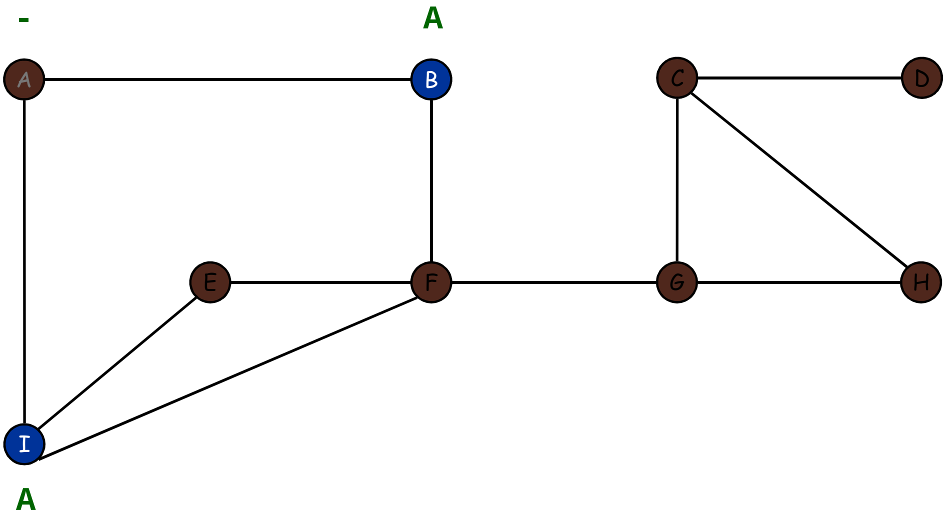


finished with A

front **B I**

FIFO Queue

Breadth First Search

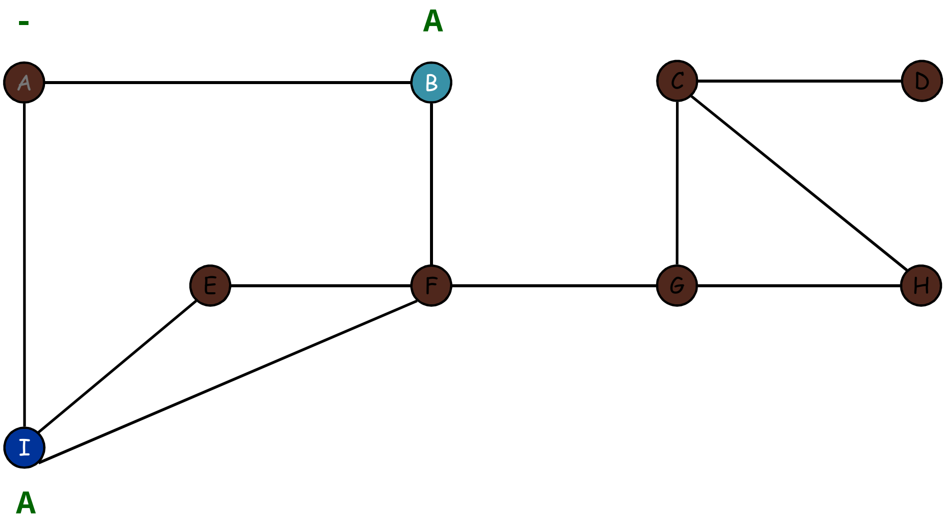


dequeue next vertex

front **B I**

FIFO Queue

Breadth First Search

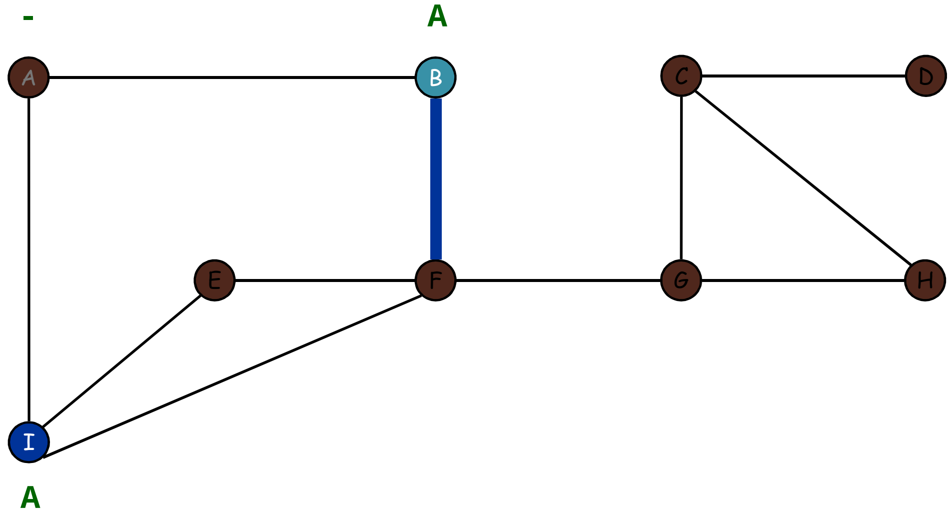


visit neighbors of B

front **I**

FIFO Queue

Breadth First Search

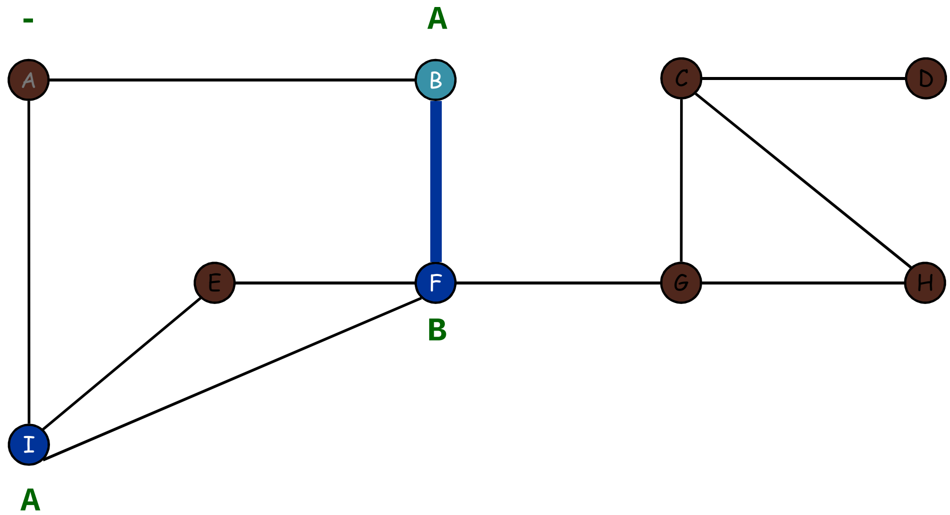


visit neighbors of B

front I

FIFO Queue

Breadth First Search

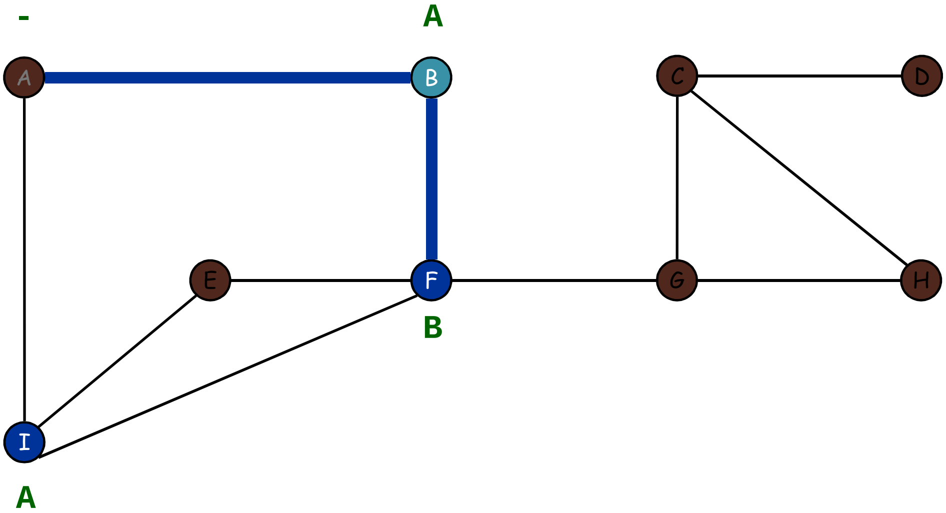


F discovered

front I F

FIFO Queue

Breadth First Search

