Chapter 5

Numerical method in steady-state heat conduction

Numerical methods

- Analytical solutions that allow for the determination of the exact temperature distribution are only available for limited ideal cases.

- Graphical solutions have been used to gain an insight into complex heat transfer problems, where analytical solutions are not available, but they have limited accuracy and are primarily used for two-dimensional problems.

- Advances in numerical computing now allow for complex heat transfer problems to be solved rapidly on computers, i.e., "numerical techniques“.

- Current numerical techniques include: finite-difference analysis; finite element analysis; and finite-volume analysis.

- In general, these techniques are routinely used to solve problems in heat transfer, fluid dynamics, stress analysis, electrostatics and magnetics, etc.

- We will show the use of finite-difference analysis to solve conduction heat transfer problems.

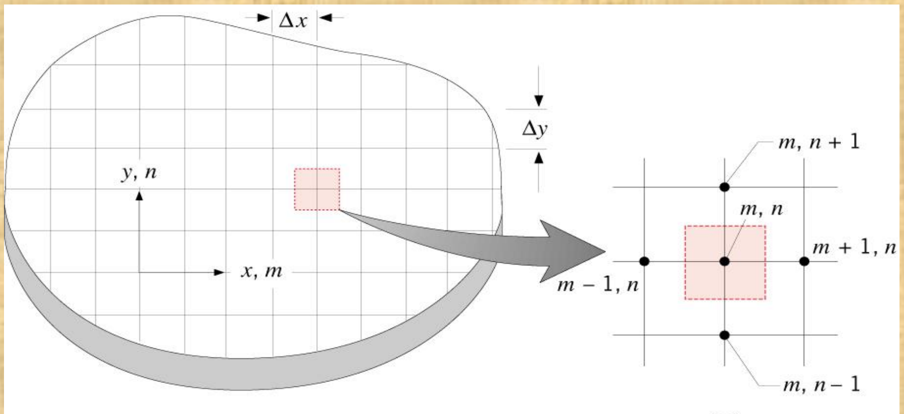
The Finite-Difference Method

- **An approximate method for determining temperatures at discrete (*nodal*) points of the physical system.**

- **Procedure:**
  - **Represent the physical system by a *nodal network*.**
  - **Use the energy balance method to obtain a finite-difference equation for each node of unknown temperature.**
  - **Solve the resulting set of algebraic equations for the unknown nodal temperatures.**

The Nodal Network and Finite-Difference Approximation

- **The** **nodal network** **identifies discrete points at which the temperature is to be determined and uses an *m,n* notation to designate their location.**



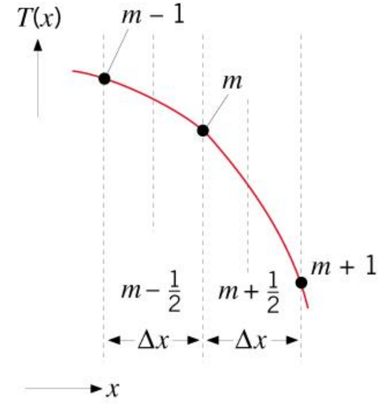
(a)

What is represented by the temperature determined at a nodal point, as for example, $T_{m,n}$?

- **A** **finite-difference approximation** **is used to represent temperature gradients in the domain.**

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,\,n} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,\,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



(b)

How is the accuracy of the solution affected by construction of the nodal network? What are the trade-offs between selection of a *fine* or a *coarse mesh*?

Finite-Difference Formulation of Differential Equation

example: 1-D steady-state heat conduction equation with internal heat generation

For a point m we approximate the 2nd derivative as

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_m \approx \frac{\left.\frac{dT}{dx}\right|_{m+½} - \left.\frac{dT}{dx}\right|_{m-½}}{\Delta x} \approx \frac{\frac{T_{m+1}-T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

$$\approx \frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2}$$

Now the finite-difference approximation of the heat conduction equation is

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{q}_m}{k} = 0$$

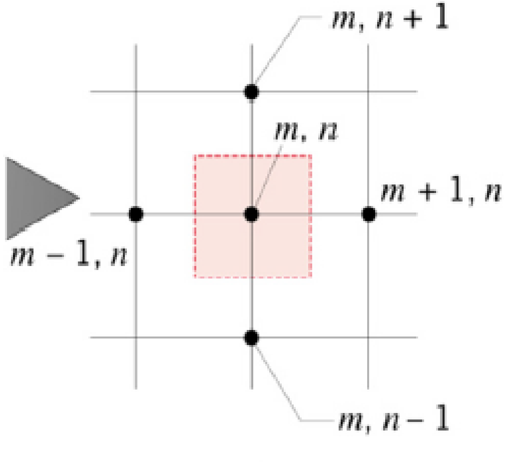
This is repeated for all the modes in the region considered

Finite-Difference Formulation of Differential Equation

If this was a 2-D problem we could also construct a similar relationship in the both the x and Y-direction at a point *(m,n)* i.e.,

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{(\Delta y)^2}$$



Now the finite-difference approximation of the 2-D heat conduction equation is

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{(\Delta y)^2} + \frac{\dot{q}}{k} = 0$$

Once again this is repeated for all the modes in the region considered. We could also derive a similar equation for the 3-D case

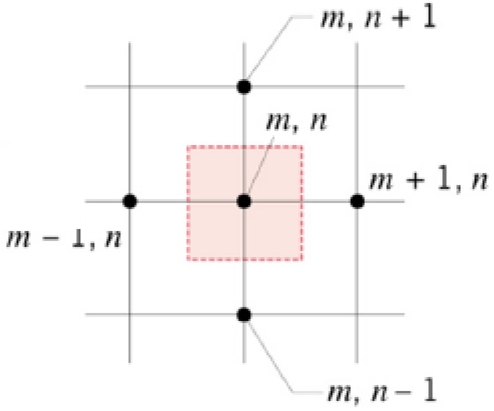
Finite-Difference Formulation of Differential Equation

