Lecture-4

Basic Feasible Solution:

Consider a system of linear equation

$Ax = b,\ A = m \times n = [a_{ij}],\ b : m \times 1; x : n \times 1.$ The matrix $[A, b]$ is called augumented matrix. The necessary and sufficient condition for a system to be consistent is that $\rho(A) = \rho([A, b])$.

Further, if $\rho(A) = \rho([A, b]) = n$(no. of unknown) then the system has a unique solution, while if $\rho(A) = \rho([A, b]) < n$ then the system has infinite solutions.

Example 1.15

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 &= 3 \\ 3x_1 + x_2 - x_3 &= 5 \end{aligned}$$

$$\begin{aligned} [A, b] &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & -1 & 5 \end{array}\right] \\ & R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1 \\ &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & -4 & 2 \end{array}\right] \\ & R_3 \to -\frac{1}{2}R_31 \\ &= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] \end{aligned}$$

Now $\rho(A) = 3 = \rho([A, b]) = n$. So the system has a unique solution i.e.

$$x_1 = 1,\ x_2 = 1, x_3 = -1$$

Consider a system $Ax = b$ of m equations in n unknown($n > m$). Let $\rho(A) = \rho([A, b]) = m$, i.e., none of the equations is redundant.

Definition1.13 Basic Solution: A solution obtained by setting exactly $n-m$ variables to zero provided the determinant formed by the columns associated to the remaining m variables is non zero is called Basic Solution.

The remaining m variables are termed as basic variables

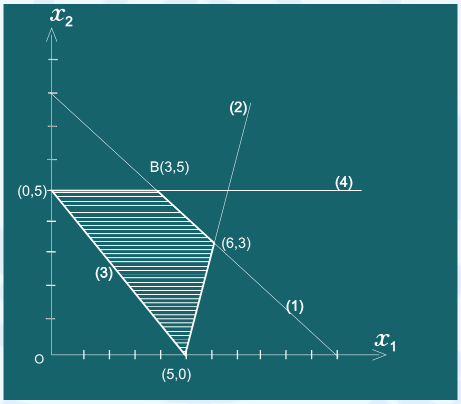
$$Ax = b \;\Rightarrow\; [B, N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b,\ |B| \neq 0 \ (B\text{-basis matrix})$$

$$x_N = 0 \;\Rightarrow\; Bx_B = b \;\Rightarrow\; x_B = B^{-1}b \ (x_N\text{-non basic variables}).$$

Thus a solution in which the vectors associated to m variables are L.I. and remaining $n-m$ variables are zero is called a basic solution. Note that for a solution to be basic, atleast $n-m$ variables must be zero.

Example 1.16 Find(Graphically) basic feasible solution of

$$\begin{aligned} 2x_1 + 3x_2 &\leq 21 \\ 3x_1 - x_2 &\leq 15 \\ x_1 + x_2 &\geq 5 \\ x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$



$$\begin{aligned} (0,5) &\rightarrow (0,5,6,20,0,0) \rightarrow \text{degenerate BFS} \\ (5,0) &\rightarrow (5,0,11,0,0,5) \rightarrow \text{degenerate BFS} \\ (6,3) &\rightarrow (6,3,0,0,4,2) \rightarrow \text{nondegenerate BFS} \\ (3,5) &\rightarrow (3,5,0,4,3,0) \rightarrow \text{nondegenerate BFS} \end{aligned}$$

Example 1.17 Find all basic solutions of the system

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 + x_4 &= 3 \\ 6x_1 + 4x_2 + 4x_3 + 6x_4 &= 2 \end{aligned}$$