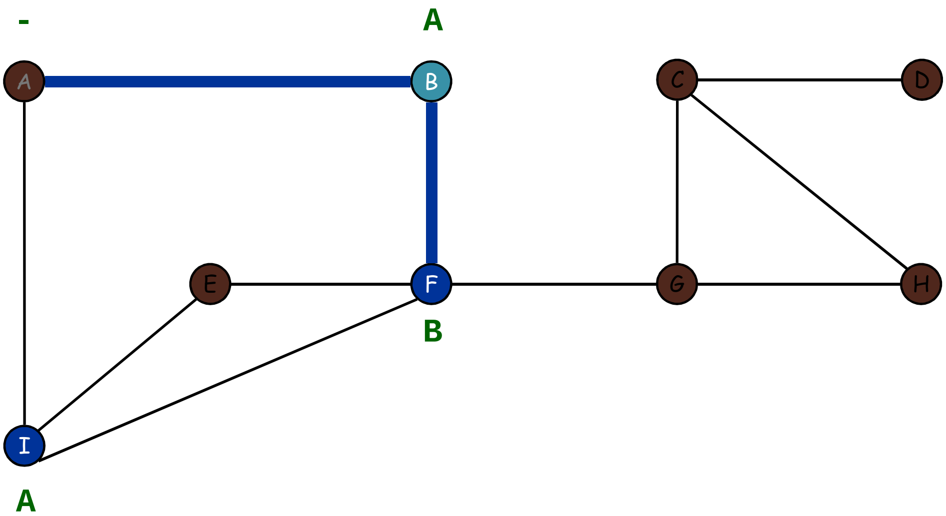


visit neighbors of B

front | **I F**

FIFO Queue

Breadth First Search

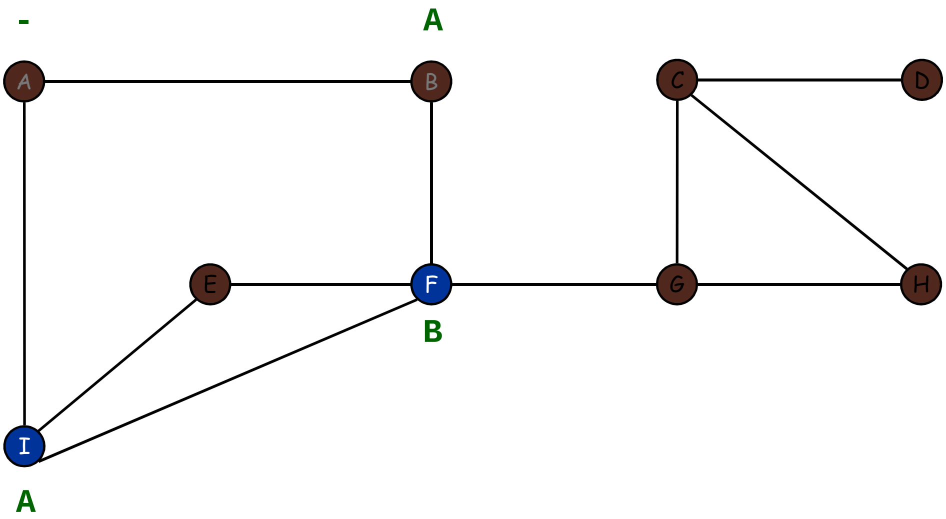


A already discovered

front | **I F**

FIFO Queue

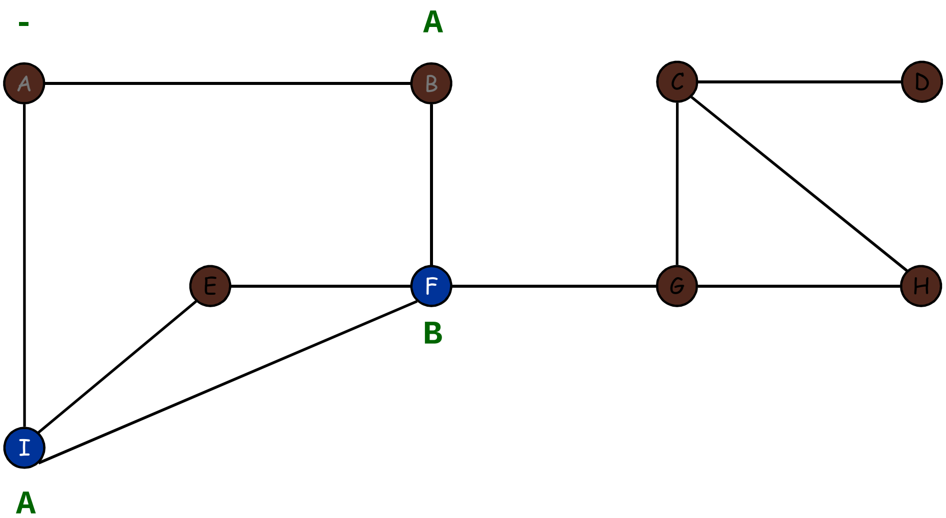
Breadth First Search



finished with B | front | I F

FIFO Queue

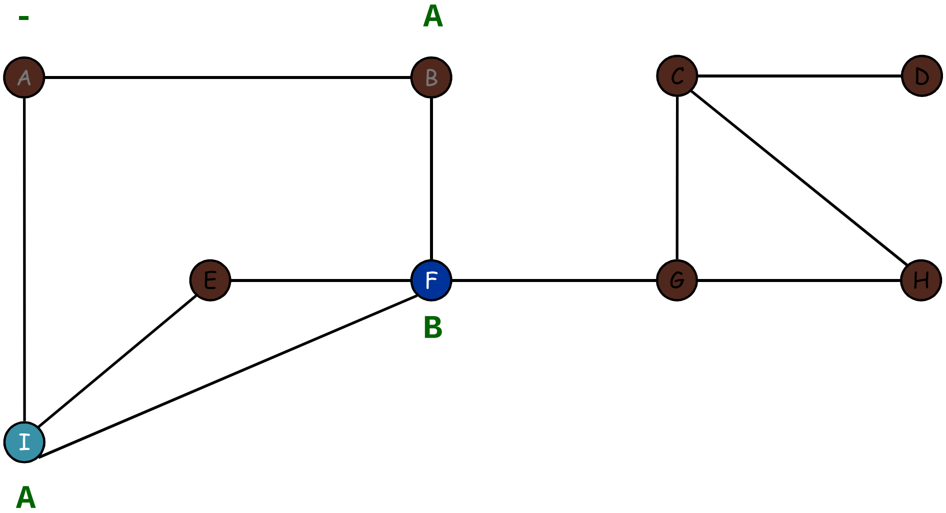
Breadth First Search



dequeue next vertex | front | I F

FIFO Queue

Breadth First Search

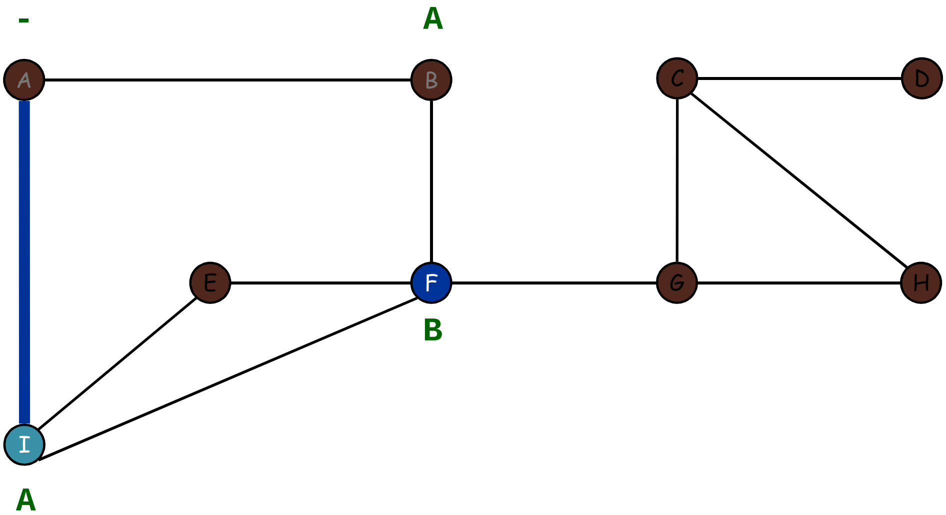


visit neighbors of I

front **F**

FIFO Queue

