We have carefully wrote this book for those interested in concepts of topology for different branches of Mathematics that are used in a variety of applications in other scientific fields. What distinguishes this book from authors is contains a very simple proofs and many new examples. This book introduces the topological concepts by a very simple method such as separation axioms, compactness, connectedness and metric spaces.

A. S. Farrag and S. E. Abbas, Professors of Mathematics, Faculty of Science, Sohag University, Egypt and Jazan University, Saudi-Arabia. Their primary research areas are General Topology and Fuzzy Topology. Where they published many papers in refereed journals.

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General Topology

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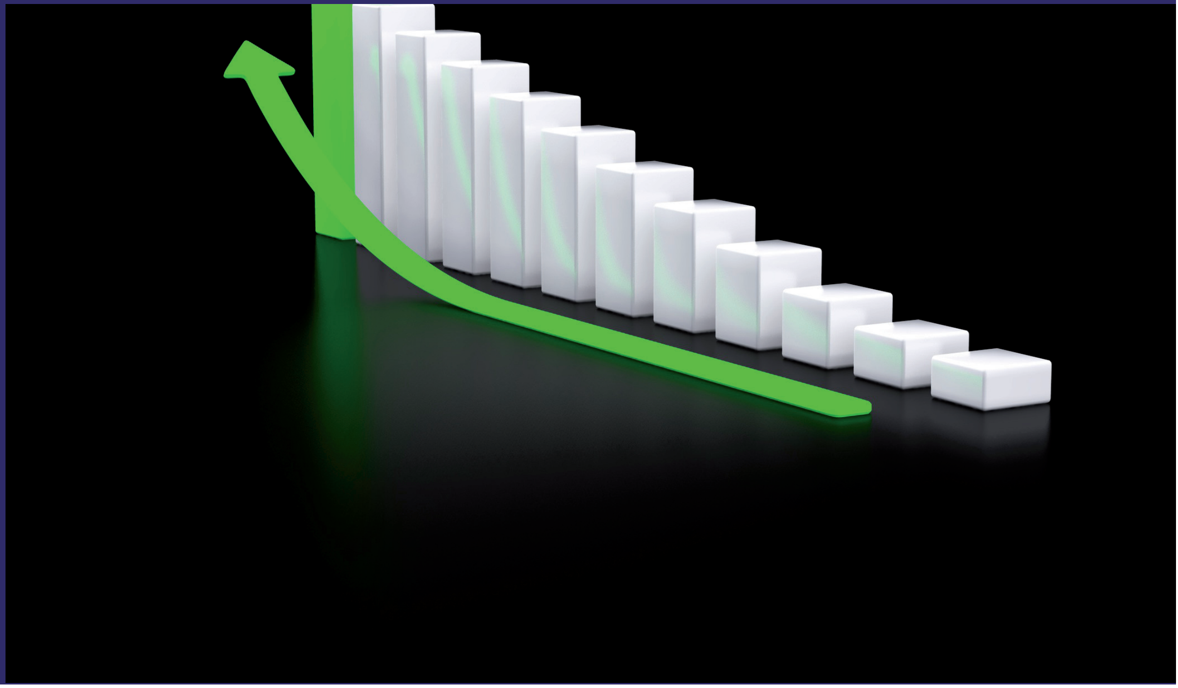

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General Topology

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Preface

In this book, my aim is to provide the reader with a foundation in general topology that will adequately prepare him for further work in a broad variety of mathematical disciplines. The arrangement of the material is such that the book can also serve as a reference for the more advanced mathematician.

The reader is assumed to have a background of at least one semester of rigorous analysis; for such persons, the treatment herein is self- contained and the material can easily be covered in a two-semester course.

Chapters I, provide an introduction to the axiomatic foundations of set theory. This chapter consists of four articles and it can consider being an introduction to the general topology which needs some mathematical concepts. One of the most important concepts is the mathematical logics and how to use its facilities in proving different problems. The second is the theory of sets and all related concepts like relations, functions and classification of sets into finite, infinite, countable and uncountable. Some concepts of the mathematical analyses like differentiation and integration must be studied before. Chapters II-V, are devoted to general topological structures; the emphasis is on the mapping, and an extended treatment of identifications is given. Separation axioms, Compactness, Connectedness and product spaces are introduced in Chapters VI-VIII. Finally, by imposing increasingly more severe conditions on the topology, the development proceeds down the hierarchy of topological spaces to the metric spaces.

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Chapter I

Introduction

Topics:

- Set Theoretic Preliminaries.
- Cartesian product, relations and functions.
- Classification of sets.
- Topology on the real line.

This chapter consists of four articles and it can consider being an introduction to the general topology which needs some mathematical concepts. One of the most important concepts is the mathematical logics and how to use its facilities in proving different problems. The second is the theory of sets and all related concepts like relations, functions and classification of sets into finite, infinite, countable and uncountable. Some concepts of the mathematical analyses like differentiation and integration must be studied before. In the first article we gives a glance about sets, cartesian product of sets, relations and functions also we gives some problems which help the reader to understand these fundamental concepts. In article two we gives an idea about the kinds of finite, infinite, countable and uncountable sets. A proof is given that $N \times N$ where N is the set of the natural numbers and the rational set Q are countable sets also a proof that the interval $(0,1)$ is an infinite uncountable set and it is equivalent to the set R of the real numbers. In the last article we give a glance about the topology on the real line i.e on the set R where the general topology is an abstract concept or a generalization of the concepts of the topology on R.

1.1 Set Theoretic Preliminaries.

Definition 1.1.1 A set is a collection of objects, called the elements or members of the set. The sets will be denoted by the capital letters $A, B, C, \ldots, X, Y, Z, \ldots$ and the elements of the sets will be denoted by the small letters $a, b, c, \ldots, x, y, z, \ldots$. Let A be a set

$x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

Definition 1.1.2 Let A and B be sets. Then, A is a subset of B, denoted $A \subset B$, if every element of A is also an element of B. In notation, $A \subset B$ is $x \in A \Rightarrow x \in B$.

Remark 1.1.1 *We accept as an axiom from set theory the existence of a set* $\emptyset = \{\ \}$ *that has no elements.* $\emptyset$ *is called the empty set or null set.*

Remark 1.1.2 *Let* A *be any set. Then* $\emptyset \subset A$. *Notice that* $\emptyset$ *satisfies the definition of subset vacuously, there no elements of* $\emptyset$ *that must be contained in the other set.*

Definition 1.1.3 Let A and B be sets. Then $A = B$ if and only if $A \subset B$ and $B \subset A$. In notation, Then $A = B$ iff $A \subset B$ and $B \subset A$

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

Remark 1.1.3

- $A \subset B$ *means* $\forall x \in A, x \in B$
- $A \subseteq B$ *means* $A \subset B$ *or* $A = B$.
- $A \subsetneq B$ *means* $A \subset B$ *and* $A \neq B$.

Definition 1.1.4 If $A \subsetneq B$, then A is called a proper subset of B.

Lemma 1.1.1 *Every set is a subset of itself.*

Proof: Let A be a set and let $x \in A$. Then, as

$$x \in A \Rightarrow x \in A, A \subseteq A.$$

Lemma 1.1.2 *If* $A \subseteq B$ *and* $B \subseteq C$, *then* $A \subseteq C$.

Proof: Suppose $A \subseteq B$ and $B \subseteq C$. Let $x \in A$. Then as $A \subseteq B, x \in B$. Now as $B \subseteq C, x \in C$. Hence, $\forall x \in A, x \in C$. Thus, $A \subseteq C$. So $A \subseteq B \subseteq C$.

Definition 1.1.5 Let A and B be sets. The union of A and B is the set which contains elements in A or in B or both,

$$A \bigcup B = \{x \in X : x \in A \ or \ x \in B\}.$$

It is clear that, $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$. Further, if $A \subseteq B$, then $A \bigcup B = B$. So, $\varnothing \bigcup A = A$.

Definition 1.1.6. Let A and B be sets. The intersection of A and B is the set containing those elements that are simultaneously in both A and B ,

$$A \bigcap B = \{x \in X : x \in A \ and \ x \in B\}.$$

It is clear that, $A \bigcap B \subseteq A$ and $A \bigcap B \subseteq B$. Further, if $A \subseteq B$, then $A \bigcap B = A$. So, $\varnothing \bigcap A = \varnothing$.

Definition 1.1.7. Sets for which their intersection is $\varnothing$ are called disjoint.

Definition 1.1.8. Let A and B be sets. The difference between A and B is the set

$$A - B = \{x \in X : x \in A \ and \ x \notin B\}.$$

If all sets under consideration are subsets of a whole set X then the complement of a subset A of X is the difference between X and A denoted through this book by A^c i.e. $A^c = X - A$.

Theorem 1.1.1. *Let* $A \subseteq B$ *and let* C *be any set. Then*

1) $A \bigcup C \subseteq B \bigcup C$.
2) $A \bigcap C \subseteq B \bigcap C$.

Proof: 1) Let $x \in A \bigcup C$. Then $x \in A$ or $x \in C$. Suppose $x \in A$, then as $A \subseteq B$, $x \in B$. So $x \in B \bigcup C$. Now suppose $x \in C$. So $x \in B \bigcup C$. Hence, in either case $A \bigcup C \subseteq B \bigcup C$.

2) Let $x \in A \bigcap C$. Then $x \in A$ and $x \in C$. Suppose $x \in A$, then as $A \subseteq B$, $x \in B$. So $x \in B \bigcap C$. Hence, $A \bigcap C \subseteq B \bigcap C$.

Theorem 1.1.2. *(Distributive Laws) Let* A, B *and* C *be sets. Then*

1) $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$.

2) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

Proof: 1) Let $x\in A\cup(B\cap C)$. Then $x\in A$ or $x\in B\cap C$. Suppose $x\in A$, then $x\in A\cup B$. It is also true that $x\in A\cup C$. So $x\in(A\cup B)\cap(A\cup C)$

Now suppose $x\in B\cap C$. Then $x\in B$ and $x\in C$. As $x\in B, x\in A\cup B$. As $x\in C, x\in A\cup C$. So $x\in(A\cup B)\cap(A\cup C)$. Hence

$$A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C).$$

Conversely, let $x\in(A\cup B)\cap(A\cup C)$. Then $x\in(A\cup B)$ and $x\in(A\cup C)$. As $x\in(A\cup B)$, $x\in A$ or $x\in B$. As $x\in(A\cup C)$, $x\in A$ or $x\in C$. Suppose $x\in A$, then $x\in A\cup(B\cap C)$. Now suppose $x\notin A$, then $x\in B$ and $x\in C$. So, $x\in B\cap C$. Thus $x\in A\cup(B\cap C)$. Hence in either case

$$(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C).$$
$$\therefore A\cup(B\cap C)=(A\cup B)\cap(A\cup C).$$

2) Let $x\in A\cap(B\cup C)$. Then $x\in A$ and $x\in B\cup C$. As $x\in B\cup C, x\in B$ or $x\in C$. Suppose $x\in B$. Then $x\in A\cap B$. So $x\in(A\cap B)\cup(A\cap C)$. Now suppose $x\in C$, then $x\in A\cap C$. So $x\in(A\cap B)\cup(A\cap C)$. Hence, in either case

$$A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C).$$

Conversely, let $x\in(A\cap B)\cup(A\cap C)$. Then $x\in(A\cap B)$ or $x\in(A\cap C)$. Suppose $x\in(A\cap B)$, then $x\in A$ and $x\in B$. As $x\in B$, $x\in B\cup C$. So, $x\in A\cap(B\cup C)$. Now suppose $x\in A\cap C$. Then $x\in A$ and $x\in C$. As $x\in C, x\in B\cup C$. So $x\in A\cap(B\cup C)$. Hence in either case

$$(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C).$$
$$\therefore A\cap(B\cup C)=(A\cap B)\cup(A\cap C).$$

Theorem 1.1.3 *Let* $A \subseteq B \subseteq X$ *. Then*

1) $B - A = B \bigcap (X - A)$.
2) $X - B \subseteq X - A$.

Proof: 1) Let $x \in B - A$. Then $x \in B$ and $x \notin A$. As $x \notin A, x \in X - A$. So $x \in B \bigcap (X - A)$.

Conversely, let $x \in B \bigcap (X - A)$. Then $x \in B$ and $x \in X - A$. As $x \in X - A, x \notin A$. So $x \in B - A$.

$$\therefore B - A = B \bigcap (X - A).$$

2) Let $x \in X - B$. Then $x \notin B$. As $A \subseteq B, x \notin A$. So $x \in X - A$. Hence $X - B \subseteq X - A$.

Theorem 1.1.4 *Let* X *be the whole set and* A *,* B *and* C *are subsets of* X *. Then*
(1) $(A - B) - C = (A - C) - B = (A - C) - (B - C)$.
(2) $A - (A - B) = A \cap B$ **.**
(3) $(A \cup B) - (B \cup C) = (A - B) \cup (A - C) \cup (B - C)$.

(4) $(A - C) \cap (B - C) = A \cap (B - C) = (A \cap B) - C$.

Proof: We prove (1) and (4) , (2) and (3) left to the reader.

Proof of (1) using (4) from Theorem 1.1.3 (1) and the distributive law

$$\begin{aligned}(A - B) - C &= (A \cap B^c) \cap C^c = (A \cap B^c) \cap (A \cap C^c)\\ &= (A - B) \cap (A - C)\\ &= (A \cap C^c) \cap B^c = (A - C) - B\end{aligned}$$

Also

$$\begin{aligned}(A - C) - (B - C) &= (A \cap C^c) \cap (B^c \cup C)\\ &= [(A \cap C^c) \cap B^c] \cup [(A \cap C^c) \cap C]\\ &= [(A \cap C^c) \cap B^c] \cup \varnothing = (A - C) - B\end{aligned}$$

Proof of (4) using (4) of Theorem 1.1.3 (1) and associative law

$$A \cap (B - C) = A \cap (B \cap C^c) = (A \cap B) \cap C^c = (A \cap B) - C$$

Definition 1.1.9. The symmetric difference or the squads between A and B is defined by the equation

$$A\Delta B=(A-B)\cup(B-A)$$

And we remarks that

(1) $x\in A\Delta B \Leftrightarrow (x\in A\wedge x\notin B)\vee(x\in B\wedge x\notin A)$.

(2) $x\notin A\Delta B \Leftrightarrow (x\in A\wedge x\in B)\vee(x\notin A\wedge x\notin B)$.

Theorem 1.1.5. *Considering the sets A and B we haves*

(1) $A\Delta B=(A\cup B)-(A\cap B)$.

(2) $A\cap(B\Delta C)=(A\cap B)\Delta(A\cap C)$.

(3) $A\Delta B=B\Delta A$.

(4) $A\Delta(B\Delta C)=(A\Delta B)\Delta C$.