If $\Delta x = \Delta y$, then the finite-difference approximation of the 2-D heat conduction equation is

$$T_{m-1,n} - 2T_{m,n} + T_{m+1,n} + T_{m,n-1} - 2T_{m,n} + T_{m,n+1} + \frac{\dot{q}(\Delta x)^2}{k} = 0$$

which can be reduced to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}(\Delta x)^2}{k} = 0$$



and the relationship reduces to

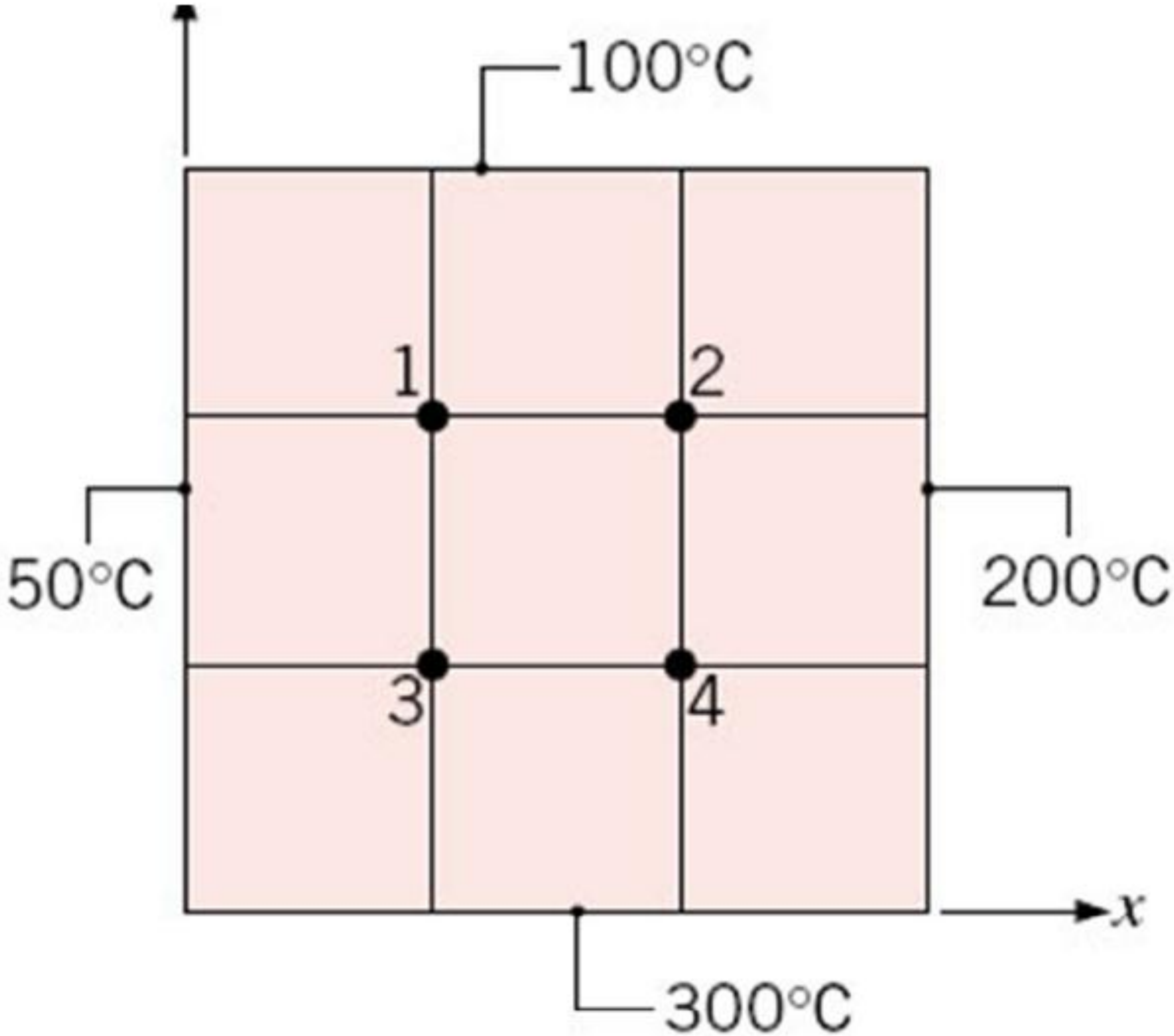
$$T_{m,n} = \frac{1}{4}\left[T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} + \frac{\dot{q}(\Delta x)^2}{k}\right]$$

if there is no internal heat generation,

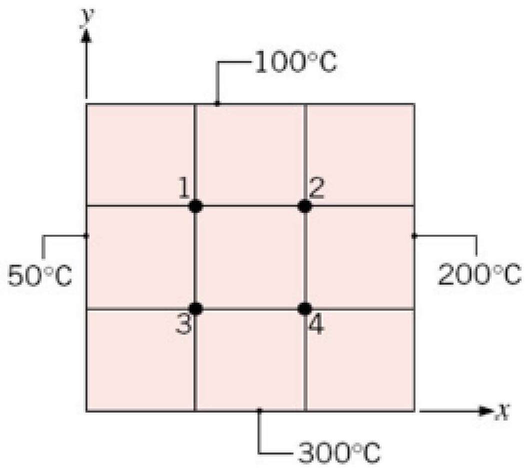
$$T_{m,n} = \frac{1}{4}\left[T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}\right]$$

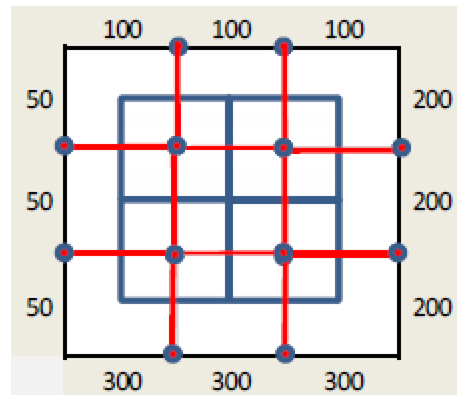
Which is just the average of the surrounding node's temperatures!