Breadth First Search

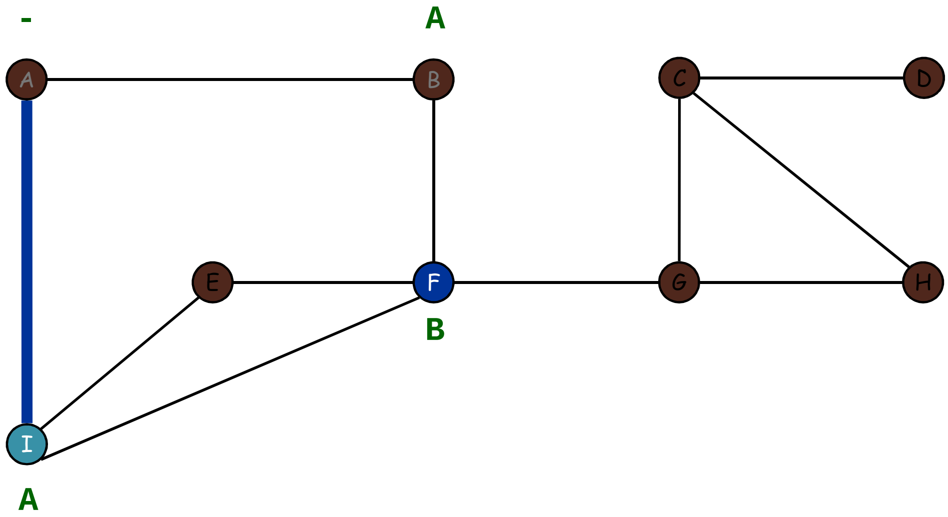


visit neighbors of I

front **F**

FIFO Queue

Breadth First Search

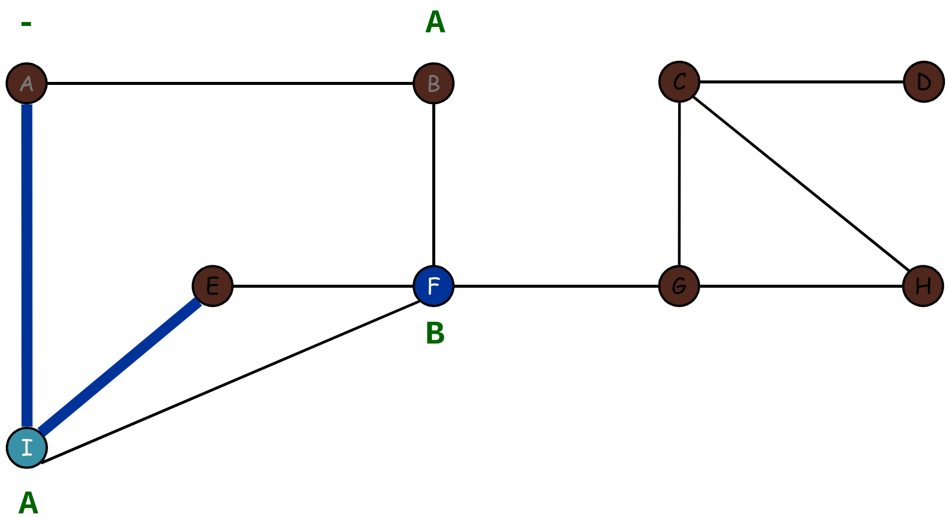


A already discovered

front F

FIFO Queue

Breadth First Search

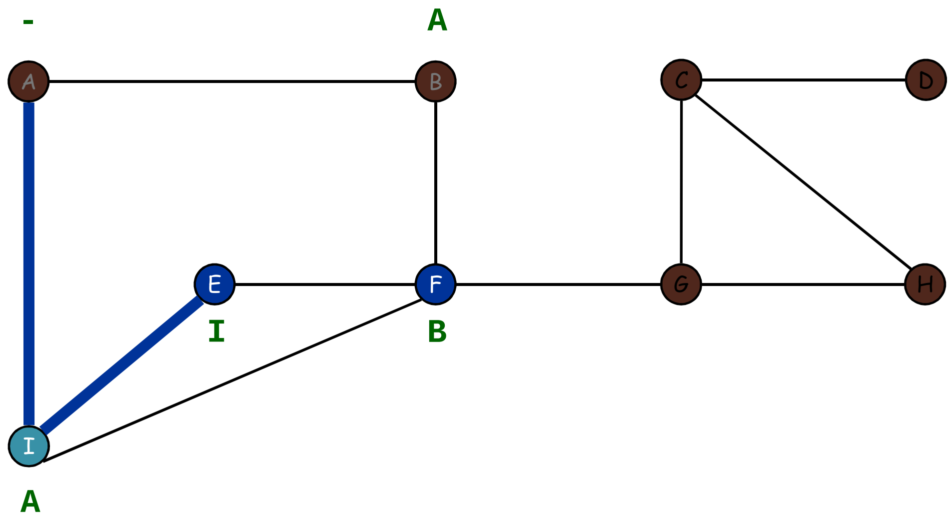


visit neighbors of I

front F

FIFO Queue

Breadth First Search

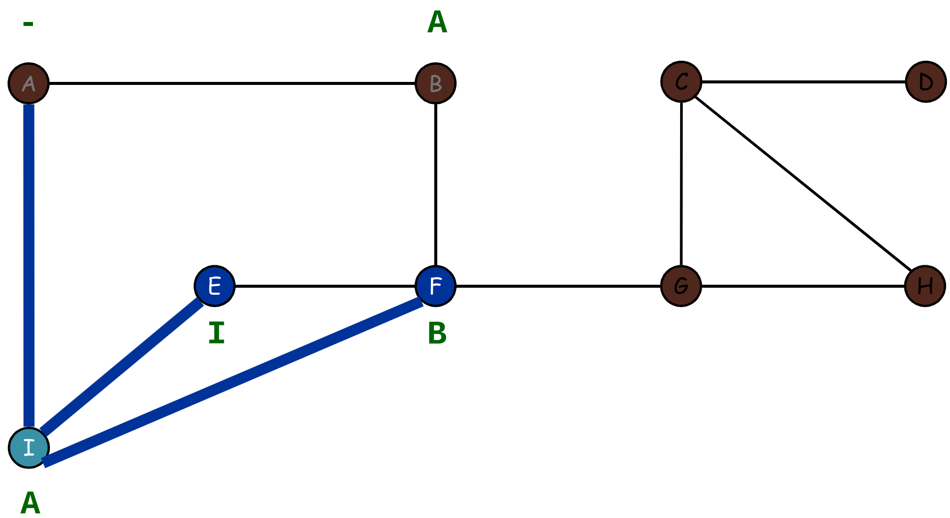


E discovered

front **F E**

FIFO Queue

Breadth First Search

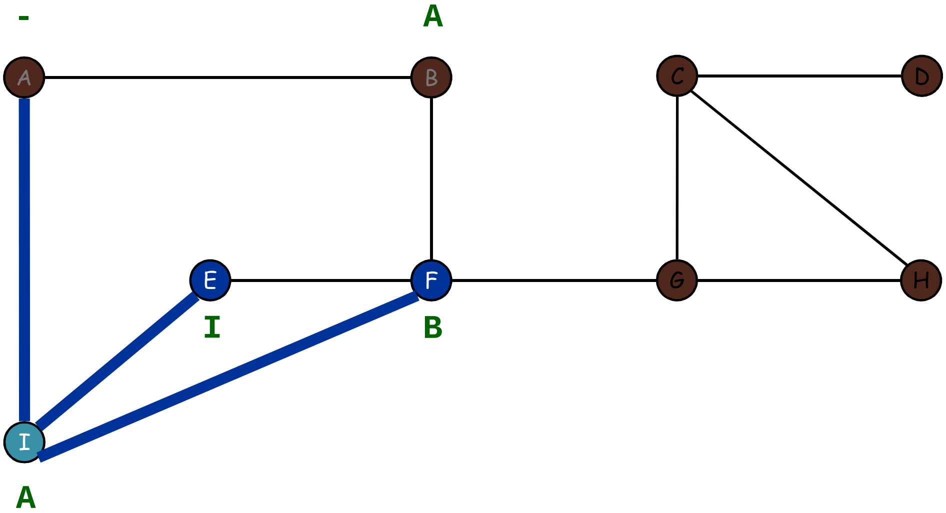


visit neighbors of I

front **F E**

FIFO Queue