Proof: Directly from the definition we prove the statements (1), (2) and (3) at follows:

Firstly (1)

$$\begin{aligned}A\Delta B &= (A\cap B^C)\cup(B\cap A^C)\\ &=[A\cup(B\cap A^C)]\cap[B^C\cup(B\cap A^C)]\\ &=[(A\cup B)\cap(A\cup A^C)]\cap[(B^C\cup B)\cap(B^C\cup A^C)]\\ &=(A\cup B)\cap(A\cap B)^C=(A\cup B)-(A\cap B)\end{aligned}$$

Secondly (2) by using the statement (4), the distributive and associative laws and Theorem 1.1.3.

$$
\begin{aligned}
&A \cap (B \Delta C) \\
&= A \cap [(B \cup C) - (B \cap C)] \\
&= A \cap [(B \cup C) \cap (B^c \cup C^c)] \\
&= [A \cap (B \cup C)] \cap [A \cap (B^c \cup C^c)] \\
&= [(A \cap B) \cup (A \cap C)] \cap [(A \cap B^c) \cup (A \cap C^c)] \\
&= [(A \cap B) \cup (A \cap C)] \cap [(A - B) \cup (A - C)] \\
&= [(A \cap B) \cup (A \cap C)] \cap [(A - (A \cap B)) \cup (A - (A \cap C))] \\
&= [(A \cap B) \cup (A \cap C)] \cap [(A \cap (A \cap B)^c) \cup (A \cap (A \cap C)^c)] \\
&= [(A \cap B) \cup (A \cap C)] \cap [(A \cap \{(A \cap B)^c \cup (A \cap C)^c\}] \\
&= [\{(A \cap B) \cup (A \cap C)\} \cap A] \cap [(A \cap B)^c \cup (A \cap C)^c]
\end{aligned}
$$

Since $(A \cap B) \cup (A \cap C) \subset A$ we gets

$$
\begin{aligned}
A \cap (B \Delta C) &= [(A \cap B) \cup (A \cap C)] \cap [(A \cap B) \cap (A \cap C)]^c \\
&= [(A \cap B) \cup (A \cap C)] - [(A \cap B) \cap (A \cap C)] \\
&= (A \cap B) \Delta (A \cap C)
\end{aligned}
$$

Thirdly (4) using Definition 1.1.9 gets

$$
\begin{aligned}
&x \in (A \Delta B) \Delta C \\
&\quad \Rightarrow (i)\ (x \in A \Delta B \wedge x \notin C) \vee (ii)\ (x \in C \wedge x \notin A \Delta B)
\end{aligned}
$$

Then from (i)

$$
\begin{aligned}
&x \in A \Delta B \wedge x \notin C \\
&\Rightarrow [(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)] \wedge x \notin C \\
&\Rightarrow (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in B \wedge x \notin A \wedge x \notin C) \\
&\qquad \Rightarrow (x \in A \wedge x \notin B \Delta C) \vee (x \in B \Delta C \wedge x \notin A) \\
&\qquad \Rightarrow x \in (A - B \Delta C) \vee x \in (B \Delta C - A) \\
&\qquad \Rightarrow x \in [(A - B \Delta C) \cup (B \Delta C - A)] \\
&\qquad \Rightarrow x \in A \Delta (B \Delta C) \qquad (I)
\end{aligned}
$$

from (ii)

$$\begin{aligned}
& x \in C \wedge x \notin A \Delta B \Rightarrow \\
& \quad \Rightarrow x \in C \wedge [(x \in A \wedge x \in B) \vee (x \notin A \wedge x \notin B)] \\
& \quad \Rightarrow (x \in A \wedge x \in B \wedge x \in C) \vee (x \notin B \wedge x \in C \wedge x \notin A) \\
& \quad \Rightarrow [x \in A \wedge x \notin (B \Delta C)] \vee [x \in (B \Delta C) \wedge x \notin A] \\
& \quad \Rightarrow [x \in A - (B \Delta C)] \vee [x \in (B \Delta C) - A] \\
& \quad \Rightarrow x \in [(A - B \Delta C) \cup (B \Delta C - A)] \\
& \quad \Rightarrow x \in A \Delta (B \Delta C) \quad (II)
\end{aligned}$$

From (I) and (II)

$$(A \Delta B) \Delta C \subset A \Delta (B \Delta C) \quad (III)$$

By using the commutative property (3) of this Theorem and the result (III) we gets

$$\begin{aligned}
A \Delta (B \Delta C) &= (B \Delta C) \Delta A = (C \Delta B) \Delta A \subset C \Delta (B \Delta A) = (B \Delta A) \Delta C \\
&= (A \Delta B) \Delta C
\end{aligned}$$

So $\qquad A \Delta (B \Delta C) \subset (A \Delta B) \Delta C \quad (VI)$

Therefore from (III) and (IV) we obtains the required result

$$A \Delta (B \Delta C) = (A \Delta B) \Delta C \,.$$

Definition 1.1.10 Let Γ be a non-empty set, called the indexing set such that $\forall i \in \Gamma$ there corresponds a set A_i. The family $\left\{A_i : i \in \Gamma\right\}$ is called an indexed family of sets.

Definition 1.1.11 Let $\left\{A_i : i \in \Gamma\right\}$ be an indexed family of sets. Then,

(1) The arbitrary union over $i \in \Gamma$ is the set

$$\bigcup_{i \in \Gamma} A_i = \left\{x : x \in A_i \ \textit{for at least one } i \in \Gamma\right\}.$$

(2) The arbitrary intersection over $i \in \Gamma$ is the set

$$\bigcap_{i \in \Gamma} A_i = \left\{x : x \in A_i \ \textit{for all } i \in \Gamma\right\}.$$

Remark 1.1.4 (1) *If* $|\Gamma| = \aleph,$ *then* $\bigcup_{i\in\Gamma} A_i = \bigcup_{i=1}^{\infty} A_i$ *and is called a countable union. Similarly,* $\bigcap_{i\in\Gamma} A_i = \bigcap_{i=1}^{\infty} A_i$ *and is called countable intersection.*

(2) If $|\Gamma| = n,$ *then* $\bigcup_{i\in\Gamma} A_i = \bigcup_{i=1}^{n} A_i$ *and is called a finite union. Similarly,* $\bigcap_{i\in\Gamma} A_i = \bigcap_{i=1}^{n} A_i$ *and is called a finite intersection.*

Theorem 1.1.6 *(Demorgan's Laws)*

Let $\left\{A_i : i \in \Gamma\right\}$ *be an indexed family of sets such that* $A_i \subseteq X$, $\forall i \in I$.

Then

(1) $X - \left(\bigcap_{i\in\Gamma} A_i \right) = \bigcup_{i\in\Gamma} (X - A_i).$

(2) $X - \left(\bigcup_{i\in\Gamma} A_i \right) = \bigcap_{i\in\Gamma} (X - A_i).$

Proof (1) Let $x \in X - \left(\bigcap_{i\in\Gamma} A_i \right)$. Then $x \notin \bigcap_{i\in\Gamma} A_i$. So $\exists i \in \Gamma$ such that $x \notin A_i$. So $x \in \left(X - A_i\right)$ for this $i \in \Gamma.$ Thus, $x \in \bigcup_{i\in\Gamma} (X - A_i)$. Hence

$$X - \left(\bigcap_{i\in\Gamma} A_i \right) \subseteq \bigcup_{i\in\Gamma} (X - A_i).$$

Conversely, let $x \in \bigcup_{i\in\Gamma} (X - A_i)$. Then $x \in \left(X - A_i\right)$ for at least one $i \in \Gamma.$ So, $x \notin A_i$ for this $i \in \Gamma.$ So, $x \notin \bigcap_{i\in\Gamma} A_i$. Thus $x \in X - \left(\bigcap_{i\in\Gamma} A_i \right)$.

Hence

$$\bigcup_{i\in\Gamma} (X - A_i) \subseteq X - \left(\bigcap_{i\in\Gamma} A_i \right).$$

(2) Let $x \in X - \left(\bigcup_{i \in \Gamma} A_i \right)$. Then $x \notin \bigcup_{i \in \Gamma} A_i$. So $x \notin A_i$ for all $i \in \Gamma$. So $x \in \left(X - A_i \right)$ for all $i \in I$. Thus, $x \in \bigcap_{i \in \Gamma} (X - A_i)$. Hence

$$X - \left(\bigcup_{i \in \Gamma} A_i \right) \subseteq \bigcap_{i \in \Gamma} (X - A_i).$$

Conversely, let $x \in \bigcap_{i \in \Gamma} (X - A_i)$. Then $x \in \left(X - A_i \right)$ for all $i \in \Gamma$. So, $x \notin A_i$ for any $i \in \Gamma$. So, $x \notin \bigcup_{i \in \Gamma} A_i$. Thus $x \in X - \left(\bigcup_{i \in \Gamma} A_i \right)$. Hence

$$\bigcap_{i \in \Gamma} (X - A_i) \subseteq X - \left(\bigcup_{i \in \Gamma} A_i \right).$$

Remark 1.1.5 *Let* $\left\{ A_i : i \in \Gamma \right\}$ *be an indexed family of subsets of* X*. Recall that arbitrary union is*

$$\bigcup_{i \in \Gamma} A_i = A_i = \left\{ x : x \in A_i \ \text{for at least one} \ i \in \Gamma \right\}.$$

If the indexing set $\Gamma = Z^+$*, we have a countable union:*

$$\bigcup_{i \in \Gamma} A_i = \bigcup_{i=1}^{\infty} A_i = \left\{ x : x \in A_i \ \text{for at least one} \ i \in Z^+ \right\}.$$

If the indexing set is $\Gamma = \left\{ 1, 2, \ldots\ldots, n \right\}$*, we have a finite union*

$$\bigcup_{i \in \Gamma} A_i = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n.$$

There are two degenerate cases:

- *Case (1):* $\Gamma = \left\{ 1 \right\}$

$$\bigcup_{i \in \Gamma} A_i = \bigcup_{i=1}^{1} A_i = A_1.$$

- *Case (2):* $\Gamma = \varnothing$

$$\bigcup_{i\in\varnothing} A_i = \left\{x : x \in A_i \ \textit{for at least one } i \in \varnothing\right\}.$$

Since $\nexists\ i \in \varnothing,\ x \notin A_i$ *for any index* $i \in \varnothing$. *So* $\bigcup_{i\in\varnothing} A_i = \varnothing$.

The degenerate case (2) is called the empty union.

Note that

$$\bigcup_{i\in\varnothing} A_i = \varnothing \Leftrightarrow \left(X - \bigcup_{i\in\varnothing} A_i\right) = X - \varnothing \Leftrightarrow \bigcap_{i\in\varnothing}\left(X - A_i\right) = X,$$

i.e., $\bigcap_{i\in\varnothing} A_i = X$ *and is called the empty intersection.*

Definition 1.1.12 Let A be a countable set. The power of A is the set of all subsets of A .

It is clear that, if $|A| = n$, then $|P(A)| = 2^n$. If $|A| = \aleph_\circ$, then $|P(A)| = 2^{\aleph_\circ} = c$. Also if $T \subset P(X) - \{\varnothing\}$ such that $\bigcup\{A : A \in T\} = X$, $\varnothing \notin T$ and $A \cap B = \varnothing$ for each two members A and B of T then T is said to be a partition of X .

Example 1.1.1. Let $A = \{a, b, c\}$. Then,

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, A\}.$$

Notice $|A| = 3$ and $|P(A)| = 2^3 = 8.$

Example 1.1.2. Let $X = \{1,2,3\}$ Then

(1) $P(X) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X,\varnothing\}$

(2) $T_1 = \{\{1\},\{2\},\{3\}\}$, $T_2 = \{\{1,2\},\{3\}\}$, $T_3 = \{\{1\},\{2,3\}\}$, $T_4 = \{\{2\},\{1,3\}\}$ and $T_5 = \{X\}$ are partitions of X and $\{T_1, T_2, T_3, T_4, T_5\}$ is the family of all partitions of X .

Example 1.1.3. Let A , B , $A-B$, $B-A$ and $A\cap B$ be nonempty sets. Then $\{A-B, B-A\}$ is a partition of $A\Delta B$ and $\{A-B, A\cap B, B-A\}$ is a partition of $A\cup B$.

1.2 Cartesian product, relations and functions:

Definition 1.2.1 If X and Y are nonempty sets, then the cartesian product of X by Y denoted $X\times Y$ is the set of all ordered pairs (x,y) such that $x\in X$ and $y\in Y$ that is

$$X\times Y=\{(x,y): x\in X, y\in Y\}$$

and it could be remarked that

(1) $X\times Y=\varnothing$ iff $X=\varnothing$ or $Y=\varnothing$ since $X\times Y\neq\varnothing$ iff there is an element $(x,y)\in X\times Y$ iff there are two elements $x\in X$ and $y\in Y$ iff $X\neq\varnothing$ and $Y\neq\varnothing$.

(2) If $(a,b),(x,y)\in X\times Y$ then $(a,b)=(x,y)$ iff $a=x$ and $b=y$.

(3) If X and Y are nonempty set then $X\times Y=Y\times X$ iff $X=Y$. For if $X\neq Y$ then there are two points $x\in X-Y$ and $y\in Y$ which implies that $x\neq y$ which implies that $(x,y)\in X\times Y$ and $(x,y)\notin Y\times X$ or there are two points $y\in Y-X$ and $x\in X$ which implies that $(x,y)\notin Y\times X$ and $(x,y)\in X\times Y$ which implies that $X\times Y\neq Y\times X$. Conversely if $X\times Y\neq Y\times X$ then here is an ordered pair (x,y) such that $(x,y)\in X\times Y-Y\times X$ which means that either $x\in X-Y$ or $y\in Y-X$ which implies that $X\neq Y$ or there is an ordered pair $(y,x)\in Y\times X-X\times Y$ which leads also to the result $X\neq Y$.

The following Theorem gives us some properties for the cartesian product of sets.

Theorem 1.2.1 *Consider the sets* X , Y , Z *and* W *and let* $A\subset X$ *and* $B\subset Y$ *, Then,*

(i) $X\times(Y\bigcup Z)=(X\times Y)\bigcup(X\times Z)$.

(ii) $X\times(Y\bigcap Z)=(X\times Y)\bigcap(X\times Z)$.

(iii) $X\times(Y-Z)=(X\times Y)-(X\times Z)$.

(iv) $(X-Y)\times Z=(X\times Z)-(Y\times Z)$.

(v) $X\times Y\bigcap Z\times W=X\bigcap Z\times Y\bigcap W$.

(vi) $(X\times Y)\bigcup(Z\times W)=(X\bigcup Z)\times(Y\bigcup W)$.

(vii) $X\times Y-A\times B=((X-A)\times Y)\bigcup(X\times(Y-B))$.

Proof: The proof is left to the reader.

Definition 1.2.2 Let X and Y be nonempty sets i.e, $X\neq\varnothing$ and $Y\neq\varnothing$. Then any family $S\subset X\times Y$ is called a relation from X to Y the elements of S are ordered pairs, $x\in X$ and $y\in Y$ such that $(x,y)\in S$ means that x is related to y by the relation S this is written also in the form xSy . Through this book we uses $(x,y)\in S$ if x is related to y and $(x,y)\notin S$ if x is not related to y by S .

If X , Y and Z are nonempty sets, $S\subset X\times Y$ and $T\subset Y\times Z$ are two relations then

(1) If $Y=X$ then the relation $S\subset X\times X$ is called a relation on X or in X and the relation $\nabla=\{(x,x):x\in X\}$ is called the diagonal relation on X .