Solution: $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = b$

$$\alpha_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix},\ \alpha_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},\ \alpha_4 = \begin{bmatrix} 1 \\ 6 \end{bmatrix},\ b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$B_1 = (\alpha_1, \alpha_2),\quad |B_1| = -28 \neq 0 \;\Rightarrow\; x_{B_1} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$$B_2 = (\alpha_1, \alpha_3), \quad |B_2| = -4 \neq 0 \quad \Rightarrow \quad x_{B_2} = \begin{bmatrix} -2 \\ \frac{7}{2} \end{bmatrix}$$

$$B_3 = (\alpha_1, \alpha_4), \quad |B_3| = 6 \neq 0 \quad \Rightarrow \quad x_{B_3} = \begin{bmatrix} \frac{8}{3} \\ \frac{-7}{3} \end{bmatrix}$$

$$B_4 = (\alpha_2, \alpha_3), \quad |B_4| = 16 \neq 0 \quad \Rightarrow \quad x_{B_4} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$B_5 = (\alpha_2, \alpha_4), \quad |B_5| = 32 \neq 0 \quad \Rightarrow \quad x_{B_5} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$B_6 = (\alpha_3, \alpha_4), \quad |B_6| = 8 \neq 0 \quad \Rightarrow \quad x_{B_6} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Basic Solution:-

$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$, - Degenerate Basic Feasible Solution

$\begin{bmatrix} -2 & 0 & \frac{-7}{3} & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{8}{3} & 0 & 0 & \frac{-7}{3} \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 2 & -1 \end{bmatrix}$ −Nondegenerate Basic Solutin

Example 1.18 Basic solutions of the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 + 5x_3 &= 5 \end{aligned} \quad \text{are}$$

$x_1 = 2, x_2 = 1, x_3 = 0$
$x_1 = 5, x_2 = 0, x_3 = -1$
$x_1 = 0, x_2 = \frac{5}{3}, x_3 = \frac{2}{3}$
All basic solutions are degenerate.

Number of Basic Solution:
If $Ax = b,\ A : m \times n,\ \rho(A) = m$ then maximum number of basic solutions is ${}^nC_m$.

In L.P.P. a feasible solution which is also basic is called a basic feasible solution. Recall that feasible solution satisfies the set of constraints and the non-negativity restriction.

A feasible solution in which $n - m$ variables are zero and the vectors associated to the remaining m variables, called basic variables, are Linearly Independent, is called a B.F.S.

Obviously a feasible solution which contains more than m positive variables is not a basic feasible solution.

Definition1.14 Degenerate Basic Solution: If any of the basic variables vanishes, the solution is called degenerate basic solution. On the other hand if none of the basic variables vanishes, the solution is called non-degenerate basic solution. Thus, a non-degenerate basic solution contains exactly m non-zero and $n-m$ zero variables.

Theorem 1.7 Every extreme point of a convex set of all feasible solution of the $L.P.P.$

$$\begin{aligned} \text{Max } z &= cx \\ \text{subject to } Ax &= b \\ x &\geq 0 \end{aligned}$$

is a basic feasible solution and vice-versa.

Example 1.19 Which of the following vectors is a basic feasible solution of system

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 + x_5 &= 9 \\ 2x_1 + x_2 + 3x_4 + x_6 &= 9 \\ -x_1 + x_2 + 3x_4 + x_7 &= 0 \\ x_i &\geq 0 \end{aligned}$$

$x_1 = (2,2,0,1,0,0,0)$, $\quad x_3 = (3,3,0,0,0,0,0)$

$x_2 = (0,0,9,0,0,9,-9)$, $\quad x_4 = (0,0,0,0,9,9,0)$

$x_5 = (1,0,0,0,8,7,1)$, $\quad x_6 = (0,0,0,3,0,0,0)$

Solution.(i) Columns associated with non-zero variable in x_1 are

$$a_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \; a_4 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

There exist scalars 1,1,-1 such that

$1.a_1 + 1.a_2 - 1.a_4 = 0 \;\Rightarrow$ Linearly Dependent

$\Rightarrow\;$ x_1 is not basic feasible solution.