Breadth First Search



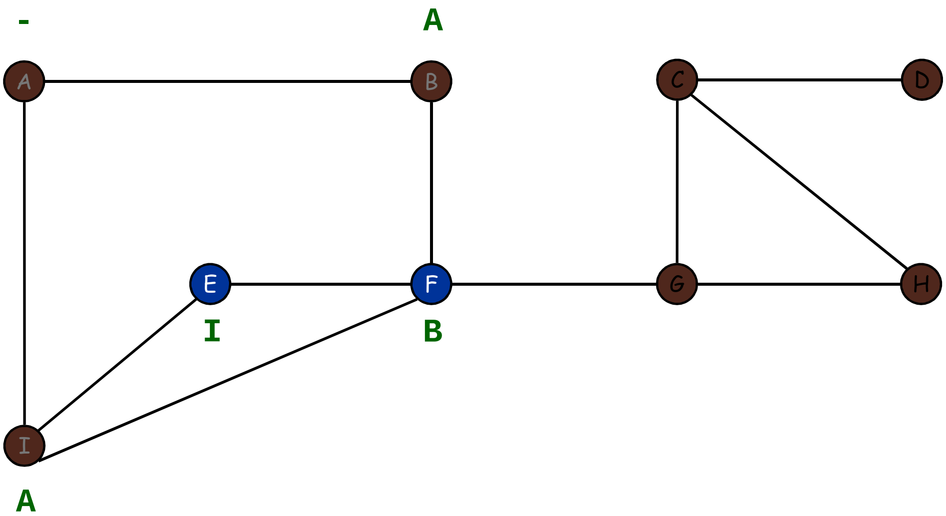
F already discovered

front

F E

FIFO Queue

Breadth First Search



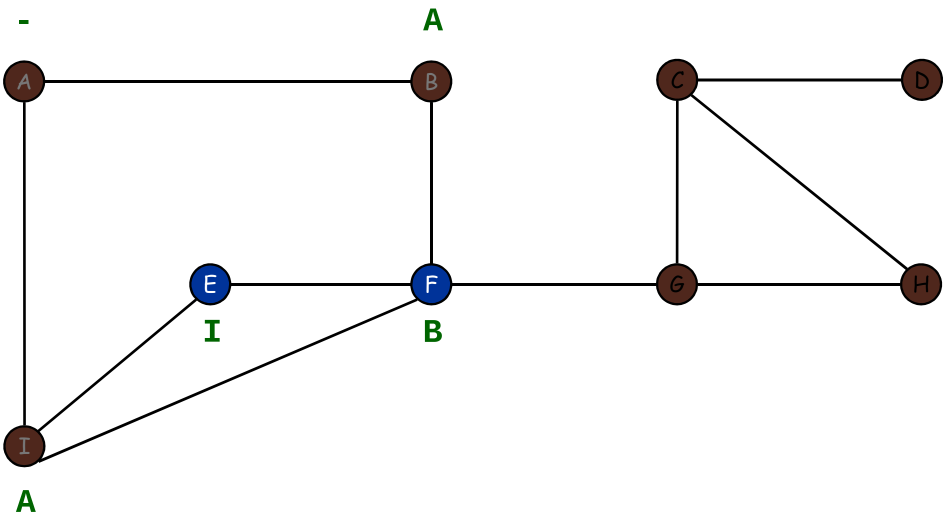
I finished

front

F E

FIFO Queue

Breadth First Search

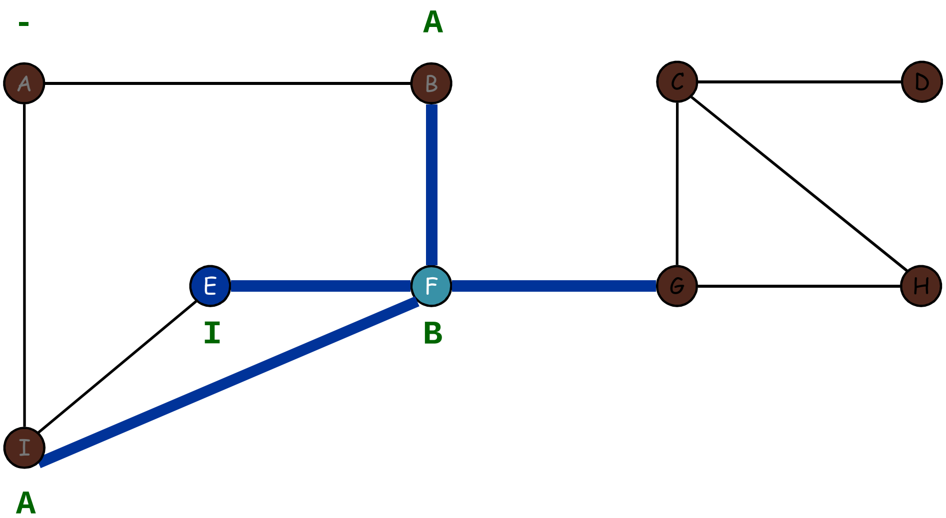


dequeue next vertex

front F E

FIFO Queue

Breadth First Search

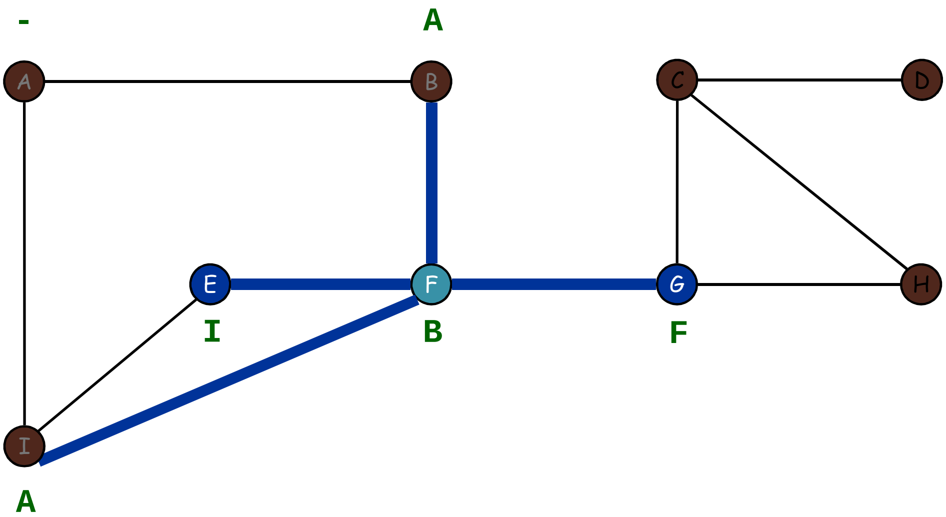


visit neighbors of F

front E

FIFO Queue

Breadth First Search



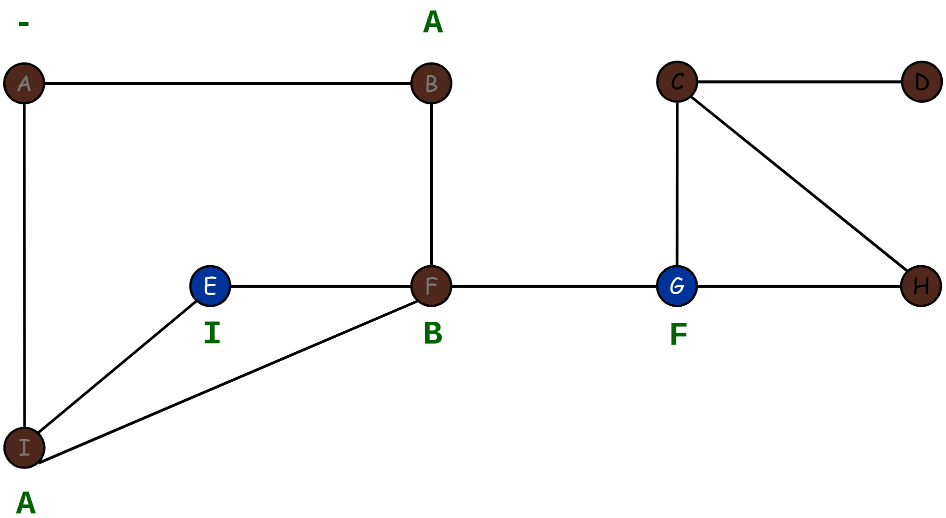
G discovered

front

E G

FIFO Queue