(2) $S^{-1}=\{(y,x):(x,y)\in S\}$ Is the inverse relation of S .

(3) $T\circ S$ Is defined to be a relation from X to Z such that $(x,y)\in S$ and $(y,z)\in T$ implies that $(x,z)\in T\circ S$ i.e.

$$(x,y)\in S\wedge(y,z)\in T\Rightarrow(x,z)\in T\circ S$$

The relation $T\circ S$ is called the composition of S and T .

The relation S on a nonempty set X is called

(1) Reflexive if $(x,y)\in S$ for each $x\in X$ iff $\nabla\subset S$.

(2) Symmetric. if $(x,y)\in S$ implies that $(y,x)\in S$ for each two points $x,y\in X$ iff $S^{-1}=S$.

(3) Transitive if $(x,y)\in S$ and $(y,x)\in S$ implies that $(x,z)\in S$ for each points $x,y,z\in X$ iff $S\circ S\subset S$.

(4) Is an equivalence relation if it is reflexive, symmetric and transitive relation.

(5) If S is an equivalence relation on a nonempty set X and $x\in X$ then the set $[x]=\{y\in X:(x,y)\in S\}$ is called the equivalence class of the point x by the relation S.

Example 1.2.1 Consider the set of the real numbers R and let $S\subset R\times R$ be a relation in R such that $(x,y)\in S$ if $y=\sqrt{x}$ then

(1) S is not reflexive since $x\neq\sqrt{x}$ for each number $x\in R-\{0,1\}$ which implies that $(x,x)\notin S$, for example $(2,2)\notin S$.

(2) S is not symmetric since for example $(4,2)\in S$ while $(2,4)\notin S$.

(3) S is not transitive since $(16,4),(4,2)\in S$ while $(16,2)\notin S$.

Example 1.2.2 Let $X=\{1,2,3,4\}$ and S be a relation on X such that $S=\{(1,1),(1,2),(2,1),(2,2),(3,3)(3,4),(4,3),(4,4)\}$. Clearly S is an equivalence relation on X because it is reflexive, symmetric and transitive. The equivalence classes of the points of X are $[1]=[2]=\{1,2\}$ and $[3]=[4]=\{3,4\}$ the family $\{\{1,2\},\{3,4\}\}$ of these equivalence classes forms a partition of X and this is generally valid as it is given by the following Theorem.

Theorem 1.2.2 *Let S be an equivalence relation on a nonempty set X. Then,*

(1) $x\in[x]$ For each point $x\in X$.

(2) $(x,y)\in S\Leftrightarrow[x]=[y]$.

(3) For each two points $x,y\in X$, $[x]=[y]$ or $[x]\cap[y]=\varnothing$.

Proof: The proof left to the reader.

Definition 1.2.3 Let X and Y be nonempty sets. Then the relation $f \subset X \times Y$ is called a function or a map if for each point $x \in X$ there is a unique point $y \in Y$ such that $(x, y) \in f$ or equivalently

(1) $x \in X \Rightarrow \exists y \in Y : (x, y) \in f$.

(2) $(x, y), (x, z) \in f \Rightarrow y = z$ for each $y, z \in Y$ and

(a) $f \subset X \times Y$ is a function from X to Y written in the form $f : X \to Y$.

(b) $(x, y) \in f$ written in the form $y = f(x)$ that is $(x, y) \in f$ and $y = f(x)$ are equivalent and $f(x)$ is called the image of the point x by the function f .

If we consider that $f(x)$ is the set of all images of x by f then f is a function if

(1) $f(x) \neq \emptyset$ for each point $x \in X$ and

(2) $y, z \in f(x) \Rightarrow y = z$ Equivalently $f(x)$ is singleton for each point $x \in X$.

Example 1.2.3 The relation $f \subset R \times R$ such that $y = f(x) = x^2$ for each $x \in R$ is a function from R to R and is written $f : R \to R$.

Example 1.2.4 Consider the set of the natural numbers N and the set $Y = \{1,2,3\}$ then $f : N \to Y$ such that $f(1) = 1$, $f(2) = 2$ and $f(n) = 3$ for each $n \in N$ such that $n \geq 3$ is a function from N to Y .

Example 1.2.5 Consider the set N and the set $Y = \{3,6,9,\ldots\ldots\} = \{3n : n \in N\}$ then $f : Y \to N$ such that $f(m) = \frac{1}{3}m$ for each $m \in Y$ is a function from Y to N .

Definition 1.2.4 Let $f : X \to Y$ be a function, $A \subset X$ and $B \subset Y$. Then,

(1) The image of the set A by the function f is

$$f(A)=\{f(x):x\in A\}$$

If $x\in X$ we write $f(\{x\})=f(x)$ which is unique.

(2) The inverse image of the set B by f is

$$f^{-1}(B)=\{x\in X:f(x)\in B\}.$$

If $y\in Y$ we write $f^{-1}(\{y\})=f^{-1}(y)$.

(3) $f(X)=\{f(x):x\in X\}$ is called the range of f .

Clearly $f(X)\subset Y$ and $f^{-1}(Y)=X$.

Theorem 1.2.3 *Let* $f:X\to Y$ *be a function,* $A,B\subset X$ *,* $g:Z\to W$ *,* $C\subset Z$ *and* $f\times g:X\times Z\to Y\times W$ *be such that*

$$(f\times g)(x,z)=(f(x),g(z))\ ;\ \forall\,(x,z)\in X\times Z\ .\ \textit{Then,}$$

(1) $A\subset B\Rightarrow f(A)\subset f(B)$ *.*

(2) $f(A\cup B)=f(A)\cup f(B)$ *.*

(3) $f(A\cap B)\subset f(A)\cap f(B)$ *but not equal.*

(4) $f(A)-f(B)\subset f(A-B)$ *but not equal.*

(5) $(f\times g)(A\times C)=f(A)\times g(C)$

Proof: We shall prove (2) and explain by examples that the equalities in (3) and (4) are not generally true.

(2) $y\in f(A\cup B)\Rightarrow\exists x\in(A\cup B):y=f(x)$ then

$$x\in(A\cup B)\Rightarrow x\in A\vee x\in B\Rightarrow y\in f(A)\vee y\in f(B)\Rightarrow$$
$$y\in(f(A)\cup f(B))\Rightarrow f(A\cup B)\subset f(A)\cup f(B)\quad (I)$$

Conversely, $y\in(f(A)\cup f(B))\Rightarrow y\in f(A)\vee y\in f(B)$,

$$y \in f(A) \Rightarrow \exists x \in A : y = f(x)$$
$$\Rightarrow x \in (A \cup B) \Rightarrow y \in f(A \cup B)$$

Similarly $y \in f(B) \Rightarrow y \in f(A \cup B)$ and so

$$f(A) \cup f(B) \subset f(A \cup B) \qquad (II)$$

From (I) and (II) we gets $f(A \cup B) = f(A) \cup f(B)$.

To explains the inequalities in (3) and (4) let $X = \{1,2,3\}$, $Y = \{x, y\}$ and $f : X \to Y$ be such that $f = \{(1,x),(2,x),(3,y)\}$, if $A = \{1,3\}$ and $B = \{2,3\}$ then $A \cap B = \{3\}$ so $f(A) = f(B) = \{x, y\}$ and $f(A \cap B) = \{y\}$ which implies that $f(A \cap B) \neq f(A) \cap f(B)$ this proved the inequality in (3). Also using this example clearly

$$f(A) - f(B) = \varnothing \wedge A - B = \{1\} \Rightarrow f(A - B) = \{x\}$$

Then the none quality in (4) is valid.

Theorem 1.2.4 *If* $f : X \to Y$ *and* $A, B \subset Y$ *then*

(1) $A \subset B \Rightarrow f^{-1}(A) \subset f^{-1}(B)$,

(2) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$,

(3) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$,

(4) $f^{-1}(A) - f^{-1}(B) = f^{-1}(A - B)$,

(5) $(f \times g)^{-1}(A \times C) = f^{-1}(A) \times g^{-1}(C)$.

Proof: It is an easy proof and is left to the reader.

Remark 1.2.1 *Let* $f : X \to Y$ *be a function. Then, the family* $P = \{f^{-1}(y) : y \in f(X)\}$ *is a partition of* X .

Proof: Clearly $\bigcup \{f^{-1}(y) : y \in f(X)\} = X$ and if $y, z \in f(X)$ such that $y \neq z$ then

$$x \in f^{-1}(y) \cap f^{-1}(z) \Rightarrow y = f(x) = z \Rightarrow f^{-1}(y) = f^{-1}(z)$$

The contra positive of this result completes the proof which is

$$f^{-1}(y) \neq f^{-1}(z) \Rightarrow f^{-1}(y) \cap f^{-1}(z) = \varnothing .$$

Definition 1.2.5 If $f : X \to Y$ is a function then it is called

(1) An injective or one to one function if two distinct points of X have two distinct images this is

$$x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Or equivalently its contra positive

$$x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

We remarks that if $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $y \in Y$ then

$$x_1, x_2 \in f^{-1}(y) \Rightarrow f(x_1) = y = f(x_2)$$

Which implies that f is not injective, the contra positive of this result is the function f is injective iff $f^{-1}(y)$ is empty or singleton for each $y \in Y$ this is another definition of the injective function.

(2) A surjective or onto function if for each point $y \in Y$ there is at least a point $x \in X$ such that $y = f(x)$ equivalently $f(X) = Y$ equivalently $f^{-1}(y) \neq \varnothing$ for each point $y \in Y$.

(3) A correspondence or bijective if it is both injective and surjective. In such case the function $f^{-1} : Y \to X$ given by $f^{-1}(y) = x \in X$ for each $y \in Y$ where $f(x) = y$ is called the inverse function of f . Clearly f^{-1} is a bijective function and $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

Example 1.2.6 The function $f:R\to R$ such that $y=f(x)=x^2$ for each $x\in R$ is not injective since for example $-2,2\in R$ while $f(-2)=4=f(2)$ and not surjective since $-3=f(x)=x^2$ implies that x is not real.

Example 1.2.7 The function in Example 1.2.4 is a surjective and not injective function and the function in Example 1.2.5 is a bijective function.

Theorem 1.2.5 *If* $f:X\to Y$, $A,B\subset X$ *and* $E\subset Y$ *then* $A\subset f^{-1}(f(A))$, $f(f^{-1}(E))\subset E$ *and*

(1) $f(A\cap f^{-1}(E))=f(A)\cap E$ *,*

(2) $f(A\cap B)=f(A)\cap f(B)$ *iff* f *is injective,*

(3) f *is injective iff* $f^{-1}(f(A))\subset A$ *,*

(4) f *is surjective iff* $E\subset f(f^{-1}(E))$*.*

(5) f *is injective iff* $f(A-B)\subset f(A)-f(B)$*.*

(6) $f(A^c)$ *and* $Y-f(A)$ *are not comparable.*

(7) $Y-f(A)\subset f(A^c)$ *iff* f *is surjective.*

(8) $f(A^c)\subset Y-f(A)$ *iff* f *is injective.*

Proof: (1) Firstly $f(f^{-1}(E))\subset E$ implies that

$$f(A)\cap f(f^{-1}(E))\subset f(A)\cap E$$

And from (3) of Theorem 1.2.3 we gets

$$f(A\cap f^{-1}(E))\subset f(A)\cap f(f^{-1}(E))\subset f(A)\cap E \;--(I)$$

Secondly $y\in f(A)\cap E\Rightarrow y\in f(A)\wedge y\in E$ from which

$y\in f(A)\Rightarrow \exists x\in A:y=f(x)$ and so

$$\begin{aligned} y \in E \Rightarrow f(x) \in E \Rightarrow x \in f^{-1}(E) \Rightarrow x \in A \cap f^{-1}(E) \\ \Rightarrow y \in f(A \cap f^{-1}(E)) \\ \Rightarrow f(A) \cap E \subset f(A \cap f^{-1}(E)) --(II) \end{aligned}$$

From (I) and (II) $f(A \cap f^{-1}(E)) = f(A) \cap E$.

(2) From statement (3) of Theorem 1.2.3,

$$f(A \cap B) \subset f(A) \cap f(B) --(I)$$

Also $y \in f(A) \cap f(B) \Rightarrow y \in f(A) \wedge y \in f(B)$ and we find that

$$\begin{aligned} y \in f(A) \Rightarrow \exists x_1 \in A : y = f(x_1), \\ y \in f(B) \Rightarrow \exists x_2 \in B : y = f(x_2) \end{aligned}$$

If the function f is injective then $f(x_1) = y = f(x_2) \Rightarrow x_1 = x_2 = x$ which implies that $y = f(x)$. So $x \in A \cap B$ and

$$\begin{aligned} x \in A \cap B \Rightarrow y = f(x) \in f(A \cap B) \\ \Rightarrow f(A) \cap f(B) \subset f(A \cap B) --(II) \end{aligned}$$

From (I) and (II) we gets $f(A \cap B) = f(A) \cap f(B)$.

Secondly Suppose that $f(A \cap B) = f(A) \cap f(B)$ for each $A, B \subset X$. If $x_1, x_2 \in X$ then

$$\begin{aligned} x_1 \neq x_2 \Rightarrow \varnothing = f(\varnothing) = f(\{x_1\} \cap \{x_2\}) \\ = f(\{x_1\}) \cap f(\{x_2\}) \Rightarrow f(x_1) \neq f(x_2) \end{aligned}$$

Therefore f is injective.

(3) Let $y \in Y$ be such that $f^{-1}(y) \neq \varnothing$ then there exists $x \in X$ such that $y = f(x)$ so if $f^{-1}(f(A)) \subset A$ for each subset A of X then $\{x\} \subset f^{-1}(f(\{x\})) \subset \{x\}$ which implies that $f^{-1}(y) = f^{-1}(f(x)) = \{x\}$ is a singleton set. Hence f is an injective function.

Conversely, let f be an injective function and A be a subset of X. Then $x\in f^{-1}(f(A))\Rightarrow f(x)\in f(A)$ and if there exists a point $x_1\in A$ such that $f(x)=f(x_1)$, the function f is injective implies that $x=x_1$ which implies that $x\in A$ and therefore $f^{-1}(f(A))\subset A$.

(4) Firstly Let $E\subset f(f^{-1}(E))$ for each subset E of Y, if $E=Y$ then $Y\subset f(f^{-1}(Y))=f(X)\Rightarrow f(X)=Y$ and so f is surjective. Secondly suppose that f is surjective and $y\in E$ then there is a point $x\in X$ such that $y=f(x)$ and we find that

$$f(x)\in E\Rightarrow x\in f^{-1}(E)\Rightarrow y=f(x)\in f(f^{-1}(E))\Rightarrow E\subset f(f^{-1}(E))$$

This is the end of the Proof.

(5) Firstly suppose that f is an injective function, if $y\in f(A-B)$ then there exists a point $x\in A-B$ such that $y=f(x)$ and

$$x\in A-B\Rightarrow x\in A, x\in B^c\Rightarrow f(x)\in f(A), f(x)\in f(B^c)$$

Since f is injective then by the definition $f^{-1}(y)=\{x\}$ from which $f(x)\notin f(B)$ implies that $y=f(x)\in f(A)-f(B)$. So, $f(A-B)\subset f(A)-f(B)$.