Consider this simple case



Consider this simple case





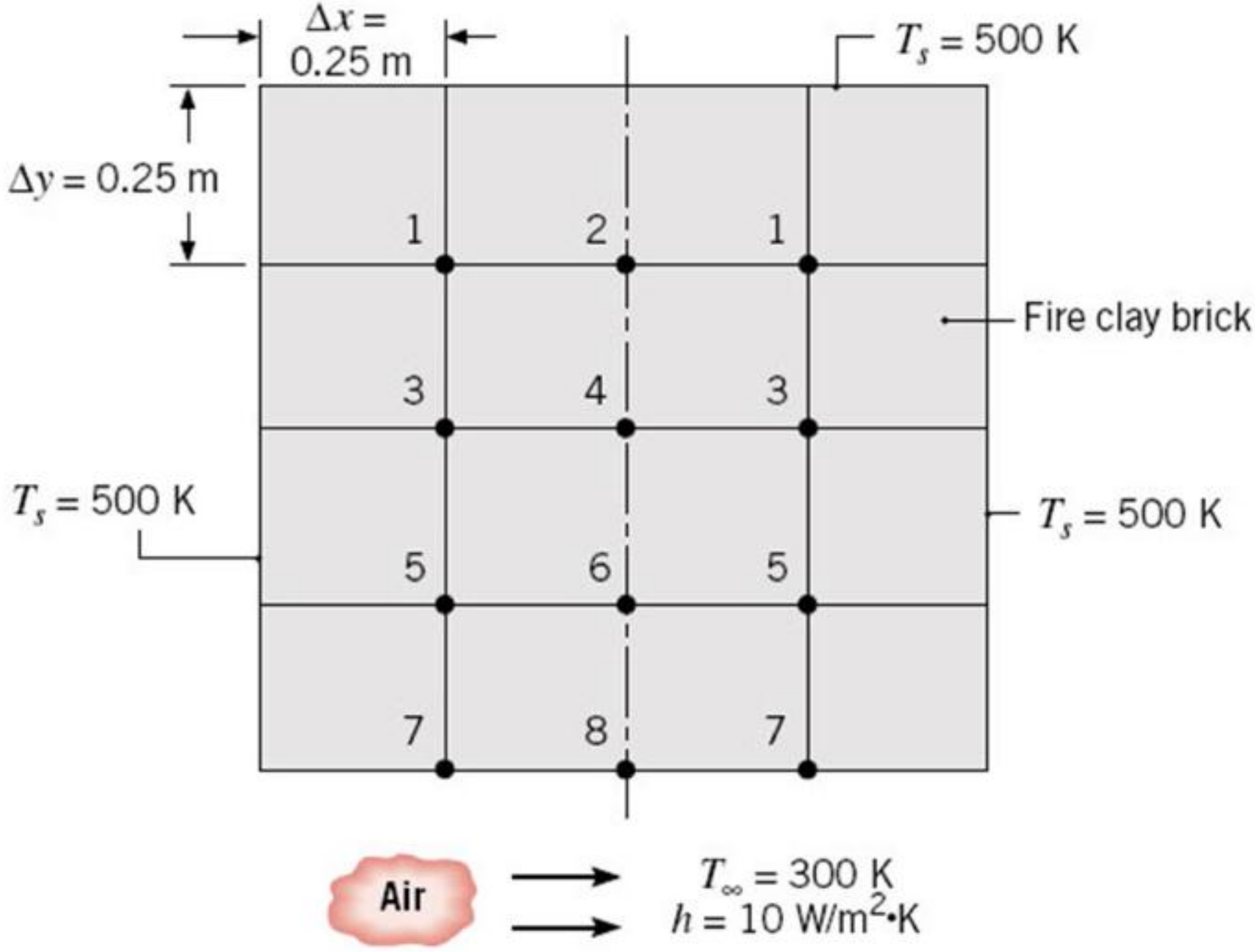
$$T_1 = \frac{1}{4}\left[100 + T_3 + 50 + T_2\right]$$

$$T_2 = \frac{1}{4}\left[100 + T_1 + 200 + T_4\right]$$

$$T_3 = \frac{1}{4}\left[300 + T_1 + 50 + T_4\right]$$

$$T_4 = \frac{1}{4}\left[300 + T_3 + 200 + T_2\right]$$

What if the boundary conditions are different

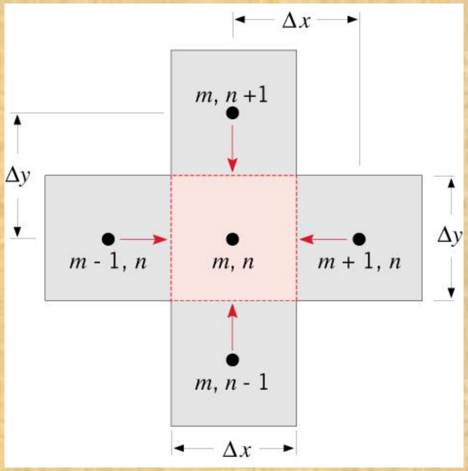


Derivation of the Finite-Difference Equations
- The Energy Balance Method -

- **As a convenience that obviates the need to predetermine the direction of heat flow, assume all heat flows are into the nodal region of interest, and express all heat rates accordingly. Hence, the energy balance becomes:**

$$\dot{E}_{in} + \dot{E}_g = 0$$

- **Consider application to an *interior nodal point* (one that exchanges heat by conduction with four, equidistant nodal points):**



$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q}\left(\Delta x \cdot \Delta y \cdot \ell\right) = 0$$

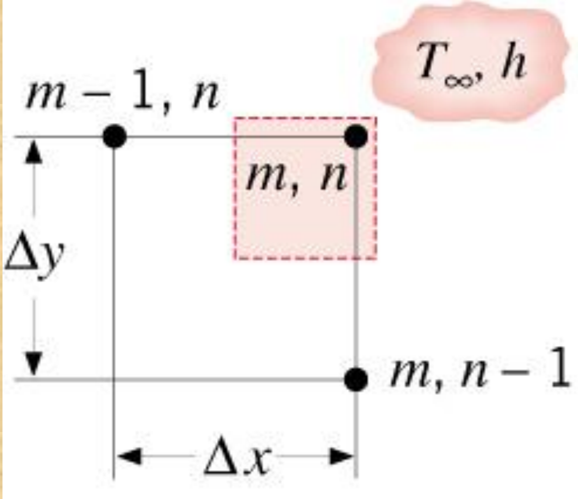
where, for example,

$$q_{(m-1,n)\to(m,n)} = k\left(\Delta y \cdot \ell\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

Is it possible for all heat flows to be into the *m,n* nodal region?

What feature of the analysis insures a correct form of the energy balance equation despite the assumption of conditions that are not realizable?