(ii) Vectors associated with non-zero variables are $a_3 = (1,0,1)^T$, $a_6 = (0,1,0)^T$, $a_7 = (0,0,1)^T$.

$$|B| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow\;$ vectors $a_3,\; a_6,\; a_7$ is Linearly Independent.

$\Rightarrow\;$ x_2 is a basic solution but not an basic feasible solution as $x_7 < 0$.

(iii) Vectors associated with non-zero variables are $a_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Now these vectors together with vector $a_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are Linearly Independent as

$$|B| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 3 \neq 0$$

This solution is a basic feasible solution taking x_1, x_2, x_3 as basic variables. Degenerate B.F.S. as $x_5 = 0$.

(iv) Columns a_5, a_6 together with a_7 are Linearly Independent. So solution x_4 is a degenerate B.F.S. as basic variable $x_7 = 0$.

(v) The solution x_5 contains more than 3 non-zero variables so it is not a B.F.S.

(vi) There is only one nonzero variable with column $a_4 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

This vector along with a_6 and a_7 is L.I. as

$$|B| = \begin{vmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \neq 0$$

This Solution is a degenerate B.F.S..

Fundamental Theorem of Linear Programming

Theorem 1.8 If there is a feasible solution to the system $Ax = b$, $x \geq 0$, where A is $m \times n$ matrix, $m < n$, $\rho(A) = m$, then there is also a B.F.S.

Proof: Let $x = (x_1, x_2, \ldots, x_n)$ be a feasible solution of given system of equations $Ax = b$, $x \geq 0$.

Suppose out of these n components of x, k are non-zero and rest $(n-k)$ are zero. Without loss of generality, we assume that first k components of x are non zero. Thus, we have $x = (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$.

Since x is a feasible solution, we have

$$Ax = b \Rightarrow \sum_{j=i}^{k} a_j x_j = b \tag{1}$$

and $x \geq 0 \ \Rightarrow \ x_j > 0, \ \forall \ j = 1, 2, \ldots, k$

Now if $\{a_1, a_2, \ldots, a_k\}$ are L.I. then x is a B.F.S.

Suppose this is not the case, i.e., the vectors are L.D. Then there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$, not all zero, such that

$$\sum_{j=i}^{k} \lambda_j a_j = 0.$$

Suppose $\lambda_r \neq 0$. Then we get

$$a_r = -\sum_{j=ij\neq r}^{k} \frac{\lambda_j}{\lambda_r} a_j = 0.$$

Substituting value of a_r in (1), we get

$$\sum_{j=ij\neq r}^{k} \left(x_j - \frac{\lambda_j}{\lambda_r} x_r \right) a_j = b.$$

$$\Rightarrow \sum_{j=1}^{k} \hat{x}_j)a_j = b.$$

Now, $\hat{x}_j = x_j - \frac{\lambda_j}{\lambda_r}x_r,\ j \neq r.$

$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, 0, \ldots, 0)$ is a feasible solution of the system $Ax = b,\ x \geq 0$ if $\hat{x} \geq 0$. That means

$$x_j - \frac{\lambda_j}{\lambda_r}x_{\geq}0r,\ \forall j = 1, \ldots, k,\ j \neq r.$$

Now, if $\lambda_j \leq 0$ then since $x_j > 0,\ x_r > 0,\ \lambda_r > 0$ we get $\hat{x}_j = x_j - \frac{\lambda_j}{\lambda_r}x_r \geq 0$. The only cases are for those $\lambda_j \geq 0$.

Then, $x_j - \frac{\lambda_j}{\lambda_r}x_r \geq 0 \ \Rightarrow \ \frac{x_j}{\lambda_j} \geq \frac{x_r}{\lambda_r}$.

Thus, $\frac{x_r}{\lambda_r} = \min_j\{\frac{x_j}{\lambda_j} : \lambda_j > 0\}$ (2)

If we choose r according to rule (2) then $\hat{x}_j \geq 0 \ \forall \ j = 1, \ldots, k$. So, $\hat{x}$ is a feasible solution of the given system with atmost $(k-1)$ non-zero components. If the columns of A associated with these $(k-1)$ non-zero components is linearly independent then $\hat{x}$ is a B.F.S.