Conversely suppose that $f(A-B)\subset f(A)-f(B)$ and $x_1,x_2\in X$ then

$$\begin{aligned} f(\{x_1\})=f(\{x_2\})&\Rightarrow f(\{x_1\})-f(\{x_2\})=\varnothing\\ &\Rightarrow f(\{x_1\}-\{x_2\})\subset f(\{x_1\})-f(\{x_2\})=\varnothing\\ &\Rightarrow f(\{x_1\}-\{x_2\})=\varnothing\Rightarrow\{x_1\}-\{x_2\}=\varnothing\Rightarrow x_1=x_2\end{aligned}$$

Therefore, f is injective.

(6) Consider the function $f:R\to R$ which is given by $f(x)=x^2$ for each $x\in R$ and $A=\{-2,3\}$ then $f(A)=\{4,9\}$ which implies that

$R-f(A)=R-\{4,9\}$ while $f(A^c)=f(R-\{-2,3\})=R^+=[0,\infty)$ which implies that $R-f(A)\not\subset f(A^c)$.

On the other hand $4,9\in f(A^c)-(R-f(A))$ which implies that

$$f(A^c)\not\subset(R-f(A)).$$

(7) $y\in Y-f(A)\Rightarrow y\notin f(A)$ And since f is surjective then there exists $x\in A^c$ such that $y=f(x)$ which implies that $y\in f(A^c)$ which implies that $Y-f(A)\subset f(A^c)$.

Conversely If $Y-f(A)\subset f(A^c)$ then by setting $A=X$ we gets

$$Y-f(X)\subset f(\varnothing)=\varnothing\Rightarrow Y-f(X)=\varnothing\Rightarrow Y=f(X).$$

Therefore f is a surjective function.

(8) From (5) by replacing A by X and B by A we gets

$$f(A^c)=f(X-A)\subset f(X)-f(A)\subset Y-f(A)$$

Another way: $y\in f(A^c)\Rightarrow\exists x\in A^c:y=f(x)$ and f is injective implies that $f^{-1}(y)=\{x\}$ from $y\notin f(A)$ which implies that $y\in Y-f(A)$. Then $f(A^c)\subset Y-f(A)$.

Conversely suppose that $f(A^c)\subset Y-f(A)$ for each subset A of X and $y\in Y$ be such that $f^{-1}(y)\neq\varnothing$ then there is a point $x\in X$ such that $y=f(x)$ then $A=\{x\}^c$ implies that

$$y=f(x)=f((\{x\}^c)^c)\subset Y-f(\{x\}^c)\Rightarrow y\notin f(\{x\}^c)$$

$$\Rightarrow f^{-1}(y)=\{x\}$$

Therefore, f is an injective function. In fact (8) is a special case of (5).

Definition 1.2.6 Consider the functions $f:X\to Y$ and $g:Y\to Z$ such that $f(X)=Y$ one can define the function $g\circ f:X\to Z$ called the composition of g and f such that $g\circ f(x)=g(f(x))$ for each point $x\in X$.

Remark 1.2.2 *From the Definition 1.2.6 one can remark that*

(1) $g\circ f(X)=g(f(X))=g(Y)$ *from which if* $g\circ f$ *is surjective then* g *is also.*
(2) If $g\circ f$ *is injective then for each two points* $x_1,x_2\in X$

$$f(x_1)=f(x_2)\Rightarrow g(f(x_1))=g(f(x_2))\Rightarrow g\circ f(x_1)$$
$$=g\circ f(x_2)\Rightarrow x_1=x_2$$

Which implies that f *is also injective*.

Theorem 1.2.6 *Consider the functions* $f:X\to Y$, $g:Y\to Z$ *and* $g\circ f:X\to Z$ *such that* $f(X)=Y$ *, if* $z\in Z$ *such that* $g^{-1}(z)\neq\emptyset$ *then*

$$(g\circ f)^{-1}(z)=f^{-1}\circ g^{-1}(z)=f^{-1}(g^{-1}(z)).$$

Proof: Firstly

$$x\in(g\circ f)^{-1}(z)\Rightarrow z=g(f(x))\Rightarrow\exists y\in Y:y=f(x)$$
$$\Rightarrow z=g(y)\Rightarrow x\in f^{-1}(y)\wedge y\in g^{-1}(z)$$

But $y\in g^{-1}(z)\Rightarrow f^{-1}(y)\subset f^{-1}(g^{-1}(z))$ which implies that $x\in f^{-1}(g^{-1}(z))$ and so

$$(f\circ g)^{-1}(z))\subset f^{-1}(g^{-1}(z))--(I)$$

Secondly, $x\in(f\circ g)^{-1}(z))=f^{-1}(g^{-1}(z))$ but $g^{-1}(z)\neq\emptyset$ which implies that there is a point $y\in Y$ such that $y\in g^{-1}(z)\subset Y$ and there is a point $x\in X$ such that $y=f(x)$ so,

$$z=g(y)=g(f(x))=g\circ f(x)\Rightarrow x\in(g\circ f)^{-1}(x)$$
$$\Rightarrow f^{-1}\circ g^{-1}(z)\subset(g\circ f)^{-1}(z)--(II)$$

From (I) and (II) we gets the required equality.

If f and g are bijective then f^{-1}, g^{-1} and $(g\circ f)^{-1}$ are bijectve and $(g\circ f)^{-1}(z)=f^{-1}\circ g^{-1}(z)$ for each $z\in Z$.

At the following we define some special and interesting functions.

(1) The constant function $f:X\to Y$ such that $f(x)=c$ for each point $x\in X$ where c is a point of Y .

(2) The inclusion function $i:X\to X$ is defined on a nonempty set X and written in the form $i:X\subset X$, it is defined such that $i(x)=x$ for each $x\in X$ **.** Clearly i is one to one and onto i.e. it is a correspondence and clearly $i^{-1}=i$ where i^{-1} is the inverse function of i .

(3) If $f:X\to Y$ and $A\subset X$, the restriction function is defined to be $f/A:A\to Y$ such that $f/A(x)=f(x)$ for each point $x\in A$. The restriction function of the inclusion function i on A is $i/A:A\subset X$ such that $i/A(x)=x$ for each $x\in A$ and we may write it $i:A\subset X$. If $f:X\to Y$ then $f/A=f\circ i$. If $f:X\to Y$ such that $g=f/A$ then f is called the extension function of the function f where $g=f\circ i$.

(4) The function $x:N\to X$ where N is the set of the natural numbers is called a sequence of points of the set X where $x(n)=x_n\in X$ for each $n\in N$, usually we write the sequence in terms of their values in the form $x=\{x_1,x_2,x_3,\ldots\}$ and shortly we write it in the form $<x_n>$**.**

(5) The indexed family of subsets of a non empty set X , if ∇ is nonempty, the function $f:\nabla\to P(X)$ such that $f(\alpha)=A_\alpha\in P(X)$ for each $\alpha\in\nabla$ where

$$f=\{A_\alpha\in P(X):\alpha\in\nabla\}$$

is said to be the indexed set.

(6) Let X be nonempty and $A\subset X$ **, t**he function which is defined by

$$\chi_A(x)=\begin{cases}1 & if\quad x\in A\\ 0 & if\quad x\notin A\end{cases}$$

And is called the characteristic function of A and it satisfies the conditions:

(1) $\chi_{A\cap B}(x)=\chi_A(x)\chi_B(x)$**,**

(2) $\chi_{A\cup B}(x)=\chi_A(x)+\chi_B(x)-\chi_A(x)\chi_B(x)$ and

(3) $\chi_{A-B}(x)=\chi_A(x)[1-\chi_B(x)]$

We shall prove (3) and left (1) and (2) to the reader. For

$$\chi_{A-B}(x)=1\Leftrightarrow x\in A-B\Leftrightarrow x\in A\wedge x\notin B$$

$$\Leftrightarrow\chi_A(x)=1\wedge\chi_B(x)=0\Leftrightarrow\chi_A(x)[1-\chi_B(x)]=1$$

Also

$$\chi_{A-B}(x) = 0 \Leftrightarrow x \notin A - B \Leftrightarrow x \notin A \vee x \in B$$
$$\Rightarrow \chi_A(x) = 0 \vee \chi_B(x) = 1 \Rightarrow \chi_A(x)[1-\chi_B(x)] = 0$$

1.3 Classification of sets.

In this article a number of an interesting properties and rules of the sets of real numbers will be given like the bounded and unbounded, the finite and infinite and the countable and uncountable sets which are important for the study of the topology. To define the bounded and unbounded sets we needs firstly to define the order relation which define as follows:

Definition 1.3.1 If X is a nonempty set i.e. $X \neq \varnothing$ then the relation "$\leq$" is called antisymetric if for each two points $x, y \in X$, $x \leq y$ and $y \leq x$ implies that $x = y$ i.e. $x \leq y \wedge y \leq x \Rightarrow x = y$ And we say that

(a) The relation $\leq$ is called a partially ordered relation if it is reflexive, antisymetric and transitive. In this case The ordered pair $(X, \leq)$ shortly The set X is a partially ordered set by the relation "$\leq$"

(b) The relation $\leq$ is called a linearly or totally order relation on X if it is a partially order relation and for each two points $x, y \in X$ either $x \leq y$ or $y \leq x$. In this case the set X is said to be totally or linearly ordered and is called a chain.

Example 1.3.1 The usual relation "$\leq$"on the set of the real numbers R is a totally order relation and so R is a chain by this relation.

Example 1.3.2 If X is a nonempty set then the inclusion "$\subset$" is a partially order relation on $P(X)$ and $P(X)$ is a partially ordered by "$\subset$".

Example 1.3.3 If X is a partially ordered set by the relation "$\leq$" and $A \subset X$ such that $A \neq \varnothing$ then A is partially ordered by the same relation "$\leq$" on X .

Example 1.3.4 If X is the set of the complex numbers and "$\leq$" is a relation on X such that for each two points $z_1, z_2 \in X$, $z_1 \leq z_2$ iff $|z_1| \leq |z_2|$ then this relation is a partially order relation on X since $\{|z| : z \in X\} \subset R$ and R is partially ordered by the relation "$\leq$".

Remark 1.3.1 *If X is a nonempty partially ordered set by the relation "$\leq$" then for each points $x, y \in X$ (1) if $x \leq y$ and $x \neq y$ we write $x < y$. (2) If $x \leq y$ we say that x precedes or weaker than y equivalently y follow or greater than x and we may write $y \geq x$.*

Definition 1.3.2 If X is a nonempty partially ordered set by the relation "$\leq$" then

(1) $a \in X$ is called the first element of X if $x \geq a$ for each point $x \in X$ and is called the minimal element if $x \leq a$ for any point $x \in X$ implies that $a = x$.

(2) $b \in X$ is called the last element of X if $x \leq b$ for each point $x \in X$ and is called the maximal element if $x \geq b$ for any point $x \in X$ implies that $b = x$. If $A \subset X$ such that $A \neq \varnothing$ then

(1) $a \in X$ is called a lower bound of A if $a \leq x$ for each $x \in A$.

(2) $b \in X$ is called an upper bound of A if $x \leq b$ for each $x \in A$.

The set A is said to be

(1) Bounded above if it has an upper bound.

(2) Bounded below if it has a lower bound.

(3) Bounded if it is bounded above and bounded below i.e. if it has an upper and lower bounds.

Example 1.3.5 considering the set N of the natural numbers, the set Z of the integers and $A = \{-2, 0, 3, 4\}$ as subsets of R which are partially ordered sets by the relation "$\leq$" then

(1) $1 \in N$ is the minimal element and a lower bound of N which has no maximal element and no upper bounds. So N is bounded below, unbound above and the set of their lower bounds is $(-\infty, 1]$.

(2) The set Z is unbounded it has no lower and no upper bounds.

(3) -2 is the minimal element and a lower bound for the set A and the set of all lower bounds of A is $(-\infty, -2]$. Also 4 is the maximal element and an upper bound of A and the set of all upper bounds of A is $[4, \infty)$.

Definition 1.3.3 Let X be a partially ordered set and $A \subset X$ be such that $A \neq \varnothing$ then

(1) The upper bound a of the set A is called the least upper bound of A denoted $\sup .A$ if it is the minimal element of the set of the upper bounds of A . That is $a = \sup .A$ iff a is an upper bound of A and $a \leq c$ for each upper bound c of A .

(2) The lower bound b of the set A is called the greatest lower bound of A denoted $\inf .A$ if it is the maximal element of the set of the lower bounds of A . That is $b = \inf .A$ iff b is a lower bound of A and $b \geq d$ for each lower bound d of A .

(3) The set X is said to be complete if there exist a $\sup .A$ for each bounded above subset A of X or there exist an $\inf .A$ for each bounded below subset A of X .

Example 1.3.6 $\inf .N = 1$ and $\sup .N$ does not exist.

Remark 1.3.2 *The real set R is a complete set, any bounded above subset of R has a suprimum and any bounded below subset of R has infimum.*

Theorem 1.3.1 *(Archimedes's axiom) If $a \in R$ such that $a > 0$ then there exists $n_o \in N$ such that $n_o a > 1$ i.e.*

$$a \in R : a > 0 \Rightarrow \exists n_o \in N : n_o a > 1$$

Proof: Suppose that $na \le 1$ for each $n \in N$ then $A = \{an : n \in N\}$ is bounded above so by Remark 1.3.1, there exists $L \in R$ such that $L = \sup.(A)$ then $an \le L$ for each $n \in N$. But

$$n \in N \Rightarrow (n+1) \in N$$

$$\Rightarrow a(n+1) \le L ; \forall n \in N \Rightarrow an \le L - a ; \forall n \in N$$

This means that $L - a$ is an upper bound of A which implies that $L - a \ge L$ which is impossible this impossibility because of the incorrect assumption that $na \le 1$ for each $n \in N$ and then there exists $n_o \in N$ such that $n_o a > 1$.

Remark 1.3.3 *There are two equivalent forms of Archimedes's axiom*

(a) $a, b \in R^+ \Rightarrow \exists n_o \in N : n_o a > b$.

(b) $x \in R^+ \Rightarrow \exists n_o \in N : n_o > x$.

The proof is easily and left to the reader.

Proposition 1.3.1 *If $A \subset R$ and $a = \sup.A$ then by the definition of $\sup.A$, any number less than a can not be an upper bound of A that is if $c < a$ then there exists $x \in A$ such that $c < x \le a$.*

Similarly if $b = \inf.A$ and $d \in R$ such that $d > b$ then there exists $x \in A$ such that $b \le x < d$.

Proposition 1.3.2 *If $a \in R$ and $A = \{x \in Q : x < a\}$ then $a = \sup.A$.*

Proof: By using Archimedes's axiom we gets

$$b < a \Rightarrow a - b > 0, 20 > 0 \Rightarrow \exists n \in N : n(a-b) > 20$$

$$\Rightarrow na > nb + 20$$

This means that na lies greater than 20 units on the right of nb that is there are 20 natural numbers between na and nb then

$$\exists k \in N : nb < k < na \Rightarrow b < \frac{k}{n} < a \Rightarrow b < q < a$$

where $\frac{k}{n} = q \in Q$. Clearly a is an upper bound of A , $q < a$ implies that $q \in A$ and $b < q$ implies that b is not an upper bound of A . Therefore $a = \sup .A$.