- **A summary of finite-difference equations for common nodal regions is provided in Table 1.2. Consider an *external corner with convection heat transfer*.**



$$q_{(m-1,n)\to(m,n)} + q_{(m,n-1)\to(m,n)} + q_{(\infty)\to(m,n)} = 0$$

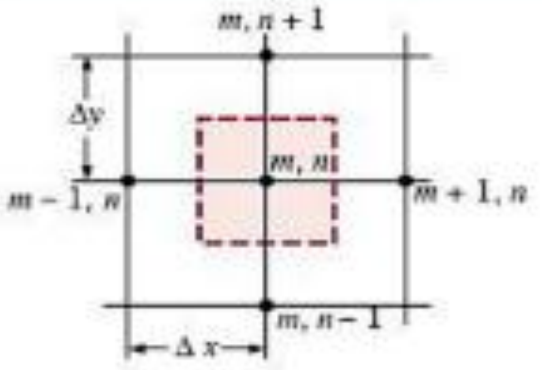
$$k\left(\frac{\Delta y}{2}\cdot\ell\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}\cdot\ell\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}$$
$$+\mathrm{h}\left(\frac{\Delta \mathrm{x}}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) + \mathrm{h}\left(\frac{\Delta \mathrm{y}}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) = 0$$

or, with $\Delta x = \Delta y$,

$$T_{m-1,n} + T_{m,n-1} + 2\frac{\mathrm{h}\Delta x}{k}T_\infty - 2\left(\frac{\mathrm{h}\Delta x}{k}+1\right)T_{m,n} = 0 \qquad (4.47)$$

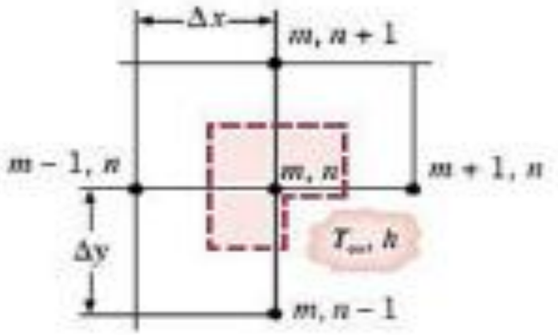
Table 1.2. A summary of finite-difference equations for common nodal regions

Configuration | **Finite-Difference Equation for $\Delta x = \Delta y$**



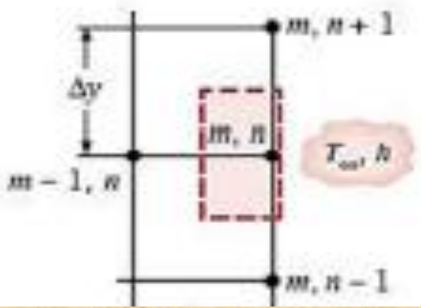
$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

Case 1. Interior node



$$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(3 + \frac{h\,\Delta x}{k}\right)T_{m,n} = 0$$

Case 2. Node at an internal corner with convection

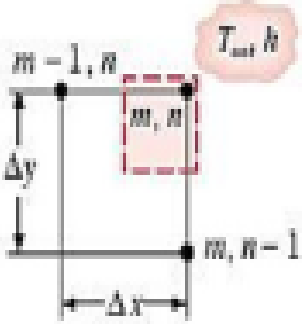


$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 2\right)T_{m,n} = 0$$

Case 3. Node at a plane surface with convection

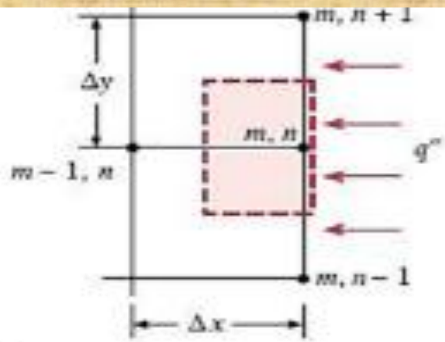
[a,b] To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

Table 1.2. A summary of finite-difference equations for common nodal regions (Cont.)



$$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 1\right)T_{m,n} = 0$$

Case 4. Node at an external corner with convection

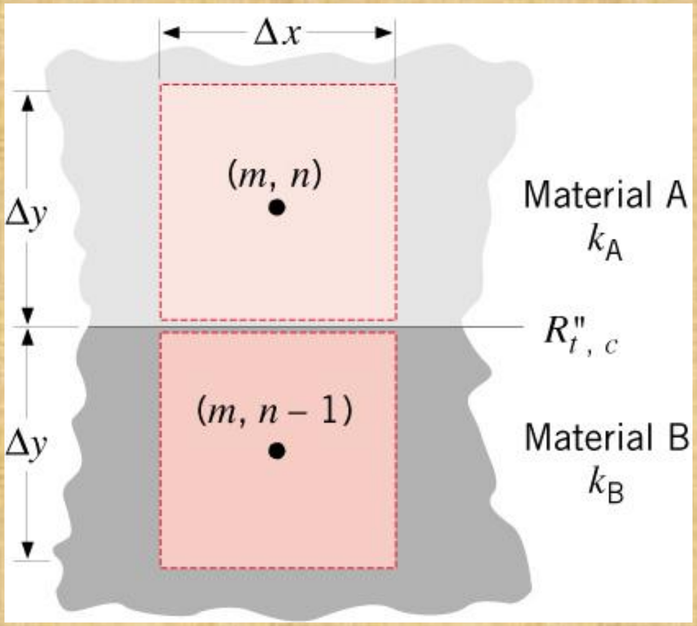


$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\,\Delta x}{k} - 4T_{m,n} = 0$$

Case 5. Node at a plane surface with uniform heat flux

[a,b] To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

- **Note potential utility of using thermal resistance concepts to express rate equations. E.g., conduction between adjoining dissimilar materials with an interfacial contact resistance.**



$$q_{(m,n-1)\to(m,n)} = \frac{T_{m,n-1} - T_{m,n}}{R_{tot}}$$

$$R_{tot} = \frac{\Delta y/2}{k_A(\Delta x \cdot \ell)} + \frac{R''_{t,c}}{\Delta x \cdot \ell} + \frac{\Delta y/2}{k_B(\Delta x \cdot \ell)} \qquad (4.50)$$

Treating Insulated Boundary Nodes as Interior Nodes: The Mirror Image Concept

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{e}_m}{k} = 0 \quad \rightarrow \quad \frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{e}_0}{k} = 0$$