Else we can continue with $\hat{x}$ in the same manner as done for x to get another feasible solution $\hat{\hat{x}}$ with atmost $(k-2)$ non-zero components.

In atmost $(k-1)$ steps we get a solution $\hat{x}$ with atmost 1 non-zero component and this is a B.F.S.

Example 1.20 If $x_1 = 2,\ x_2 = 3,\ x_3 = 1$ be a feasible solution of the LPP

$$\begin{aligned} &\text{Max } z = x_1 + 2x_2 + 4x_3 \\ \text{subject to } 2x_1 + x_2 + 4x_3 &= 11 \\ 3x_1 + x_2 + 5x_3 &= 14 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

then find the basic feasible solution.

Solution.the constraints can be written as

$$a_1x_2 + a_2x_2 + a_3x_3 = b,\ x_i \geq 0 \quad (1)$$

$$a_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ a_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix},\ b = \begin{bmatrix} 11 \\ 14 \end{bmatrix}$$

Since the vectors $\{a_1, a_2, a_3\}$ are linearly dependent, we get

$\Rightarrow \ x_1$ is not basic feasible solution. $1a_1 + 2a_2 - a_3 = 0$

$$\text{i.e.}\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -1.$$

Select r such that

$$\begin{aligned} \frac{x_r}{\lambda_r} &= \min\left\{\frac{x_i}{\lambda_i},\ \lambda_i > 0\right\} \\ &= \min\left\{\frac{x_1}{\lambda_1},\ \frac{x_2}{\lambda_2}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \min\left\{2,\ \frac{3}{2}\right\}\\
&= \frac{x_2}{\lambda_2}\\
\Rightarrow r &= 2\\
\Rightarrow a_2 &= -\frac{a_1}{2}+\frac{a_3}{2}
\end{aligned}
$$

Substitute in (1), we get

$$
\begin{aligned}
a_1x_1+\left(\frac{-a_1}{2}+\frac{a_3}{2}\right)x_2+a_3x_3 &= b\\
\Rightarrow \quad \left(x_1-\frac{x_2}{2}\right)a_1+\left(x_3+\frac{x_2}{2}\right)a_3 &= b\\
\Rightarrow \quad \left(2-\frac{3}{2}\right)a_1+\left(1+\frac{3}{2}\right)a_3 &= b\\
\Rightarrow \quad \frac{1}{2}a_1+\frac{5}{2}a_3 &= b
\end{aligned}
$$

$\Rightarrow \ x_1=\frac{1}{2},\ x_2=0,\ x_3=\frac{5}{2}$ is a feasible solution which is also a B.F.S. as $\{a_1,a_3\}$ is linearly independent because

$$
\begin{vmatrix} 2 & 4\\ 3 & 5 \end{vmatrix} = 10-12=-2 \ \neq \ 0.
$$

Example 1.21 Consider the system of constraints

$$
\begin{aligned}
x_1 &\geq 6\\
x_2+x_3 &\geq 2\\
x_1,x_2,x_3 &\geq 0
\end{aligned}
$$

The point $\bar{x}=(7,2,0)^T$ is a feasible solution of the system, and the set of column vectors corresponding to positive x_j in the system is $\{(1,0)^T,(0,1)^T\}$ which is linearly independent. However $\bar{x}$ is not a B.F.S. of this system. If we introduce the slack variables.

$$
\begin{aligned}
x_1-s_1 &= 6,\\
x_2+x_3-s_2 &= 2,\\
x_1,x_2,x_3,s_1,s_2 &\geq 0\\
\text{and } (\bar{x},\bar{s}) &= (7,2,0,1,0)^T
\end{aligned}
$$