Proposition 1.3.3 $a,b \in R : a < b \Rightarrow \exists q \in Q : a < q < b$ *i.e. the set of the rational numbers is dense in* R *i.e between any two real numbers there is a rational number.*

Proof: Clearly $a,b \in Q : a < b \Rightarrow a < \frac{a+b}{2} < b$ and generally by using Proposition 1.3.2, we gets $A = \{x \in Q : x < b\} \Rightarrow b = \sup.(A)$ and by using Proposition 1.3.1, if $a < b$ there exists $q \in A$ such hat $a < q \le b$ but $q \in A$ and $b \notin A$ implies that $a < q < b$.

Proposition 1.3.4 *From Proposition 1.3.3, and Archimedes's axiom we finds*

$$a < b \Rightarrow \exists q \in Q : a < q < b \Rightarrow b - q > 0 \Rightarrow \frac{b-q}{\sqrt{2}} > 0 \Rightarrow$$

$$\Rightarrow \exists n_o \in N : \frac{b-q}{\sqrt{2}} . > \frac{1}{n_o} \Rightarrow b - q > \frac{\sqrt{2}}{n_o} \Rightarrow b > q + \frac{\sqrt{2}}{n_o} \Rightarrow$$

$$\Rightarrow a < q < q + \frac{\sqrt{2}}{n_o} < b \Rightarrow a < p < b : p = q + \frac{\sqrt{2}}{n_o} \in Q^c$$

The two Propositions 1,3,2 and Proposition 1.3.3, say that between any two real numbers there exists a real number. This property is also common well known to be Archimedes's axiom.

Theorem 1.3.2 *If* $\{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ *is an indexed family of open intervals such that* $p \in \bigcap\{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ *then* $A = \bigcup\{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ *is an open interval containing the point* p *.*

Proof: Clearly $p \in A$ and A can be in one of the following cases:

(1) Bounded below and above,
(2) Bounded above and unbounded below,
(3) Unbounded above and bounded below and
(4) Unbounded above and unbounded below.

We prove that A is an open interval in (1) and (2) and we left the cases (3) and (4) to the reader.

Case (1): Since A is bounded above then there exists $b \in R$ such that $b = \sup .A$ and we finds $b \in A \Rightarrow \exists \alpha \in \Delta : b \in (a_\alpha, b_\alpha) \Rightarrow a_\alpha < b < b_\alpha$ and from Proposition 1.3.3, there exists $x \in Q$ such that $a_\alpha < b < x < b_\alpha$ and so $x \in A, b < x$ which contradicts that b is an upper bound of A which means that $b \notin A$ from which $p < b$. If $x \in R$ such that $p < x < b$ then $x \geq b_{\alpha}; \forall \alpha \in \Delta \Rightarrow x > y; \forall y \in A$ which implies that x is an upper bound of A implies that $x \geq b$ because $b = \sup A$ this contradicts the assumption $x < b$ which implies that there exists $\alpha \in \Delta$ such that $p < x < b_{\alpha} \leq b$. Therefore $x \in R, p < x < b \Rightarrow x \in A$. Since A is bounded below then there exists $a \in R$ such that $a = \inf A$ and in a similar way $a \notin A$, $a < p$ and $x \in R, a < x < p \Rightarrow x \in A$. Hence $A = (a,b)$

Case (2): Since A is bounded above then similar to that in case (1) where $b = \sup A$ and $x \in R, p < x < b \Rightarrow x \in A$. If A is unbounded below then $x \in R, x < p$ implies that x is not a lower bound of A implies that there exists $y \in A$ such that $y < x$. But

$$y \in A \Rightarrow \exists \alpha \in \Delta : a_{\alpha} < y < b_{\alpha} \Rightarrow a_{\alpha} < y < x < b_{\alpha}$$
$$\Rightarrow x \in (a_{\alpha}, b_{\alpha}) \Rightarrow x \in A$$

Therefore, $A = (-\infty, b)$

Definition 1.3.4 The two sets A and B are equivalent if there is a correspondence function between them i.e. an injective and surjective function $f : A \to B$.

Example 1.3.7 The set of the positive integers N and the set $A = \{2n-1 : n \in N\}$ are equivalent, the function $f : N \to A$ given by $f(n) = 2n-1; \forall n \in N$ is 1 – 1 and onto i.e. is a correspondence. Also the sets $A = \{1,2,3\}$ and $B = \{a,b,c\}$ are equivalent where $f : A \to B$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$ is a correspondence.

Definition 1.3.5 A set A is finite if $A = \varnothing$ or if it is equivalent to the set $B = \{1,2,3,\ldots\ldots,n\}$ where $n \in N$ equivalently if the cardinal number of A is either $|A| = 0$ or $|A| = n \in N$ The cardinal number of a set is the number of its elements if it is finite. The set A is an infinite set if it is not finite i.e. if there is no any correspondence between the set A and the set B equivalently if $|A| > n; \forall n \in N$.

Remark 1.3.4 *since R is complete then any finite subset A of R can be written in the form $A = \{a_1, a_2, a_3, \ldots, a_n\}$ such that $a_1 < a_2 < \ldots < a_n$.*

Remark 1.3.5 *Another definition of the infinite set as follows: A set A is infinite if there exists a correspondence between A and a proper subset of A . Equivalently A is infinite if there is an injective function $f : A \to f(A)$ such that $f(A)$ is a proper subset of A i.e. $f(A) \neq A$. According to this definition and by Example 1.3.7, N is an infinite set.*

Definition 1.3.6 A set A is countable if it is finite or there exists a correspondence between A and the set of the natural numbers N in this case A is infinite countable otherwise it is uncountable.

Remark 1.3.6 *In fact a set A is countable if there exists a surjective function $f : N \to A$ by this definition A can be finite or infinite and the cardinal number of A less than or equal to the cardinal number of N. For example the set $A = \{a, b, c\}$ is countable because of the function $f : N \to A$ which is given by $f(1) = a$, $f(2) = b$ and $f(n) = c$ for each $n \in N - \{1, 2\}$.*

Example 1.3.8 Clearly (i) N is countable and the set A given in Example 1.3.7, is also.

(ii) The set Z of the integers is countable, since the function $f : Z \to N$ which is given by $f(n) = \begin{cases} 2n; & n > 0 \\ 1 - 2n; & n < 0 \end{cases}$ is bijective.

(iii) The set $N^* = N \cup \{0\} = \{0, 1, 2, ...\}$ is countable because of the correspondence $f : N^* \to N$ which is given by $f(n) = n + 1$ for each $n \in N^*$. Also the set $N^* \times N^*$ is countable because of the correspondence $f : N^* \times N^* \to N$ which is given by

$$f(n, m) = 2^n (2m + 1); \forall (n, m) \in N^* \times N^*.$$

Theorem 1.3.3 *A subset of a countable set is countable. Equivalently any set contains uncountable set is uncountable*

Proof: Let X be an infinite countable set and A be a subset of X, if $A = \emptyset$ then it is countable. Suppose that $A \neq \emptyset$ and $n_1 \in N$ be the smallest positive integer such that $a_1 \in A$, if $A - \{a_1\} \neq \emptyset$ let $n_2 \in N$ be the smallest positive integer such that $a_2 \in A - \{a_1\}$. Continue these processes, the set A may be of the form $A = \{a_1, a_2, ...\}$ and if the set $B = \{n_1, n_2, ...\}$ is bounded above then A is finite and if B is unbounded then A is infinite in both cases A is countable.

Corollary 1.3.1 *From Example 1.3.8(iii) and Theorem 1.3.3, the set $N \times N$ is a countable set since $N \times N \subset N^* \times N^*$.*

Remark 1.3.7 *A subset of a finite set is finite. Equivalently any set contains an infinite set is infinite".*

Theorem 1.3.4 *If A is a countable set and $f : A \to B$ is a correspondence i.e. A and B are equivalent then B is countable.*

Proof: If A is a countable set then there exist a correspondence $g:N\to A$ and $f\circ g:N\to B$ is correspondence which implies that B is countable.

Theorem 1.3.5 *An infinite countable union of infinite countable sets is countable.*

Proof: Let $\{A_n : n\in N\}$ be a family of infinite countable sets then for each $n\in N$, $A_n=\{a_{n_1},a_{n_2},...\}$. Then $f:N\times N\to \bigcup_{n\in N} A_n$ given by $f(i,j)=a_{ij}\in A_i;\forall(i,j)\in N\times N$ is a correspondence. Hence $\bigcup_{n\in N} A_n$ is a countable set.

Corollary 1.3.2 *A finite union of finite or infinite countable sets is a finite or infinite countable set.*

Corollary 1.3.3 *The set of the rational numbers is a countable set.*

Proof: The functions $f:N\times N\to Q^+$ which is given by $f(n,m)=\frac{n}{m};\ \forall(n,m)\in N\times N$ and $g:N\times N\to Q^-$ which is given by $g(n,m)=-\frac{n}{m};\ \forall(n,m)\in N\times N$ are injective and surjective functions and so Q^+,Q^- are countable sets. Hence by Theorem 1.3.5, $Q=Q^+\cup\{0\}\cup Q^-$ is a countable set.

Theorem 1.3.6 *If A and B are two countable sets then $A\times B$ is countable.*

Proof: Let $A=\{a_1,a_2,...\}$ and $B=\{b_1,b_2,...\}$ then

$$\begin{aligned}A\times B&=\{(a_1,b_1),(a_1,b_2),(a_1,b_3),...,\\&(a_2,b_1),(a_2,b_2),(a_2,b_3),...,(a_3,b_1),(a_3,b_2),(a_3,b_3),...,...,...\}\\&=\bigcup_{n\in N}(A\times B)_n:(A\times B)_n=\{(a_n,b_1),(a_n,b_2),...,...\}\}\end{aligned}$$

which a countable union of countable sets, $\{(A\times B)_n : n\in N\}$ and it is a countable set.

Theorem 1.3.7 *Any infinite set contains an infinite countable subset.*

Proof: Let A be an infinite set then $A\neq\varnothing$ and so there is an element $a_1\in A$ such that $A-a_1\neq\varnothing$ since otherwise $A=\{a_1\}$ is finite. Then there is an element $a_2\in A-\{a_1\}$ such that $A-\{a_1,a_2\}=\varnothing$ since otherwise $A=\{a_1,a_2\}$ is finite. One can continue these processes to choose elements from the set A such that for each positive integer n, $A-\{a_1,a_2,...,a_n\}\neq\varnothing$ since otherwise $A=\{a_1,a_2,...,a_n\}$ is a finite set which means that there exists $a_{n+1} \in A - \{a_1, a_2, ..., a_n\}$ and so A contains a countable set $\{a_1, a_2, a_3, ...\}$.

Theorem 1.3.8 *Any family* $\{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ *of pair wise disjoints open intervals is countable.*

Proof: From Proposition 1.3.3, $\{q \in Q : q \in (a_\alpha, b_\alpha)\} \neq \emptyset$ and so for each $\alpha \in \Delta$ by selecting a rational number $q_\alpha \in (a_\alpha, b_\alpha)$ from which we obtains the set $A = \{q_\alpha : \alpha \in \Delta\}$ which is a set of rational numbers i.e. $A \subset Q$ and so it is countable then there exists a correspondence $f : N \to A$. Now consider the function $g : A \to \{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ such that $q \in A \Rightarrow \exists \alpha \in \Delta : q = q_\alpha \Rightarrow g(q) = (a_\alpha, b_\alpha)$ where $q_\alpha \in A$ is the unique which is selected from the interval (a_α, b_α), this is a correspondence function since

$$q_1, q_2 \in A : q_1 \neq q_2 \Rightarrow \exists \alpha, \beta \in \Delta : \alpha \neq \beta, q_1 = q_\alpha, q_2 = q_\beta$$

$$\Rightarrow g(q_1) = g(q_\alpha) = (a_\alpha, b_\alpha), g(q_2) = g(q_\beta) = (a_\beta, b_\beta)$$

and $\alpha \neq \beta \Rightarrow (a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \emptyset \Rightarrow g(q_1) \neq g(q_2)$. This means that g is injective.

Secondly $(a_\alpha, b_\alpha) \neq \emptyset$ for each $\alpha \in \Delta$, implies that

$\exists q \in A : q = q_\alpha \in (a_\alpha, b_\alpha), g(q) = (a_\alpha, b_\alpha)$ This means that g is a surjective function. Therefore we gets the correspondence $gof : N \to \{(a_\alpha, b_\alpha) : \alpha \in \Delta\}$ see the figure this completes the proof.

Theorem 1.3.9 *The intervals* $(0,1), [0,1), (0,1], (a,b), [a,b), (a,b], [a,b]$ *are equivalent and equivalent to the interval* $[0,1]$ *that is they have the same cardinal number.*

Proof: Firstly $f : [0,1] \to (0,1)$ such that $f(0) = \frac{1}{2}$, $f(x) = x$ when $x \notin \{0, \frac{1}{n} : n \in N\}$ and $f(x) = \frac{1}{n+2}$ when $x \in \{\frac{1}{n} : n \in N\}$ this functions injective and surjective.

Secondly $f : [0,1] \to [0,1)$ given $f(x) = x$ when $x \notin \{\frac{1}{n} : n \in N\}$ and $f(x) = \frac{1}{n+1}$ when $x \in \{\frac{1}{n} : n \in N\}$ this function is injective and surjective.

Thirdly $f:[0,1]\to[a,b]:a<b$ given by the rule $f(x)=a+(b-a)x;\forall x\in[0,1]$ is a correspondence since

(1) $x_1,x_2\in[0,1]\Rightarrow f(x_1)=f(x_2)\Rightarrow x_1=x_2$ Which implies that f is injective.
(2) $y\in[a,b]\Rightarrow a<y<b\Rightarrow x=\dfrac{y-a}{b-a}\in[0,1]:f(x)=y$ Which implies that f is surjective.

Theorem 1.3.10 *The interval $A=(0,1)$ is an uncountable set.*

Proof: Suppose that A is a countable set, it can be write as $A=\{x_1,x_2,x_3,\ldots\}$ where $x_n=o.x_{n1}x_{n2}x_{n3}\ldots x_{nn}\ldots$ for each $n\in N$ and $x_{nm}\in\{0,1,2,3,4,5,6,7,8,9\};\forall n,m\in N$. Now we construct the number $y=0.y_1y_2y_3\ldots y_n\ldots$ such that $y_n=3$ when $x_{nn}=7$ and $y_n=7$ when $x_{nn}\neq 7$ for each $n\in N$, clearly $y\in(0,1)$, $y_n\notin\{0,9\}$ for each $n\in N$ and $y_n=x_n;\forall n\in N$ that is $y\notin A$ which contradicts that $A=(0,1)$ this contradiction because of the incorrect that A is countable and hence A is uncountable.