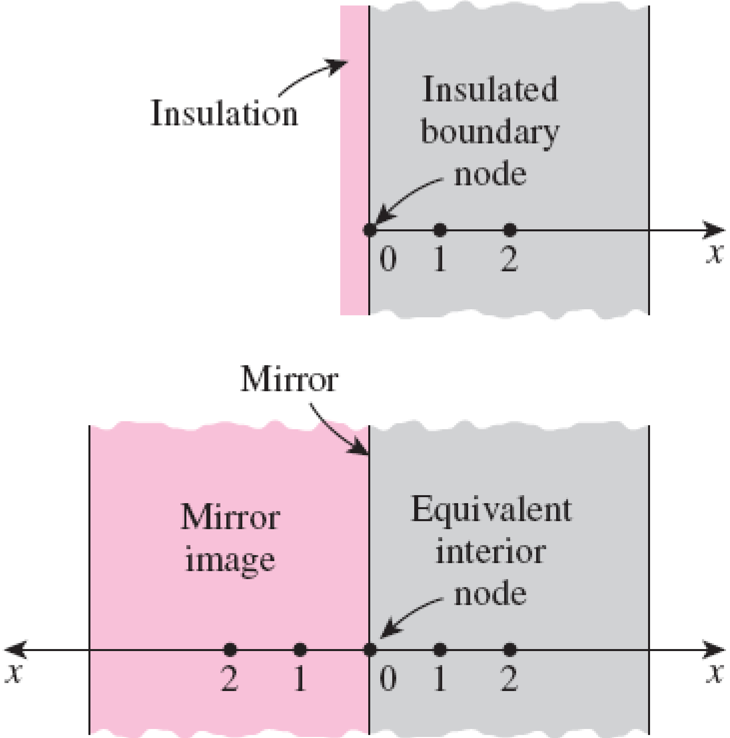


FIGURE 5–17

A node on an insulated boundary can be treated as an interior node by replacing the insulation by a mirror.

The mirror image approach can also be used for problems that possess thermal symmetry by replacing the plane of symmetry by a mirror.

Alternately, we can replace the plane of symmetry by insulation and consider only half of the medium in the solution.

The solution in the other half of the medium is simply the mirror image of the solution obtained.

Solutions Methods

- **Matrix Inversion: Expression of system of N finite-difference equations for N unknown nodal temperatures as:**

$$[A][T]=[C] \tag{4.52}$$

Coefficient Matrix (NxN)

Solution Vector ($T_1, T_2, \ldots T_N$)

Right-hand Side Vector of Constants ($C_1, C_2 \ldots C_N$)

Solution →

$$[T]=[A]^{-1}[C] \tag{4.53}$$

Inverse of Coefficient Matrix

- **Gauss-Seidel Iteration: Each finite-difference equation is written in explicit form, such that its unknown nodal temperature appears alone on the left-hand side:**

$$T_i^{(k)}=\frac{C_i}{a_{ii}}-\sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}T_j^{(k)}-\sum_{j=i+1}^{N}\frac{a_{ij}}{a_{ii}}T_j^{(k-1)} \tag{4.55}$$

where *i* =1, 2,…, N and *k* is the level of iteration.

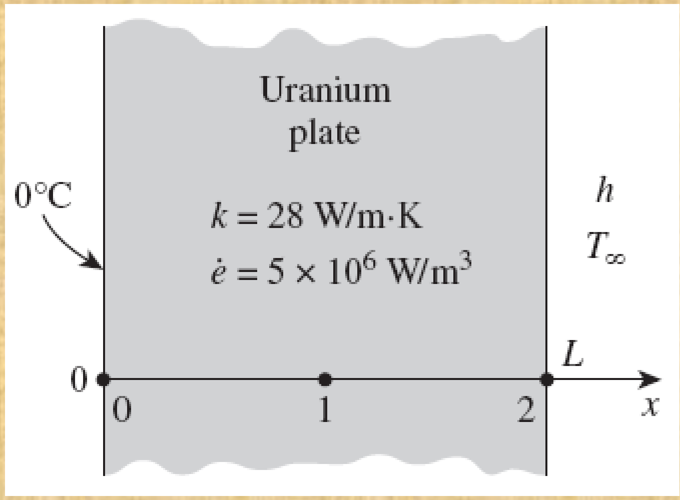
Iteration proceeds until satisfactory convergence is achieved for all nodes:

$$\left|T_i^{(k)}-T_i^{(k-1)}\right|\le\varepsilon$$

- **What measures may be taken to insure that the results of a finite-difference solution provide an accurate prediction of the temperature field?**

Example1.3: Steady heat conduction in a large Uranium plate

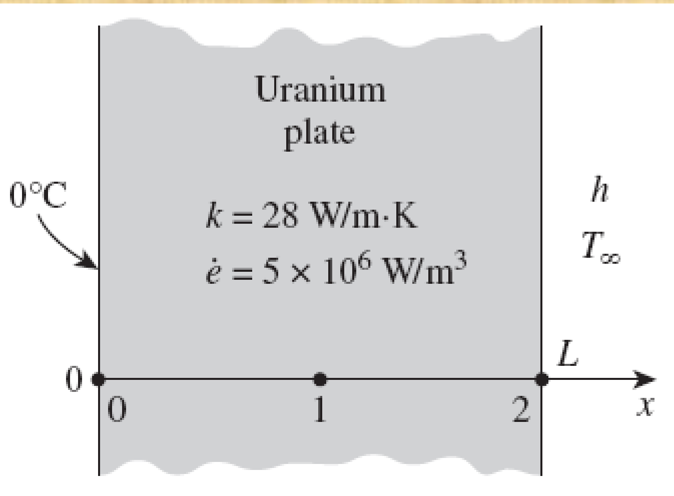
Consider a large uranium plate of thickness *L* = 4 cm and thermal conductivity *k* = 28 W/m °C in which heat is generated uniformly at a constant rate of . One side of the plate is maintained at 0 °C by iced water while the other side is subjected to convection to an environment at with a heat transfer coefficient of *h* = 45 W/m^2°C, as shown in the figure bellow. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using finite difference approach.



h = 45 W/m^2·K

T_∞ = 30°C

Solution:



$T_\infty = 30°C$ $h = 45$ W/m²·K

$$\Delta x = \frac{L}{M-1} = \frac{0.04 \text{ m}}{3-1} = 0.02 \text{ m}$$

Node 1

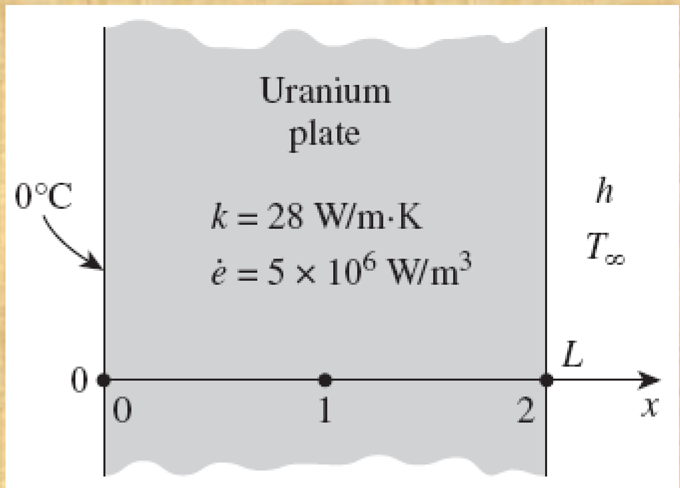
$$\frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}_1}{k} = 0 \rightarrow \frac{0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{e}_1}{k} = 0 \rightarrow 2T_1 - T_2 = \frac{\dot{e}_1 \Delta x^2}{k}$$