Breadth First Search



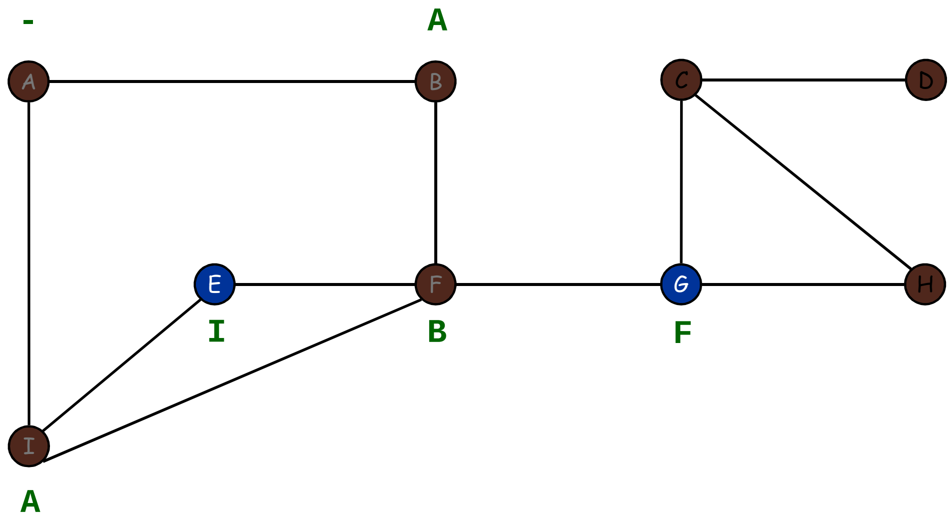
F finished

front

E G

FIFO Queue

Breadth First Search

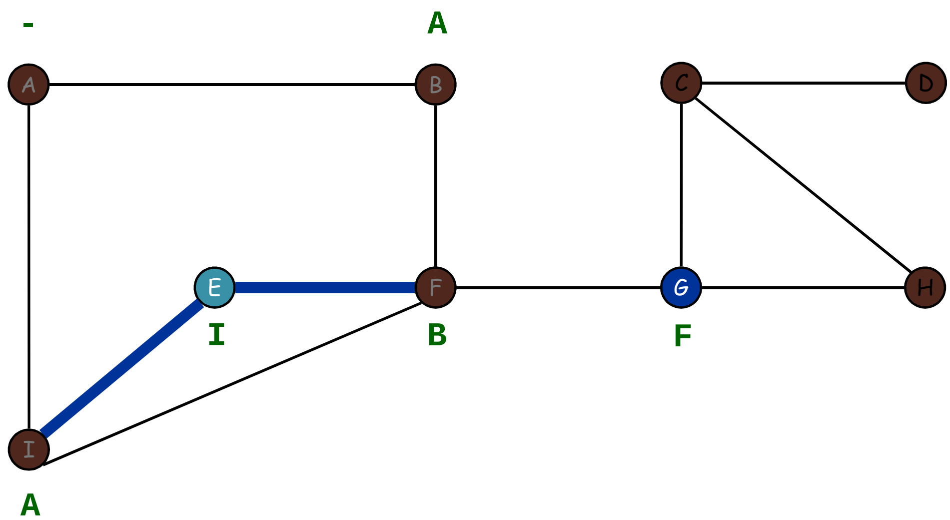


dequeue next vertex

front E G

FIFO Queue

Breadth First Search

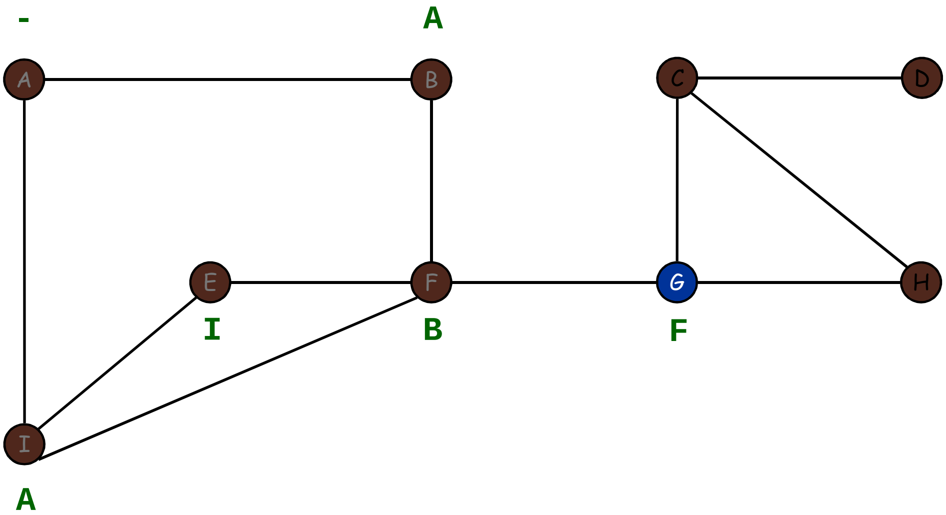


visit neighbors of E

front G

FIFO Queue

Breadth First Search



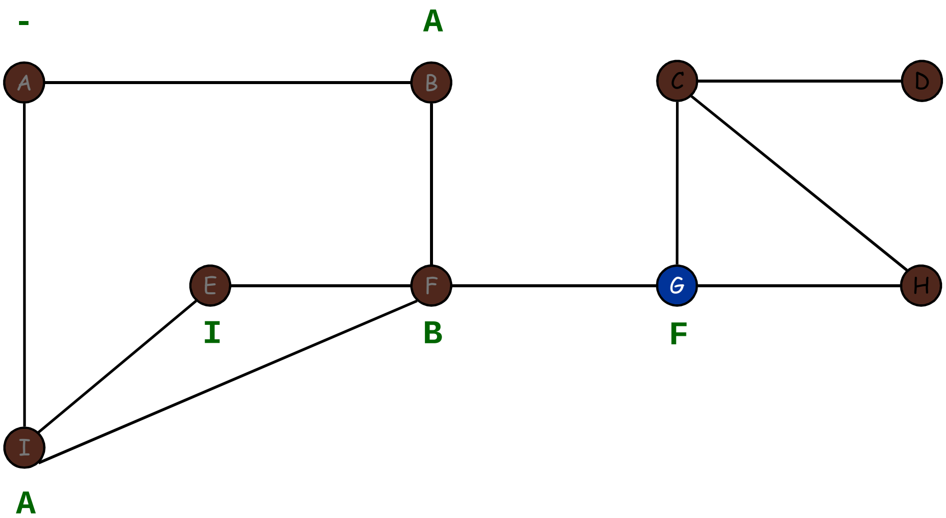
E finished

front

G

FIFO Queue

Breadth First Search



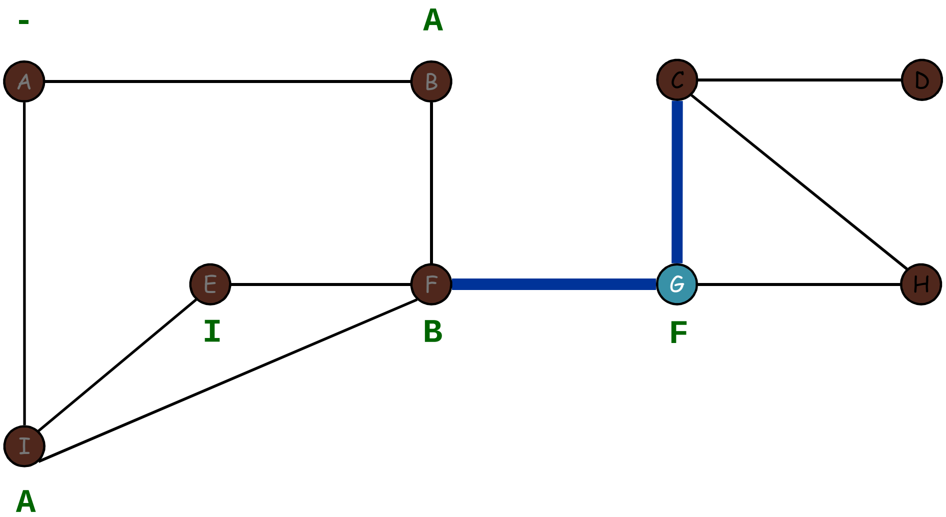
dequeue next vertex

front

G

FIFO Queue

Breadth First Search