Corollary 1.3.4 *According to Theorem 1.3.9, the set R of the real numbers is an uncountable set since $(0,1)\subset R$.*

Proposition 1.3.5 *The set R and the interval $(-1,1)$ are equivalent because of the correspondence $f:(-1,1)\to R$ which is given by*

$$f(x)=\frac{x}{1-|x|};\forall x\in(-1,1).$$

Corollary 1.3.5 *By Theorem 1.3.10, the intervals which are given in Theorem 1.3.9, are equivalent to the set of the real numbers R.*

Definition 1.3.7 The cardinal number of the set N denoted by $\aleph_o$ (aleph null) , the cardinal number of the set R denoted by c (Continuums) and it should be remarked that

$$1<2<3<\ldots<\aleph_o<c$$

Remark 1.3.8 *We know that if the cardinal number of a set X is $|X|=n$ then the cardinal number of $P(X)$ is $|P(X)|=2^n$. Generally the cardinal number of $P(X)$ is $|P(X)|=2^{|X|}$ where $|X|$ is the cardinal number of X and the family $P(X)$ is denoted by 2^X.*

Remark 1.3.9 *If* X *is a nonempty set then the family of all characteristic functions* X *is denoted by* $C(X)$ *i.e.* $C(X)=\{f : f : X \to \{0,1\}\}$ *that is there exists* $A \subset X$ *such that* $f \in C(X)$ *and* $f = \chi_A$.

Theorem 1.3.11 *If* X *is a nonempty set then* 2^X *and* $C(X)$ *are equivalent i.e. they have the same cardinal number.*

Proof: It is an easily, the function $f : 2^X \to C(X)$ which is given by $f(A)=\chi_A ; \forall A \in 2^X \ (A \subset X)$ is an injective and surjective function.

Theorem 1.3.12 *The family* 2^N *where* $2^{\aleph_o} = c$.

Proof: From Corollary 1.3.3, the sets Q and N are equivalent , then consider the function $f : R \to 2^Q$ given by $f(x)=\{y \in Q : y < x\}$ for each $x \in R$ then we finds $x_1, x_2 \in R : x_1 \neq x_2 \Rightarrow (i) x_1 < x_2 \vee (ii) x_2 < x_1$ and

$$x_1 < x_2 \Rightarrow \exists q \in Q : x_1 < q < x_2 \Rightarrow q \in f(x_2), q \notin f(x_1)$$
$$\Rightarrow f(x_1) \neq f(x_2)$$

Similarly one can show that $x_2 < x_1 \Rightarrow f(x_1) \neq f(x_2)$. Therefore f is injective which implies that $c \leq 2^{|Q|} = 2^{\aleph_o} \ ----(1)$

Secondly from Theorem 1.3.11, $C(N)=\{f : f : N \to \{0,1\}\}$ and so $|C(N)|= 2^{\aleph_o}$. Consider the function $g : C(N) \to [0,1]$ which is given by $g(f)=0.f(1)f(2)f(3)... ; \forall f \in C(N)$ from which we finds that

$$f_1, f_2 \in C(N) : f_1 \neq f_2 \Rightarrow \exists A, B \in 2^N : A \neq B,$$
$$f_1 = \chi_A \wedge f_2 = \chi_B$$

But $A \neq B \Rightarrow \chi_A \neq \chi_B \Rightarrow f_1 \neq f_2 \Rightarrow g(f_1) \neq g(f_2)$ which means that f is injective which implies that $2^{\aleph_o} \leq c \ --(2)$.

From (1) and (2), $2^{\aleph_o} = c$ this completes the proof.

Definition 1.3.8 (Convergence of sequences) The real sequence $<x_n>$ convergent to a point $x \in R$ or x is a limit point of the sequence, written in the form $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ if for each positive real number ε whatever may be small there is a natural number $n_o \in N$ such that $\left| x_n - x \right| < \varepsilon$ whenever $n \geq n_o$ i.e.

$$n \geq n_o \Rightarrow \left| x_n - x \right| < \varepsilon \Rightarrow x - \varepsilon < x_n < x + \varepsilon$$
$$\Rightarrow x_n \in (x - \varepsilon, x + \varepsilon)$$

Example 1.3.9 The sequence $<\frac{1}{n}>$ convergent to zero i.e. $\frac{1}{n}\to 0$ by using Archimedes's property

$$\varepsilon > 0 \Rightarrow \exists n_o \in N : \varepsilon > \frac{1}{n_o}$$

So

$$n \geq n_o \Rightarrow \frac{1}{n} \leq \frac{1}{n_o} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

which means that $\frac{1}{n}\to 0$.

Remark 1.3.10 *If the real sequence* $<x_n>$ *convergent to a point* x *i.e.* $x_n \to x$ *and* $a,b \in R$ *such that* $x \in (a,b)$ *then for each* $\varepsilon > 0$ *there exists* $n_o \in N$ *such that* $n \geq n_o \Rightarrow x - \varepsilon < x_n < x + \varepsilon$. *If* $\varepsilon = \min.\{x-a, b-x\}$ *then*

$$n \geq n_o \Rightarrow a \leq x - \varepsilon < x_n < x + \varepsilon \leq b \Rightarrow x_n \in (a,b)$$

Accordingly the Definition 1.2.7, of the convergence sequence $<x_n>$ *to* x *can be reformed as follows: For each numbers* $a,b \in R$ *such that* $x \in (a,b)$ *there exists* $n_o \in N$ *such that* $x_n \in (a,b)$ *whenever* $n \geq n_o$ *i.e.*

$$n \geq n_o \Rightarrow x_n \in (a,b)$$

Then we can say that the sequence $<x_n>$ *convergent to* x *if each open interval* (a,b) *containing* x *contains all except a finite number of elements of the sequence* $<x_n>$ *depended on the number* ε*, whenever* ε *decreases the number of the outside elements of the interval* (a,b) *increases.*

1.4 The topology on the real line.

In this section the relation between the points on the real line and the subsets of the real set R will be discuss then a classification of the points into limit, interior and exterior points of the real sets according to the relation between the points and sets. Also a classification of the real sets into open and closed and studies some of their properties. This leads us in the second chapter to define the general topology.

Definition 1.4.1 If A is a subset of R i.e. $A \subset R$ and $x \in A$ then x is called an interior point of A if there are two real numbers $a,b \in R$ such that $x \in (a,b) \subset A$. The set of all interior points of a set A well be denoted either by A^o or by $\operatorname{int}(A)$, through this book we shall used the symbol A^o i.e.

$$x \in A^o \Leftrightarrow \exists a,b \in R : x \in (a,b) \subset A .$$

Remark 1.4.1 *Directly from the definition,* $A^o \subset A$ *for each* $A \subset X$ *.*

Definition 1.4.2 A subset A of R is called an open set if each point $x \in A$ is an interior point of A i.e. $x \in A \Rightarrow x \in A^o$.

According to Definition 1.4.1, a subset A of R is an open set if for each point $x \in A$ there are $a,b \in R$ such that $x \in (a,b) \subset A$ i.e. if

$$x \in A \Rightarrow \exists a,b \in R : x \in (a,b) \subset A .$$

Theorem 1.4.1 *A subset* A *of* R *is an open set iff* $A = A^o$ *.*

The proof can be obtained directly from Remark 1.4.1, and Definition 1.4.1.

Example 1.4.1 The open intervals subsets of the set R of the real numbers (i) (a,b), (ii) (a,∞) and (iii) $(-\infty,a)$ are open sets where

$$x \in (a,b) \subset (a,b) \Rightarrow x \in (a,b)^o ,$$

$x \in (a,x+1) \subset (a,\infty) \Rightarrow x \in (a,\infty)^o$ and

$$x \in (x-1,a) \subset (-\infty,a) \Rightarrow x \in (-\infty,a)^o .$$

Then x is an interior point of the interval in all cases.

Example 1.4.2 If $a,b \in R$ then the point a is not interior to the interval $[a,b]$ i.e. $a \notin [a,b]^o$ since if $c,d \in R$ then

$$a \in (c,d) \Rightarrow c < a < d \Rightarrow (c,a) \subset R - [a,b]$$

i.e. $(c,d) \not\subset [a,b]$ and this means that each open interval containing the point a contains other points outside $[a,b]$. Then $[a,b]$ is not open.

Example 1.4.3 $N^o = \varnothing$, $Z^o = \varnothing$, $Q^o = \varnothing$ and $Q^{co} = \varnothing$. We shall prove the first and the forth and left the second and the third to the reader.

(1) $x \in N : x \in (a,b) \subset R \Rightarrow \exists p \in Q^c : p \in (a,b) \Rightarrow (a,b) \not\subset N$ **.**

Then x can not be an interior point of N for each point $x \in N$ i.e. $N^o = \varnothing$.

(2) $x \in Q^c : x \in (a,b) \subset R \Rightarrow \exists q \in Q : q \in (a,b) - \{x\} \Rightarrow (a,b) \not\subset Q^c$. Then x can not be an interior point of Q^c for each point $x \in Q^c$ i.e. $Q^{co} = \varnothing$.

Therefore according to Theorem 1.4.1, N, Z, Q, Q^c are not open sets.

Theorem 1.4.2 *The family of the open subsets of the set* R *of the real numbers satisfies the following axioms:*

(1) R and $\varnothing$ are open sets,
(2) The intersection of two open sets is an open set and
(3) Arbitrary unions i.e. the union of any number of open sets is open.

Proof: (1) Clearly for each point $x \in R$,

$$a,b \in R : x \in (a,b) \Rightarrow x \in (a,b) \subset R \Rightarrow x \in R^{o}$$

$$\Rightarrow R \subset R^{o} \Rightarrow R^{o} = R$$

And there is no points in $\varnothing$ which is not interior of $\varnothing$ and so $\varnothing^{o} = \varnothing$.

(2) Let $G_1, G_2 \subset R$ be two open sets then

$$x \in G_1 \cap G_2 \Rightarrow x \in G_1, x \in G_2 \Rightarrow \exists a,b,c,d : x \in (a,b) \subset G_1,$$
$$x \in (c,d) \subset G_2 \Rightarrow x \in (a,b) \cap (c,d) \subset G_1 \cap G_2 \Rightarrow$$
$$\Rightarrow x \in (e,f) \subset G_1 \cap G_2 \Rightarrow x \in (G_1 \cap G_2)^{o}$$

where $x \in (a,b) \cap (c,d) = (e,f)$. Hence

$$(G_1 \cap G_2)^{o} = (G_1 \cap G_2).$$

(3) Let $\{G_\alpha : \alpha \in \Delta\} \subset P(R)$ be a family of open sets then $x \in (\bigcup\limits_{\alpha \in \Delta} G_\alpha) \Rightarrow \exists \alpha \in \Delta : x \in G_\alpha$ and since G_α is open then

$$x \in G_\alpha \Rightarrow \exists a,b \in R : x \in (a,b) \subset G_\alpha$$

$$\Rightarrow x \in (a,b) \subset (\bigcup_{\alpha \in \Delta} G_\alpha) \Rightarrow x \in (\bigcup_{\alpha \in \Delta} G_\alpha)^{o}$$

$$\Rightarrow (\bigcup_{\alpha \in \Delta} G_\alpha)^{o} = \bigcup_{\alpha \in \Delta} G_\alpha$$

This completes the proof of the theorem.

Remark 1.4.2 *By using the mathematical induction one rewrite (2) of Theorem 1.4.2, to be The intersection of any finite number of open sets is open that is if $\{U_1, U_2, \ldots, U_n : n \in N\}$ is a family of open sets then $\bigcap\limits_{i=1}^{n} U_i$ is open. The following example explained that the intersection of infinite sets may be not open, if $a \in R$ then $\bigcap\limits_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) = \{a\}$ which is the intersection of infinite number of open sets (open intervals $(a - \frac{1}{n}, a + \frac{1}{n})$) is not open to prove this firstly*

$$\{a\} \subset \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) ----(1)$$

Secondly $x \in R : x \neq a \Rightarrow x < a \vee x > a$ and we finds

$$x < a \Rightarrow a - x > o \Rightarrow \exists n_1 \in N : x < a - \frac{1}{n_1} \Rightarrow x \notin (a - \frac{1}{n_1}, a + \frac{1}{n_1})$$

$$\Rightarrow x \notin \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) -----(2)$$

$$x > a \Rightarrow x - a > o \Rightarrow \exists n_2 \in N : x > a + \frac{1}{n_2}$$

$$\Rightarrow x \notin (a - \frac{1}{n_2}, a + \frac{1}{n_2})$$

$$\Rightarrow x \notin \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) -----(3)$$

(1) and (3) $\Rightarrow x \notin \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}); \forall x \in R - \{a\} ----(4)$

From (1) and (4), $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, a + \frac{1}{n}) = \{a\}$ *which is not open.*

Theorem 1.4.3 *Let* $G \subset R$ *be a nonempty open subset of* R *and* S *be a relation on* G *such that*

$$(x, y) \in S \Leftrightarrow \exists a, b \in R : x, y \in (a, b) \subset G$$

Then

(i) S *is an equivalence relation on* G *,*

(ii) $[x] = \bigcup \{(a, b) : x \in (a, b) \subset G\}$ *is the equivalence class of* x *for each point* $x \in G$ *where* $[x] = \{y \in G : (x, y) \in S\}$*,*

(iii) $\{[x] : x \in G\}$ *is a family of pair wise disjoint open intervals and*

(iv) $G = \bigcup \{[x] : x \in G\}$*.*

Proof: (i)

(1) Since $G \subset R$ is an open set then

$$\forall x \in G; \exists a, b \in R : x \in (a, b) \subset G \Rightarrow (x, x) \in S$$

which implies that S is reflexive.

(2) $(x, y) \in S \Rightarrow \exists a, b \in R : x, y \in (a, b) \subset G \Rightarrow (y, x) \in S$

which implies that S is symmetric and

(3) $(x, y), (y, z) \in S \Rightarrow \exists a_1, a_2, b_1, b_2 \in R : x, y \in (a_1, b_1) \subset G \wedge y,$ $z \in (a_2, b_2) \subset G \Rightarrow x, z \in (a_1, b_1) \bigcup (a_2, b_2) = (a, b) \subset G$

$y \in (a_1, b_1) \bigcap (a_2, b_2) \Rightarrow (a_1, b_1) \bigcup (a_2, b_2) = (a, b)$ is an open interval which implies that S is transitive.

Therefore from (1), (2) and (3), S is an equivalence relation.

(ii) $y \in [x] \Rightarrow (x, y) \in S \Rightarrow \exists a, b \in R : x, y \in (a, b) \subset G \Rightarrow$

$$y \in \bigcup\{(a,b): x \in (a,b) \subset G\} \Rightarrow$$
$$[x] \subset \bigcup\{(a,b): x \in (a,b) \subset G\} ----(1)$$

Also $$y \in \bigcup\{(a,b): x \in (a,b) \subset G\} \Rightarrow$$
$$\Rightarrow \exists a,b \in R : x, y \in (a,b) \subset G \Rightarrow (x,y) \in S \Rightarrow y \in [x] \Rightarrow$$
$$\bigcup\{(a,b): x \in (a,b) \subset G\} \subset [x] ----(2)$$

(1) and (2) implies that

$$[x] = \bigcup\{(a,b): x \in (a,b) \subset G\}.$$

(iii) Clearly from (ii) that x is a common point of all intervals the union of which have the union $[x]$. So, $\{[x]: x \in G\}$ is a family of pair wise disjoint open intervals .