Node 2

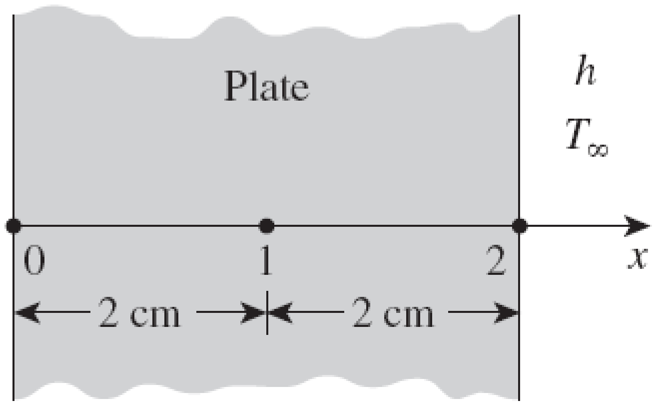
$$hA(T_\infty - T_2) + kA\frac{T_1 - T_2}{\Delta x} + \dot{e}_2(A\Delta x/2) = 0$$

$$T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\dot{e}_2 \Delta x^2}{2k}$$

$$2T_1 - T_2 = 71.43 \qquad \text{(in °C)}$$
$$T_1 - 1.032T_2 = -36.68 \qquad \text{(in °C)}$$



$T_\infty = 30°C$ $h = 45\ W/m^2{\cdot}K$



Finite difference solution:

$$T_2 = 136.1°C$$

Exact solution:

$$T_2 = 136.0°C$$

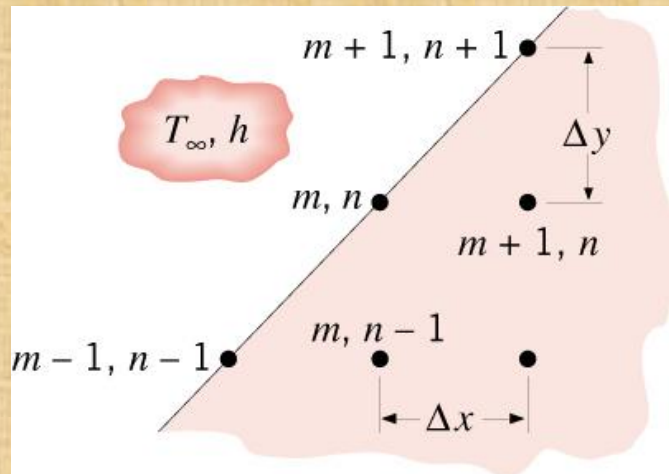
FIGURE

Despite being approximate in nature, highly accurate results can be obtained by numerical methods.

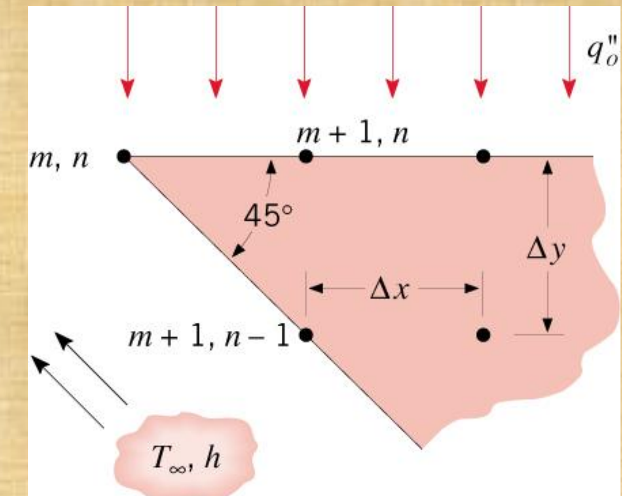
Exact solution:

$$T(x) = \frac{0.5\dot{e}hL^2/k + \dot{e}L + T_\infty h}{hL + k}x - \frac{\dot{e}x^2}{2k}$$

Example 1.4: Finite-difference equations for (a) nodal point on a diagonal surface and (b) tip of a cutting tool.



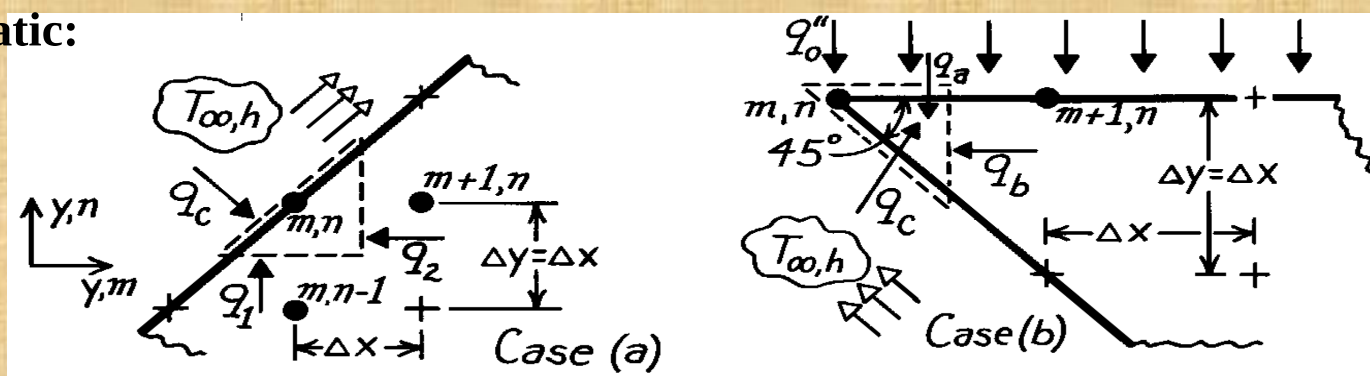
(a) Diagonal surface



(b) Cutting tool.