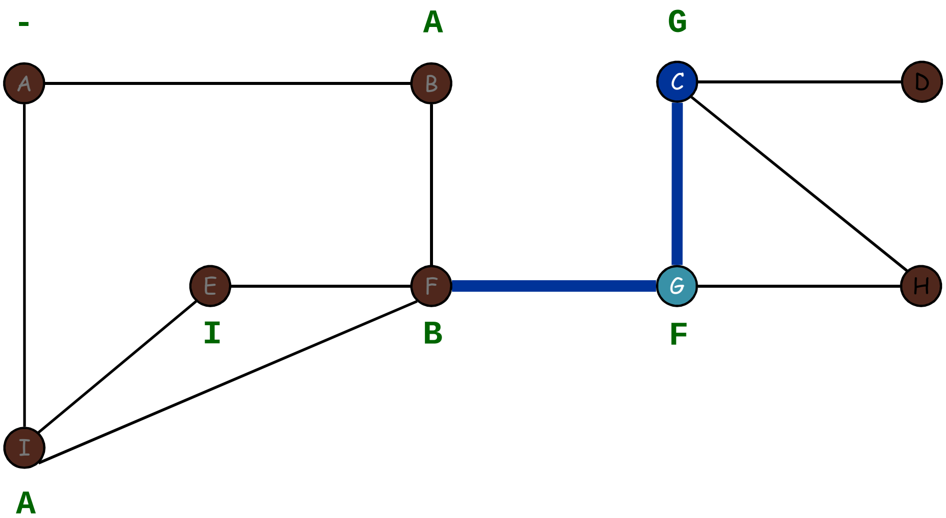


visit neighbors of G

front

FIFO Queue

Breadth First Search

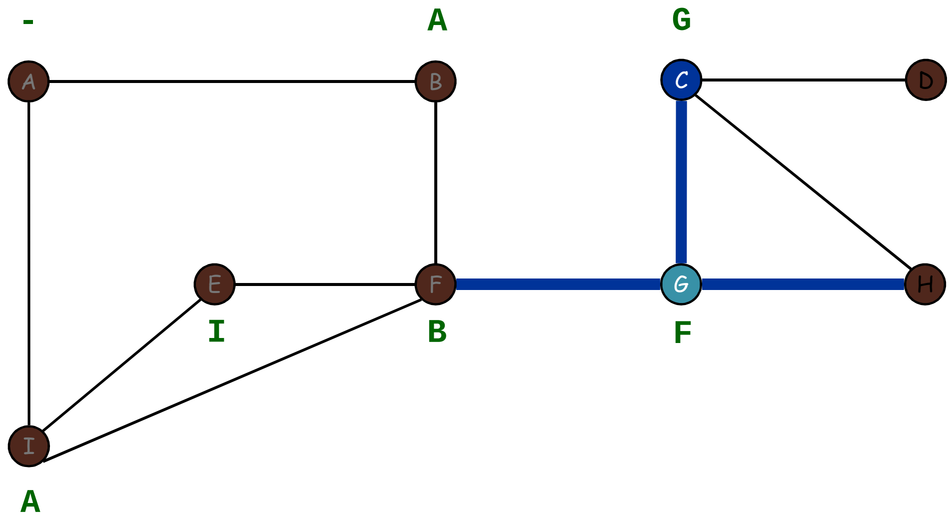


C discovered

front **C**

FIFO Queue

Breadth First Search



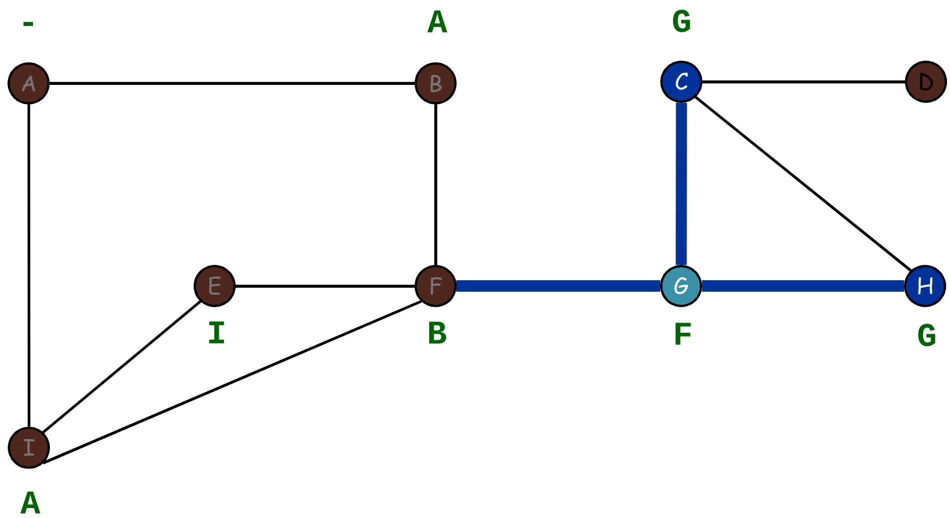
visit neighbors of G

front

C

FIFO Queue

Breadth First Search



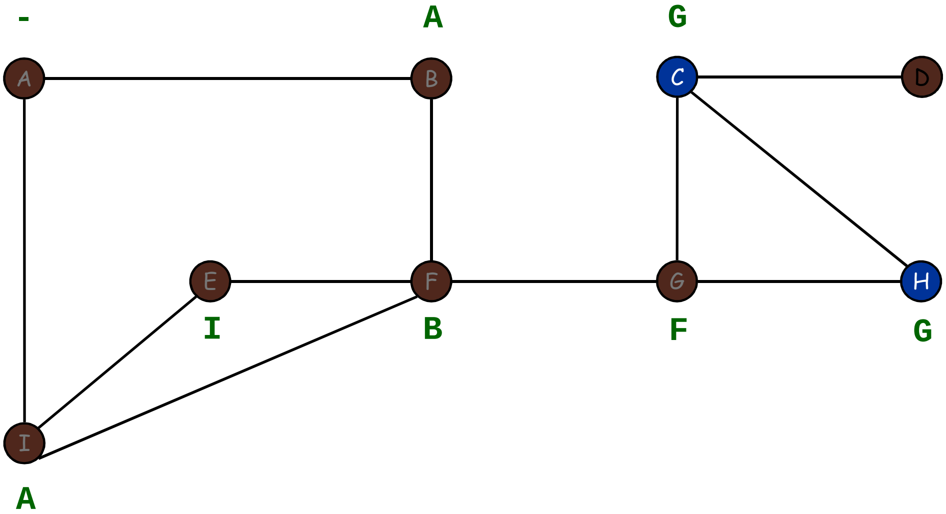
H discovered

front

C H

FIFO Queue

Breadth First Search



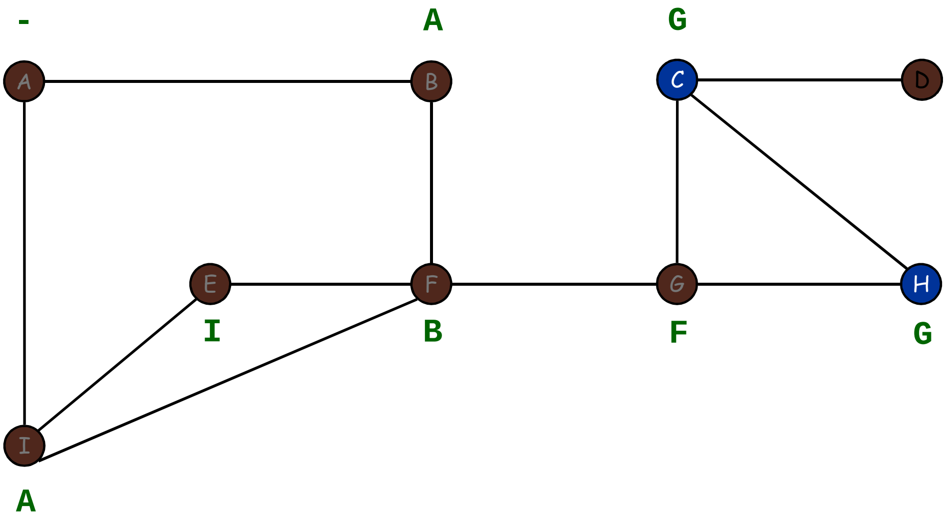
G finished

front

C H

FIFO Queue

Breadth First Search



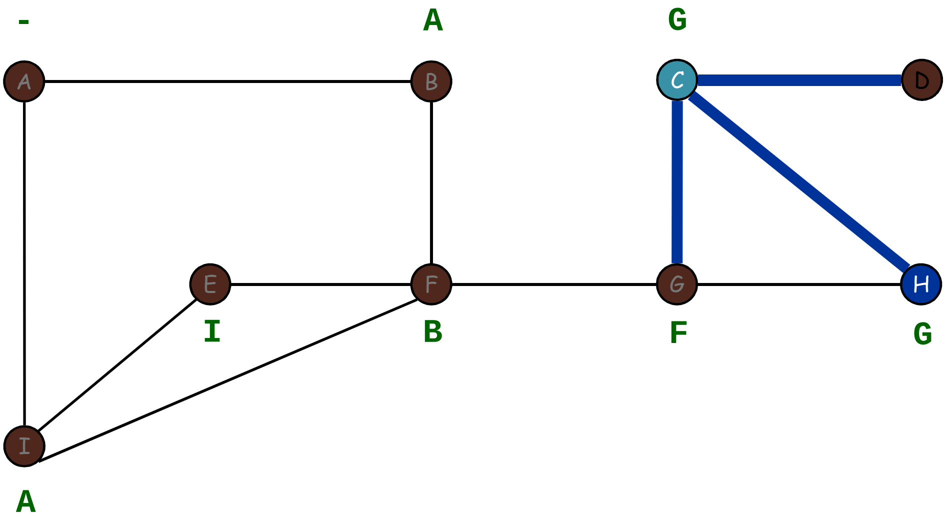