(iv) Clearly

$$x \in G \Rightarrow x \in [x] \Rightarrow x \in \bigcup\{[x]: x \in G\} \quad (I)$$
$$t \in \bigcup\{[x]: x \in G\} \Rightarrow \exists x \in G : t \in [x] \Rightarrow \exists a,b \in R :$$
$$t \in (a,b) \subset G \Rightarrow t \in G \quad (II)$$

From (I) and (II), $G = \bigcup\{[x]: x \in G\}$.

Corollary 1.4.1 *As a direct consequence of Theorem 1.4.3 and Theorem 1.3.8, a nonempty subset G of the set R is open iff it can be written as a countable union of pair wise disjoint open intervals.*

Definition 1.4.3 If $A \subset R$ then $x \in R$ is said to be a limit point or accumulation point of A if each open interval (a,b) containing x contains at least another point of A different from the point x i.e.

$$a,b \in R : x \in (a,b) \Rightarrow (a,b) - \{x\} \bigcap A \neq \varnothing.$$

This definition can be rewrite equivalently in terms of the open sets as follows: $x \in R$ is a limit point of a set $A \subset R$ iff each open set $G \subset R$ such that $x \in G$, contains at least another point of A i.e. $\left(G - \{x\}\right) \bigcap A \neq \varnothing$. The set of all limit points of a set $A \subset R$ denoted by A' and is called the derived set of A and so

$$x \in A' \Leftrightarrow (a,b) - \{x\} \bigcap A \neq \varnothing; \forall a,b \in R : x \in (a,b)$$

Remark 1.4.3 *An equivalent definition of the limit point given in the form*

$$x \notin A' \Leftrightarrow \exists a,b \in R : x \in \left((a,b) \wedge (a,b) - \{x\}\right) \bigcap A = \varnothing$$

equivalently $(a,b) \bigcap A = \{x\}$ or $(a,b) \bigcap A = \varnothing$. In this remark we can replaced the open interval (a,b) by the open set $G \subset R$ such that $x \in G$ and the remark still valid.

Example 1.4.4 $[a,b]' = (a,b)' = [a,b]; \forall a,b \in R$ since for each $x \in (a,b)$ and each $c,d \in R$ such that $x \in (c,d)$, $\left((c,d) - \{x\}\right) \bigcap (a,b) \neq \varnothing$ which implies that $(a,b) \subset (a,b)'$. Also

$c,d \in R : a \in (c,d)$ implies that $(c,d)\bigcap(a,b)=[a,d)$ if $d \le b$ and $(a,b)\bigcap(c,d)=[a,b]$ if $d \ge b$ which implies that $a \in (a,b)'$ and similarly $b \in (a,b)'$. Hence $(a,b)' = [a,b]$.

Example 1.4.5 $Q' = R$ and $Q^{C\,'} = R$, we shall prove one of them and left the other to the reader. For let $x \in R$ then

$$x \in (a,b) \subset R \Rightarrow \exists q \in Q : a < q < x < b \Rightarrow$$

$$\Rightarrow \big((a,b) - \{x\} \big) \bigcap Q \ne \varnothing \Rightarrow x \in Q' \Rightarrow R \subset Q' \Rightarrow Q' = R$$

We used Archimedes's axiom. In a similar way one can proved $Q^{C\,'} = R$ **.**

Definition 1.4.4 A subset A of R is said to be a closed set if it contains all of its limit i.e. if $A' \subset A$.

Theorem 1.4.4 *A subset A of R is a closed set iff $A^C = R - A$ is open*.

Proof: Suppose firstly that A is a closed set i.e. $A' \subset A$ then

$$x \in A^C \Rightarrow x \notin A \Rightarrow x \notin A'$$

$$\Rightarrow \exists a,b : x \in (a,b),\ (a,b) - \{x\} \bigcap A = \varnothing$$

But $x \notin A$ implies that $(a,b)\bigcap A = \varnothing$ which implies that

$$x \in (a,b) \subset A^C \Rightarrow x \in A^{CO} \Rightarrow A^C \subset A^{CO} \Rightarrow A^{CO} = A^C$$

which implies that A^C is an open set.

Conversely suppose that A^C is open i.e. $x \in A^C \Rightarrow x \in A^{CO}$ and so

$$x \notin A \Rightarrow x \in A^C \Rightarrow x \in A^{CO} \Rightarrow \exists\, a,b \in R : x \in (a,b) \subset A^C$$

$$\Rightarrow (a,b) \bigcap A = \varnothing \Rightarrow x \notin A'$$

From which $x \notin A \Rightarrow x \notin A'$ equivalently $x \in A' \Rightarrow x \in A$ equivalently $A' \subset A$ which implies that A is a closed set.

Theorem 1.4.5 *(The properties of the closed sets) The closed subsets of the real set R satisfy the following properties*

(1) R and $\varnothing$ are closed sets,

(2) The union of two and so the union of any finite number of closed sets is a closed set and

(3) Arbitrary intersection i.e. the intersection of any number of closed sets is a closed set.

Proof: It is easily and it left to the reader.

Theorem 1.4.6 *The singleton subsets of R i.e. $\{\{x\} : x \in R\}$ is a family of closed sets.*

Proof: Let $x \in R$ be any point. Then $(-\infty, x)$, (x, ∞) are two open sets and so $\{x\}^c = R - \{x\} = (-\infty, x) \bigcup (x, \infty)$ is an open set which implies by Theorem 1.4.4, that $\{x\} = (\{x\}^c)^c$ is a closed set.

Corollary 1.4.2 *Any finite subset A of R is a closed set.*

Proof: If A is a finite subset of R then it can be written in the form $A = \{x_1, x_2, \ldots, x_n\}$ where $n \in N$ and we can write $A = \bigcup_{i=1}^{n} \{x_i\}$ which is a union of finite number of closed sets then by Theorem 1.4.5 and Theorem 1.4.6, A is a closed set.

Theorem 1.4.7 *Let $A \subset R$, $G \subset R$ be an open set and $p \in G$. Then $p \in A'$ implies that $G \bigcap A$ is an infinite set.*

Proof: Suppose that $G \bigcap A$ is finite then $F = (G \bigcap A) - \{p\}$ is a finite set and so by Corollary 1.4.2, F is a closed set. Then by Theorem 1.4.2, $U = G \bigcap F^c$ is an open set and $p \in U$ from which

$$U = G \cap [(G \cap A) - \{p\}]^c = G \cap [(G \cap A)^c \cup \{p\}]$$

$$\Rightarrow U \cap A = (G \cap A) \cap [(G \cap A)^c \cup \{p\}] = (G \cap A) \cap \{p\}$$

$$\Rightarrow U - \{p\} \cap A = \varnothing \Rightarrow p \notin A'$$

Therefore $G \bigcap A$ is finite implies that $p \notin A'$ and its contra- positive is $p \in A'$ implies that $G \bigcap A$ is infinite.

Theorem 1.4.8 *For each subset A of R, A' is a closed set.*

Proof: According to the definition of the closed set we need to prove that $A'' \subset A'$, for let $x \in A''$, $G \subset R$ be an open set such that $x \in G$. Then $\left(G - \{x\}\right) \bigcap A' \neq \varnothing$ but $\{x\}$ is a closed set and Theorem 1.4.6, implies that $U = G - \{x\} = G \bigcap \{x\}^c$ is an open set and so

$$U \bigcap A' = \left(G - \{x\}\right) \bigcap A' \neq \varnothing \Rightarrow \exists y \in U \bigcap A' \Rightarrow y \in U \wedge y \in A'$$

and from Theorem 1.4.7, $(G - \{x\}) \bigcap A = U \bigcap A$ is an infinite set which implies that $(G - \{x\}) \bigcap A \neq \varnothing$ which implies that $x \in A'$. Therefore,

$$x \in A'' \Rightarrow x \in A' \Rightarrow A'' \subset A'$$

which implies that A' is a closed set.

Theorem 1.4.9 *Let A be a closed and bounded above subset of R and $p = \sup.A$. Then $p \in A$.*

Proof: Let A be a closed and bounded above subset of R, and $a, b \in R$ such that $p \in (a, b)$. Then $a < p < b$ and by Theorem 1.3.1, and because $p = \sup.A$, $a < p$ implies that there exists $x \in A$ such that

$a < x \leq p$ and either $x = p$ which implies that $p \in A$ or $x \neq p$ which implies that $a < x < p$ in this case

$$a < x < p < b \Rightarrow x \in (a,b) - \{p\}$$

$$\Rightarrow \big((a,b) - \{p\}\big) \cap A \neq \varnothing \Rightarrow p \in A' \Rightarrow p \in A$$

Since A is closed implies that $A' \subset A$.

Remark 1.4.4 *If $G \subset R$ is an open set and $x \in G$ then there are $a,b \in R$ such that $x \in (a,b) \subset G$ according to which and the definition of the convergence of sequences Remark 1.3.10, if a sequence $< x_n >$ convergent to the point x then for each open set G containing x there exists a positive integer $n_o \in N$ such that*

$$n \geq n_o \Rightarrow x_n \in (a,b) \subset G \Rightarrow x_n \in G$$

That is G contains all elements of the sequence $< x_n >$ except a finite number of them.

Exercises

(1) If $\{X_k : k \in N, 1 < k < n\}$ and $\{Y_k : k \in N, 1 < k < n\}$ are indexed families of nonempty sets where $n \in N$, Prove that

(i) $X_k \subset Y_k \,; \forall k \Rightarrow \prod_{k=1}^{n} X_k \subset \prod_{k=1}^{n} Y_k$,

(ii) $(\prod_{k=1}^{n} X_k) \cap (\prod_{k=1}^{n} Y_k) = \prod_{k=1}^{n} (X_k \cap Y_k)$ and

(iii) $(\prod_{k=1}^{n} X_k) \cup (\prod_{k=1}^{n} Y_k) = \prod_{k=1}^{n} (X_k \cup Y_k)$.

(2) Consider the families $\{A_\alpha : \alpha \in \Delta\}$ and $\{B_\beta : \beta \in \Omega\}$ and let B be any set then prove that

(i) $B \cup (\bigcup_{\alpha \in \Delta} A_\alpha) = \bigcup_{\alpha \in \Delta} (B \cup A_\alpha)$,

(ii) $B \cap (\bigcap_{\alpha \in \Delta} A_\alpha) = \bigcap_{\alpha \in \Delta} (B \cap A_\alpha)$ and

(iii) $(\bigcup_{\alpha \in \Delta} A_\alpha) \cap (\bigcup_{\beta \in \Omega} B_\beta) = \bigcup_{\alpha \in \Delta, \beta \in \Omega} (A_\alpha \cap B_\beta)$ and

(iv) $(\bigcap_{\alpha \in \Delta} A_\alpha) \cup (\bigcap_{\beta \in \Omega} (B_\beta) = \bigcap_{\alpha \in \Delta, \beta \in \Omega} (A_\alpha \cup B_\beta)$.

(3) Let $f: X \to Y$ be any function, $\{A_\alpha : \alpha \in \Delta\} \subset P(X)$ and $\{B_\alpha : \alpha \in \Delta\} \subset P(Y)$. Then prove that

(i) $f(\bigcup_{\alpha\in\Delta} A_\alpha) = \bigcup_{\alpha\in\Delta} f(A_\alpha)$,

(ii) $f \bigcap_{\alpha\in\Delta} (A_\alpha) \subset \bigcap_{\alpha\in\Delta} f(A_\alpha)$ and may be not equals,

(iii) $f^{-1}(\bigcup_{\alpha\in\Delta} B_\alpha) = \bigcup_{\alpha\in\Delta} f^{-1}(B_\alpha)$ and

(iv) $f^{-1} \bigcap_{\alpha\in\Delta} (B_\alpha) = \bigcap_{\alpha\in\Delta} f^{-1}(B_\alpha)$.

(4) Consider the function $f: X \to Y$ and prove that

(i) $f^{-1} \circ f = i = f \circ f^{-1}$ iff f is bijective

(ii) If $g: Y \to X$ such that $g \circ f = i$ then f is injective.

(5) Consider the function $f: R \to R$ such that $f(x) = x^2$ and find $f((-3,1])$, $f([-1,2])$, $f((-\infty,0))$, $f^{-1}([-2,0))$, $f^{-1}((-2,2))$, $f^{-1}([0,\infty))$ and $f^{-1}(0,3))$.

(6) Check that any of the functions injective, surjective or bijective:

(a) $f: R \to R$ such that

$$(i)\ f(x) = 3 - 2x \qquad (ii)\ f(x) = x^3.$$

(b) $f: (-1,1) \to R$ such that

$$(i)\ f(x) = \frac{x}{1-|x|} \quad (ii)\ f(x) = \frac{x}{1-x^2} \quad (iii)\ f(x) = \tan\frac{\pi}{2}x\ .$$

(c) $f: N^* \times N^* \to N$ Such that $f((n,m) = 2^n(2m+1)$ fot each $n, m \in N^*$ where $N^* = N \cup \{0\}$.

(7) Check that any of the given functions injective, surjective or neither injective nor surjective

(a) $f: R \to R$ such that

$$(i)\ f(x) = x^2 + 2 \quad (ii)\ f(x) = |x| \quad (iii)\ f(x) = \begin{cases} x+2; & x \le 0 \\ x\ ; & x > 0 \end{cases}.$$

(b) $f: [0,\infty) \to R$ such that $f(x) = \sqrt{x}$.

(8) Give an example of a family of finite sets the union of their members is infinite set.

(9) Prove that the union of two sets one of them is uncountable is uncountable.

(10) Prove that the family of the intervals $\{(a,b):a,b\in Q\}$ is countable.

(11) Prove that the intersection of sets one of them is finite (countable) is finite (countable).

(12) Prove that if X is an infinite set and $X=A\cup B$ then either A or B is infinite and the converse is also true.

(13) Prove that if $A\subset X$ is an infinite set, discuss the cases $X-A$ is finite? $X-A$ is infinite?.

(14) If $A=\{a,b,c,d,e\}$, define a surjective function $f:N\to A$.

(15) Prove that (i) $\chi_o c=c$ (ii) $\chi_0\chi_0=\chi_0$ (iii) $|(0,1)|=c$

(iv) $|A\times B|=|A||B|$ (v) $|\{f:f:A\to B\}|=|A|^{|B|}$

a. If $A\bigcap B=\varnothing$, prove that $|A\bigcup B|=|A|+|B|$.

(16) Prove that $(-\infty,a)\bigcup(b,\infty)$ is an open set for each $a,b\in R$.

(17) Find A^o and A' if $A=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$ and show that A us neither open nor closed? Show that $B=A\bigcup\{0\}$ is a closed set.

(18) Find A' if $A=\{a,b,c,d,e\}\subset R$.

(19) Show that $\bigcap\{(-\frac{1}{n},\frac{1}{n}):n\in N\}$ is not open? and $\bigcup\{[a,b-\frac{1}{n}]:n\in N\}$ is not closed?.

(20) If $A,U\subset R$ such that U is open and A is finite show that $A-U$ is a closed set?.

(21) If $a,b\in R$, find two disjoint open intervals one of them contains a and the other contains b.

(22) Give an example for two subsets $A,B\subset R$ such that A and $A-B$ are open while B is a closed set.