Solution:

Schematic:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties

ANALYSIS: (a) The control volume about node m,n is triangular with sides Δx and Δy and diagonal (surface) of length $\sqrt{2}\ \Delta x$.

The heat rates associated with the control volume are due to conduction, q_1 and q_2, and to convection, q_c. An energy balance for a unit depth normal to the page yields

$$\dot{E}_{in} = 0$$

$$q_1 + q_2 + q_c = 0$$

$$k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h\left(\sqrt{2}\ \Delta x \cdot 1\right)\left(T_\infty - T_{m,n}\right) = 0.$$

With $\Delta x = \Delta y$, it follows that

$$T_{m,n-1} + T_{m+1,n} + \sqrt{2}\ \cdot \frac{h\Delta x}{k} T_\infty - \left[2 + \sqrt{2}\ \cdot \frac{h\Delta x}{k}\right] T_{m,n} = 0.$$

(b) The control volume about node m,n is triangular with sides $\Delta x/2$ and $\Delta y/2$ and a lower diagonal surface of length $\sqrt{2}\left(\Delta x/2\right)$.

The heat rates associated with the control volume are due to the uniform heat flux, q_a, conduction, q_b, and convection q_c. An energy balance for a unit depth yields

$$\dot{E}_{in} = 0$$

$$q_a + q_b + q_c = 0$$

$$q_o'' \cdot \left[\frac{\Delta x}{2} \cdot 1\right] + k \cdot \left[\frac{\Delta y}{2} \cdot 1\right]\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h \cdot \left[\sqrt{2} \cdot \frac{\Delta x}{2}\right]\left(T_\infty - T_{m,n}\right) = 0.$$

or, with $\Delta x = \Delta y$,

$$T_{m+1,n} + \sqrt{2} \cdot \frac{h\Delta x}{k} \cdot T_\infty + q_o'' \cdot \frac{\Delta x}{k} - \left(1 + \sqrt{2} \cdot \frac{h\Delta x}{k}\right) T_{m,n} = 0.$$

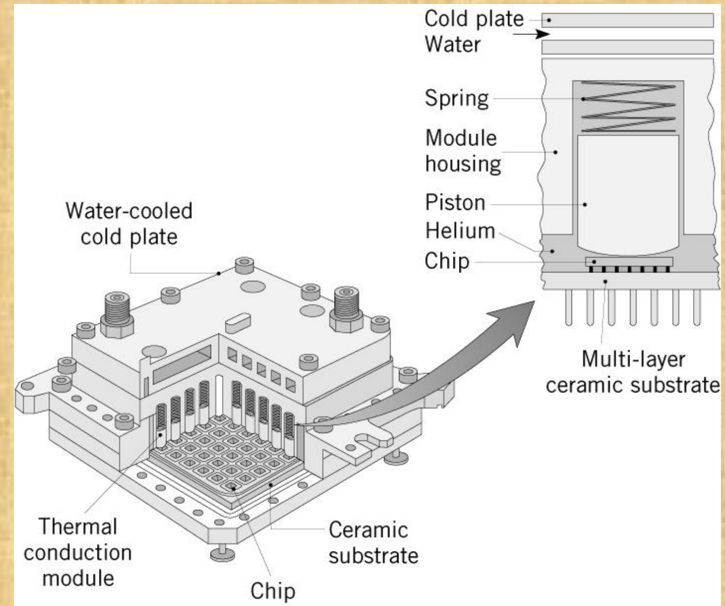
Example 1.5: Analysis of cold plate used to thermally control IBM multi-chip, thermal conduction module.

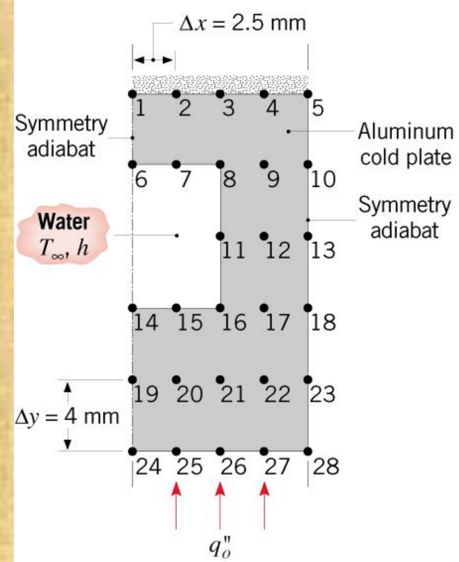
Features:

- **Heat dissipated in the chips is transferred by conduction through spring-loaded aluminum pistons to an aluminum cold plate.**

- **Nominal operating conditions may be assumed to provide a uniformly distributed heat flux of at the base of the cold plate** $q''_o = 10^5\ \mathrm{W/m^2}$

- **Heat is transferred from the cold plate by water flowing through channels in the cold plate.**