dequeue next vertex

front

C H

FIFO Queue

Breadth First Search

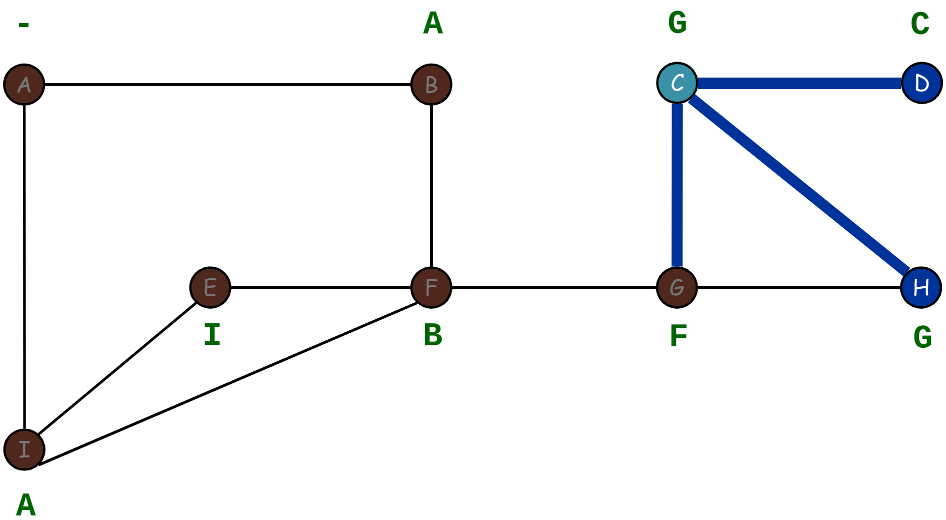


visit neighbors of C

front **H**

FIFO Queue

Breadth First Search

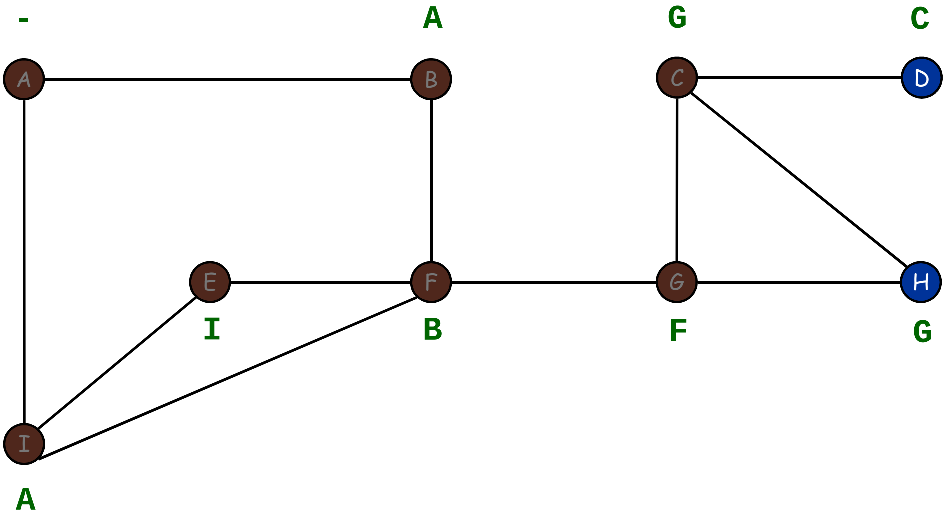


D discovered

front **H D**

FIFO Queue

Breadth First Search



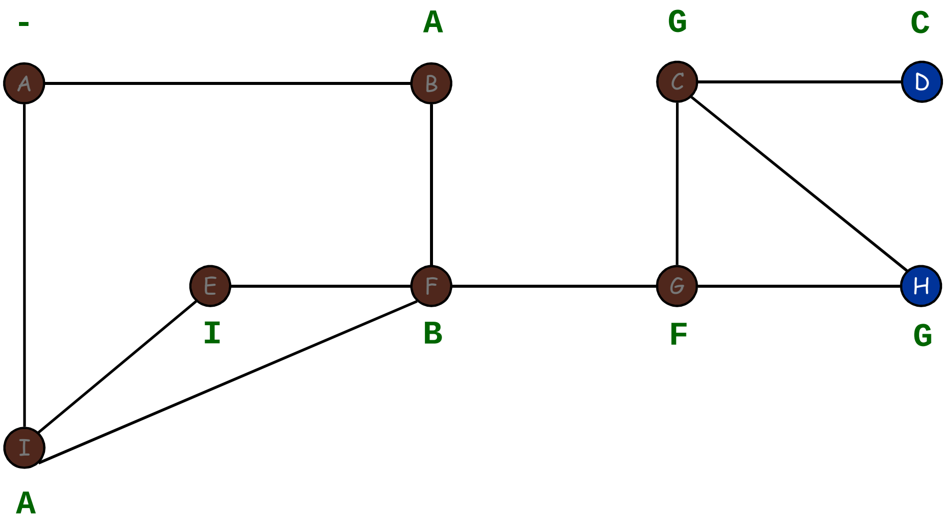
C finished

front

H D

FIFO Queue

Breadth First Search



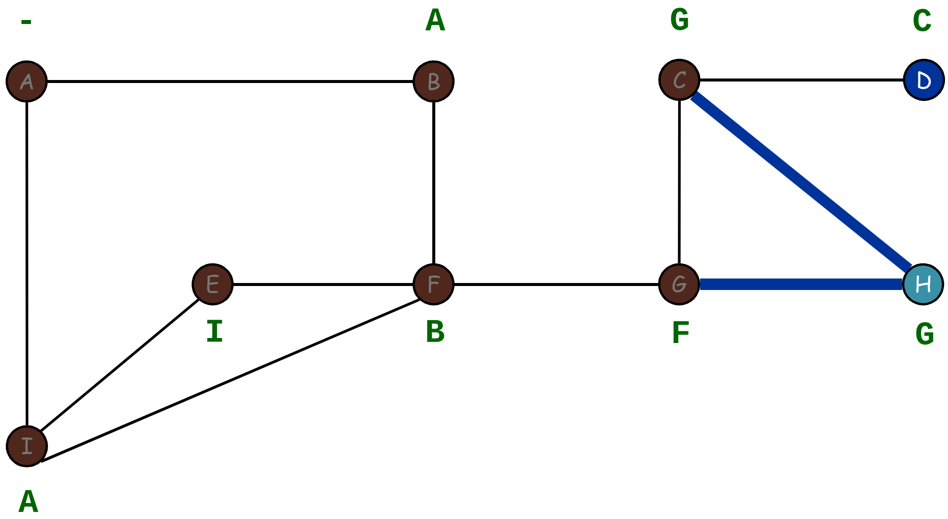
get next vertex

front

H D

FIFO Queue

Breadth First Search



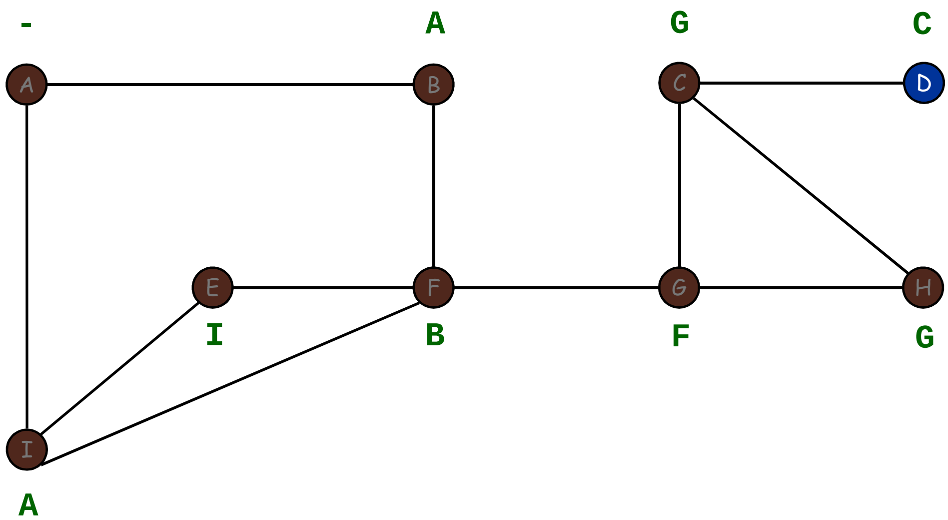