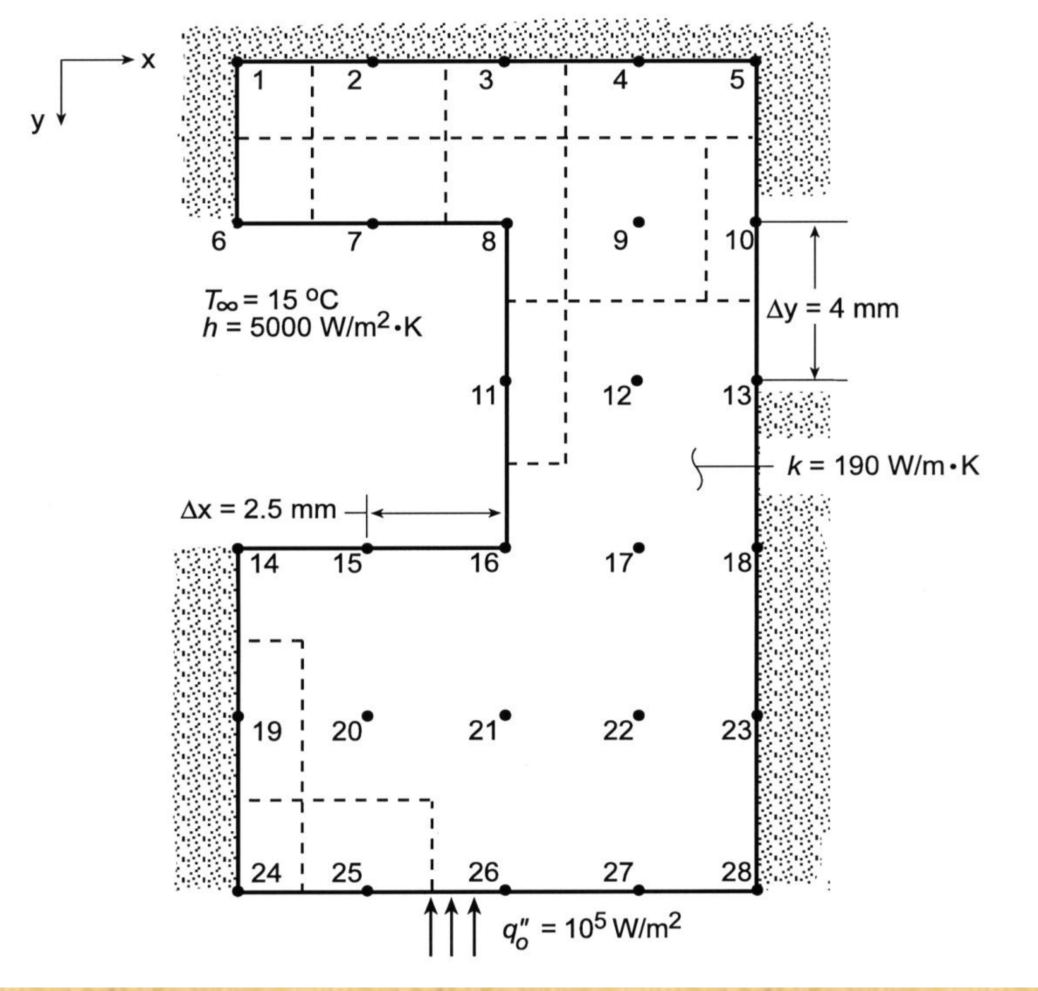
Find:

(a) Cold plate temperature distribution for the prescribed conditions.

(b) Options for operating at larger power levels while remaining within a maximum cold plate temperature of 40°C.

Solution:

Schematic:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties

ANALYSIS: Finite-difference equations must be obtained for each of the 28 nodes. Applying the energy balance method to regions 1 and 5, which are similar, it follows that

Node 1: $(\Delta y/\Delta x)T_2 + (\Delta x/\Delta y)T_6 - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_1 = 0$

Node 5: $(\Delta y/\Delta x)T_4 + (\Delta x/\Delta y)T_{10} - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_5 = 0$

Nodal regions 2, 3 and 4 are similar, and the energy balance method yields a finite-difference equation of the form

Nodes 2,3,4:

$$(\Delta y/\Delta x)\left(T_{m-1,n} + T_{m+1,n}\right) + 2(\Delta x/\Delta y)T_{m,n-1} - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_{m,n} = 0$$

Energy balances applied to the remaining combinations of similar nodes yield the following finite-difference equations.

Nodes 6, 14:

$$(\Delta x/\Delta y)T_1 + (\Delta y/\Delta x)T_7 - \left[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)\right]T_6 = -(h\Delta x/k)T_\infty$$

$$(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{15} - \left[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)\right]T_{14} = -(h\Delta x/k)T_\infty$$

Nodes 7, 15:

$$(\Delta y/\Delta x)(T_6 + T_8) + 2(\Delta x/\Delta y)T_2 - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)\right]T_7 = -(2h\Delta x/k)T_\infty$$

$$(\Delta y/\Delta x)(T_{14} + T_{16}) + 2(\Delta x/\Delta y)T_{20} - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)\right]T_{15} = -(2h\Delta x/k)T_\infty$$

Nodes 8, 16:

$$(\Delta y/\Delta x)T_7 + 2(\Delta y/\Delta x)T_9 + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_3 - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y) + (h/k)(\Delta x + \Delta y)]T_8 = -(h/k)(\Delta x + \Delta y)T_\infty$$

$$(\Delta y/\Delta x)T_{15} + 2(\Delta y/\Delta x)T_{17} + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_{21} - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y) + (h/k)(\Delta x + \Delta y)]T_{16} = -(h/k)(\Delta x + \Delta y)T_\infty$$

Node 11:

$$(\Delta x/\Delta y)T_8 + (\Delta x/\Delta y)T_{16} + 2(\Delta y/\Delta x)T_{12} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta y/k)]T_{11} = -(2h\Delta y/k)T_\infty$$

Nodes 9, 12, 17, 20, 21, 22:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + (\Delta x/\Delta y)T_{m,n+1} + (\Delta x/\Delta y)T_{m,n-1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Nodes 10, 13, 18, 23:

$$(\Delta x/\Delta y)T_{n+1,m} + (\Delta x/\Delta y)T_{n-1,m} + 2(\Delta y/\Delta x)T_{m-1,n} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Node 19:

$$(\Delta x/\Delta y)T_{14} + (\Delta x/\Delta y)T_{24} + 2(\Delta y/\Delta x)T_{20} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{19} = 0$$

Nodes 24, 28:

$$(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{25} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{24} = -(q_o''\Delta x/k)$$

$$(\Delta x/\Delta y)T_{23} + (\Delta y/\Delta x)T_{27} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{28} = -(q_o''\Delta x/k)$$

Nodes 25, 26, 27:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + 2(\Delta x/\Delta y)T_{m,n+1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = -(2q_o''\Delta x/k)$$

Evaluating the coefficients and solving the equations simultaneously, the steady-state temperature distribution (°C), tabulated according to the node locations, is:

23.77	23.91	24.27	24.61	24.74
23.41	23.62	24.31	24.89	25.07
		25.70	26.18	26.33
28.90	28.76	28.26	28.32	28.35
30.72	30.67	30.57	30.53	30.52
32.77	32.74	32.69	32.66	32.65

(b) For the prescribed conditions, the maximum allowable temperature ($T_{24} = 40°C$) is reached when

$$q_o'' = 1.407 \times 10^5 \text{ W/m}^2 \ (14.07 \text{ W/cm}^2).$$

Options for extending this limit could include use of a copper cold plate ($k \approx 400$ W/m·K) and/or increasing the convection coefficient associated with the coolant.

With $k = 400$ W/m·K, a value of $q_o'' = 17.37 \text{ W/cm}^2$ may be maintained.

. With $k = 400$ W/m·K and $h = 10{,}000 \text{ W/m}^2\cdot\text{K}$ (a practical upper limit), $q_o'' = 28.65 \text{ W/cm}^2$.

Additional, albeit small, improvements may be realized by relocating the coolant channels closer to the base of the cold plate.