visit neighbors of H

front

D

FIFO Queue

Breadth First Search



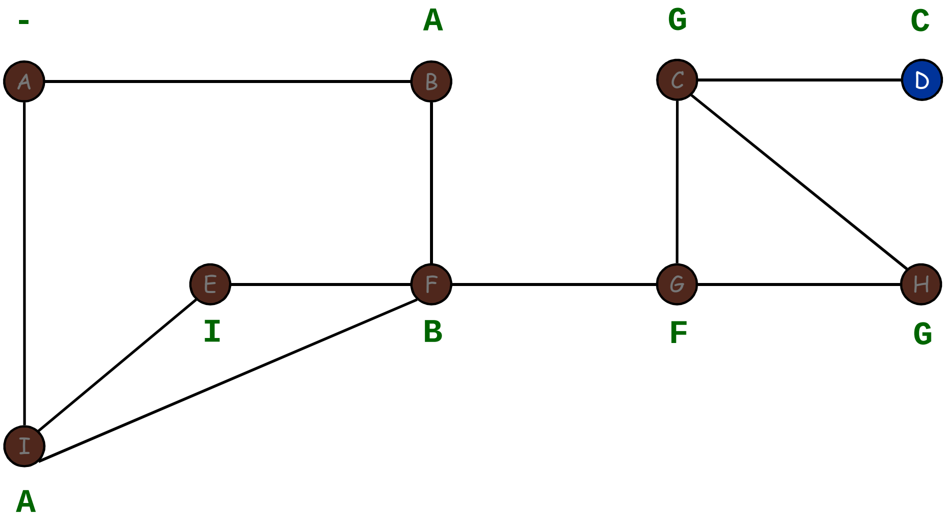
finished H

front

D

FIFO Queue

Breadth First Search

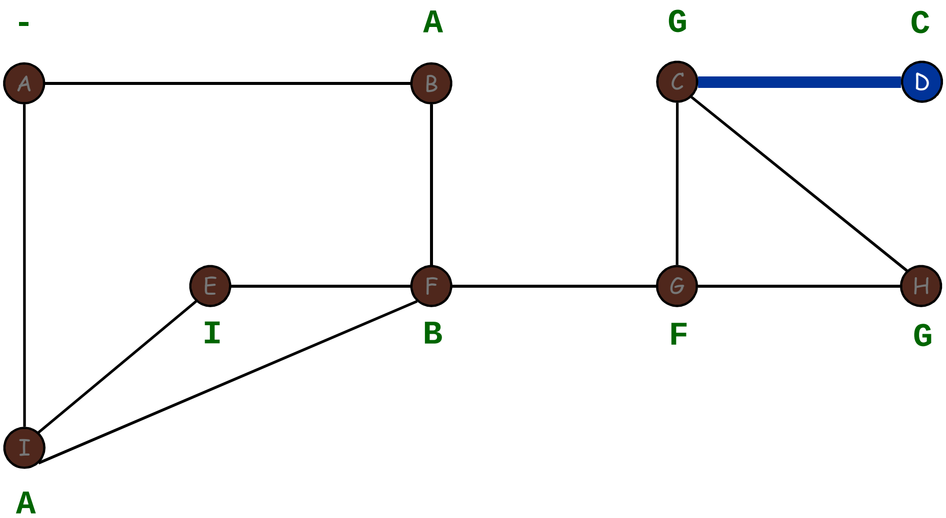


dequeue next vertex

front D

FIFO Queue

Breadth First Search

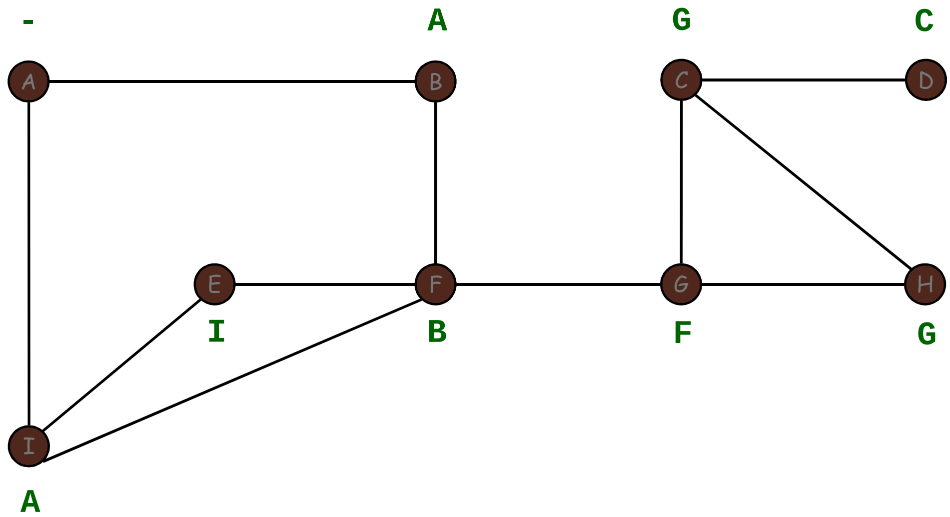


visit neighbors of D

front

FIFO Queue

Breadth First Search

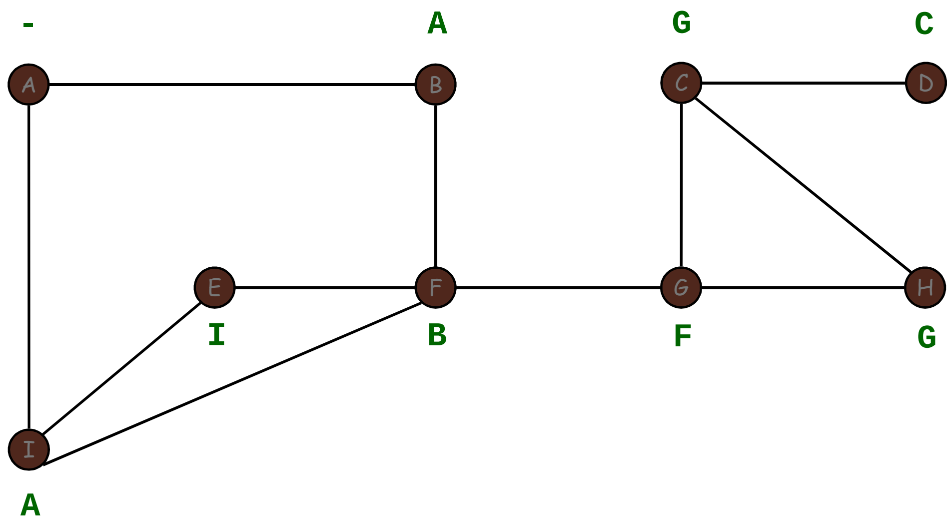


D finished

front

FIFO Queue

Breadth First Search



dequeue next vertex

front

FIFO Queue

Breadth First Search