(23) Prove that if $A\in P(R)-\{R,\varnothing\}$ then $A\in u\Rightarrow A^c\notin u$ i.e. the unique subset of R which are clopen i.e. both open and closed are R and $\varnothing$ only.

Chapter II

Topological Spaces

Topics:

- Topology on a Set.
- Comparison between topologies.
- Open and Closed Sets.
- Convergence of sequences in topological spaces.

In this chapter the topology is defined to be a family of subsets of a nonempty set with some special conditions such subsets called open sets which can be considered as a generalization of the topology on the set of the real numbers. Some interesting examples of topologies will be given.

2.1 Topology on a Set.

Definition 2.1.1 Let X be any arbitrary set. A topology on X is a collection τ of subsets of X such that

(O1) $X, \varnothing \in \tau$.
(O2) Given $G_1, G_2 \in \tau, G_1 \cap G_2 \in \tau$.
(O3) Given $\{G_i \in \tau : i \in \Delta\}, \bigcup\limits_{i \in \Delta} G_i \in \tau$.

A set X together with a specified topology is called a topological space and is sometimes denoted (X, τ).

Remark 2.1.1 *From the above definition we can remark that:*

(1) $\tau \subseteq P(X)$ *and* $\tau \in P(P(X))$.
(2) the condition (O1) means that X *and* $\varnothing$ *are belongs to* τ
(3) the condition (O2) means that the intersection of any finite members of τ *is a member of* τ.
(4) the condition (O3) means that the union of any members of τ *is a member of* τ.

(5) Axiom (O2) can be generalized by using the mathematical induction to be: The finite intersection of members of τ *is a member of* τ *i.e.*

$$\{G_i \ : i=1,2,3,\dots,n\}\subset\tau \Rightarrow \left(\bigcap_{i=1}^{n} G_i\right)\in\tau .$$

Example 2.1.1 If $X=\{a,b,c,d\}$, determine whether or not each of the following collections of X is a topology on X :

(1) $\tau_1=\{X,\varnothing,\{a\},\{b\},\{a,b\},\{b,c\}\}$.

(2) $\tau_2=\{X,\varnothing,\{a\},\{b\},\{a,b\},\{b,d\},\{a,b,d\}\}$.

(3) $\tau_3=\{X,\varnothing,\{a,b\},\{a,c\},\{a,b,c\}\}$.

(4) $\tau_4=\{X,\varnothing,\{a\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$.

Solution: (1) $\tau_1=\{X,\varnothing,\{a\},\{b\},\{a,b\},\{b,c\}\}$.

(O1) $X,\varnothing\in\tau_1$.

(O2)

$\cap$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
X	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$
$\{b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a,b\}$	$\{a,b\}$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b\}$
$\{b,c\}$	$\{b,c\}$	$\varnothing$	$\varnothing$	$\{b\}$	$\{b\}$	$\{b,c\}$

(O3)

$\cup$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
X	X	X	X	X	X	X
$\varnothing$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
$\{a\}$	X	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$
$\{b\}$	X	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$	$\{b,c\}$
$\{a,b\}$	X	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$
$\{b,c\}$	X	$\{b,c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c\}$

Then τ_1 is not a topology on X, because $\{a\},\{b,c\}\in\tau_1$ *but* $\{a\}\cup\{b,c\}=\{a,b,c\}\notin\tau_1$.

(2) $\tau_2 = \{X, \varnothing, \{a\}, \{b\}, \{a,b\}, \{b,d\}, \{a,b,d\}\}$.

(O1) $X, \varnothing \in \tau_2$.

(O2)

$\cap$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$
X	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$	$\{a\}$
$\{b\}$	$\{b\}$	$\varnothing$	$\varnothing$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a,b\}$	$\{a,b\}$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{b,d\}$	$\{b,c\}$	$\varnothing$	$\varnothing$	$\{b\}$	$\{b\}$	$\{b,d\}$	$\{b,d\}$
$\{a,b,d\}$	$\{a,b,d\}$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$

(O3)

$\cup$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$
X	X	X	X	X	X	X	X
$\varnothing$	X	$\varnothing$	$\{a\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$
$\{a\}$	X	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$	$\{a,b,d\}$
$\{b\}$	X	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$	$\{b,d\}$	$\{a,b,d\}$
$\{a,b\}$	X	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$	$\{a,b,d\}$
$\{b,d\}$	X	$\{b,d\}$	$\{a,b,d\}$	$\{b,d\}$	$\{a,b,d\}$	$\{b,d\}$	$\{a,b,d\}$
$\{a,b,d\}$	X	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$

Then τ_2 is a topology on X .

(3) $\tau_3 = \{X, \varnothing, \{a,b\}, \{a,c\}, \{a,b,c\}\}$.

(O1) $X, \varnothing \in \tau_3$.

(O2)

$\cap$	X	$\varnothing$	$\{a,b\}$	$\{a,c\}$	$\{a,b,c\}$
X	X	$\varnothing$	$\{a,b\}$	$\{a,c\}$	$\{a,b,c\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{a,b\}$	$\{a,b\}$	$\varnothing$	$\{a,b\}$	$\{a\}$	$\{a,b\}$
$\{a,c\}$	$\{a,c\}$	$\varnothing$	$\{a\}$	$\{a,c\}$	$\{a,c\}$

$\{a,b,c\}$	$\{a,b,c\}$	$\varnothing$	$\{a,b\}$	$\{a,c\}$	$\{a,b,c\}$

Then τ_3 is not a topology on X, because $\{a,c\},\{a,b\}\in\tau_3$ *but* $\{a,c\}\cap\{a,b\}=\{a\}\notin\tau_3$.

(4) $\tau_4=\{X,\varnothing,\{a\},\{b,c\},\{a,b,c\},\{b,c,d\}\}$.

(O1) $X,\varnothing\in\tau_4$.

(O2)

$\cap$	X	$\varnothing$	$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c,d\}$
X	X	$\varnothing$	$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c,d\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{a\}$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$	$\{a\}$	$\varnothing$
$\{b,c\}$	$\{b,c\}$	$\varnothing$	$\varnothing$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\varnothing$	$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c\}$
$\{b,c,d\}$	$\{b,c,d\}$	$\varnothing$	$\varnothing$	$\{b,c,d\}$	$\{b,c\}$	$\{b,c,d\}$

(O3)

$\cup$	X	$\varnothing$	$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c,d\}$
X	X	X	X	X	X	X
$\varnothing$	X	$\varnothing$	$\{a\}$	$\{b,c\}$	$\{a,b,c\}$	$\{b,c,d\}$
$\{a\}$	X	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a,b,c\}$	X
$\{b,c\}$	X	$\{b,c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$	X
$\{a,b,c\}$	X	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	X
$\{b,c,d\}$	X	$\{b,c,d\}$	X	$\{b,c,d\}$	X	$\{b,c,d\}$

Then τ_4 is a topology on X.

At the following a number of common and interesting examples of topologies will be given, these examples with other examples will be use in future to explain the properties of the topological spaces.

Example 2.1.2. (Indiscrete topology) Let X be any set. Then $I=\{X,\varnothing\}$ is the indiscrete topology on X.

Proof: O1) $X,\varnothing\in I$.

O2) $X\cap\varnothing=\varnothing\in I$.

O3) $X \bigcup \varnothing = X \in I$.

Then $I = \{X, \varnothing\}$ is a topology on X and is called the trivial topology.

Example 2.1.3. (Discrete topology) Let X be a countable set. Then $D = P(X)$ is the discrete topology on X.

Proof: O1) Since $\varnothing \subseteq X$ and $X \subseteq X$, $X, \varnothing \in D$.

O2) For any G_1 and G_2 of subsets of X, then $G_1 \bigcap G_2$ is a subset of X

O3) An arbitrary union of subsets of X is itself a subset of X.

Then $D = P(X)$ is a topology on X.

$$\therefore \tau \text{ is a topology on } X.$$

Example 2.1.4. (Particular point topology) Let X be any set and let $p \in X$. Then

$$P_p = \{G \subseteq X : p \in G\} \bigcup \{\varnothing\}$$

is the particular point topology on X.

Proof: O1) Since $\varnothing \in \{\varnothing\} \subseteq P_p, \varnothing \in P_p$.

As $X \in \{G \subseteq X : p \in G\} \subseteq P_p$, $X \in P_p$.

O2) Let $G_1, G_2 \in P_p$. Given $G_1, G_2 = \varnothing$ or $G_1, G_2 \in \{G \subseteq X : p \in G\} \subseteq P_p$, suppose $p \in G_1, p \in G_2$ then $p \in G_1 \bigcap G_2$. So $G_1 \bigcap G_2 \in \{G \subseteq X : p \in G\} \subseteq P_p$.

Otherwise, $G_1 \bigcap G_2 = \varnothing \in P_p$. In either case, $G_1 \bigcap G_2 \in P_p$.

O3) Let $\{G_i \subseteq X : i \in \Delta\}$ be an indexed family of sets such that $G_i \in P_p$ $\forall i \in \Delta$. Notice $G_i = \varnothing$ or $G_i \in \{G \subseteq X : p \in G\}$. Suppose $p \in G_i$ for at least one $i \in \Delta$. Then $p \in \bigcup_{i \in \Delta} G_i$. So $\bigcup_{i \in \Delta} G_i \in \{G \subseteq X : p \in G\} \subseteq P_p$.

Otherwise, $\bigcup_{i\in\Delta} G_i = \varnothing \in P_p$. In either case, $\bigcup_{i\in\Delta} G_i \in P_p$.

$$\therefore P_p \text{ is a topology on } X.$$

Example 2.1.5. (Excluding point topology) Let X be any set and let $p \in X$. Then

$$E_p = \{G \subseteq X : p \notin G\} \bigcup \{X\}$$

is the excluded point topology on X.

Proof: O1) Since $X \in \{X\} \subseteq E_p, X \in E_p$.

As $\varnothing \in \{G \subseteq X : p \notin G\} \subseteq E_p$, $\varnothing \in E_p$.

O2) Let $G_1, G_2 \in E_p$. Given $G_1, G_2 = X$ or $G_1, G_2 \in \{G \subseteq X : p \notin G\} \subseteq E_p$, suppose $p \notin G_i$ for at least one $i = 1,2$ then $p \notin G_1 \bigcap G_2$. So $G_1 \bigcap G_2 \in \{G \subseteq X : p \notin G\} \subseteq E_p$.

Otherwise, $G_1 \bigcap G_2 = X \in E_p$. In either case, $G_1 \bigcap G_2 \in E_p$.

O3) Let $\{G_i \subseteq X : i \in \Delta\}$ be an indexed family of sets such that $G_i \in E_p \ \forall i \in \Delta$. Notice $G_i = X$ or $G_i \in \{G \subseteq X : p \notin G\}$. Suppose $p \notin G_i$ for any $i \in \Delta$. Then $p \notin \bigcup_{i\in\Delta} G_i$. So $\bigcup_{i\in\Delta} G_i \in \{G \subseteq X : p \notin G\} \subseteq E_p$.

Otherwise, $\bigcup_{i\in\Delta} G_i = X \in E_p$. In either case, $\bigcup_{i\in\Delta} G_i \in E_p$.

$$\therefore E_p \text{ is a topology on } X.$$

Example 2.1.6 (Co-finite topology) Let X be a non empty set then $C = \{G \subset X : G^c$ is finite $\} \cup \{\varnothing\}$ is a topology on X.

Proof: O1) $\varnothing \in C$ by the definition, $X^c = \varnothing$ is finite which implies that $X \in C$.

O2) $G_1, G_2 \in C - \{\emptyset\}$ Implies that G_1^c and G_2^c are finite implies that $(G_1 \cap G_2)^c = G_1^c \cup G_2^c$ and $G \cap \emptyset = \emptyset$ for each $G \in C$ which implies that $(G_1 \cap G_2) \in C$ for each $G_1, G_2 \in C$.

O3) If $\{G_i : i \in \Delta\} \subset C$ then

$$(\bigcup_{i \in \Delta} G_i)^c = (\bigcap_{i \in \Delta} G_i^c) \subset G_i^c \;;\forall i \in \Delta : G_i \neq \emptyset$$

Then $(\bigcup_{i \in \Delta} G_i)^c$ is finite since G_i^c is finite for each $i \in \Delta$ and so $\bigcup_{i \in \Delta} G_i \in C$.

Example 2.1.7 (Co-countable topology) Let X be a non empty set then $C = \{G \subset X : G^c$ is countable $\} \cup \{\emptyset\}$ is a topology on X .

Example 2.1.8. (Usual topology) Let R be a real number . Then

$$\mho = \left\{G \subseteq R : \forall x \in G\ \exists \delta > 0 \ \ \textit{such that}\ \ (x-\delta, x+\delta) \subseteq G\right\} \bigcup \{R, \emptyset\}$$

is the usual topology on R .

Proof: O1) It is clear that $R, \emptyset \in \mho$.

O2) Suppose that $G_1, G_2 \in \mho$ and $x \in G_1 \bigcap G_2$. Then $x \in G_1$ and $x \in G_2$. Then there exists $\delta_1, \delta_2 > 0$ such that

$$(x-\delta_1, x+\delta_1) \subseteq G_1 \ \textit{and}\ (x-\delta_2, x+\delta_2) \subseteq G_2.$$

If we choose $\delta = \min\{\delta_1, \delta_2\}$ we have

$$(x-\delta, x+\delta) \subseteq G_1 \ \textit{and}\ (x-\delta, x+\delta) \subseteq G_2.$$

Then,

$$(x-\delta, x+\delta) \subseteq G_1 \bigcap G_2 \quad \forall x \in G_1 \bigcap G_2.$$

Thus $G_1 \bigcap G_2 \in \mho$.

O3) Suppose that $\{G_i : i \in \Delta\} \subseteq \mho$ and $x \in \bigcup_{i \in \Delta} G_i$. Then there exists at least one $G_{i_o} \in \{G_i : i \in \Delta\}$ such that $x \in G_{i_o}$. Since $G_{i_o} \in \mho$, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq G_{i_o}$. Also, since $G_{i_o} \subseteq \bigcup_{i \in \Delta} G_i$, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq \bigcup_{i \in \Delta} G_i$, $\forall x \in \bigcup_{i \in \Delta} G_i$. Hence $\bigcup_{i \in \Delta} G_i \in \mho$.

Example 2.1.9. (Right Ray topology) Let R be a real number . Then

$$\tau = \{L_a = (a, \infty) : a \in R\} \cup \{R, \varnothing\}$$

is the right ray topology on R .

Proof: O1) It is clear that $R, \varnothing \in \tau$.

O2) Let $L_a, L_b \in \tau$ for $a, b \in R$. Then,

$$L_a \cap L_b = \begin{cases} L_b, & \mathit{if} \quad a \le b \\ L_a, & \mathit{if} \quad a \ge b. \end{cases}$$

So, $L_a \cap L_b \in \tau$. Also, $L_a \cap \varnothing = \varnothing$ and $L_a \cap R = L_a$.

O3) Let $\left\{ L_{a_i} : a_i \in R \ \ \mathit{and} \ \ i \in \Delta \right\} \subseteq \tau$. Then

$$\bigcup_{i \in \Delta} L_{a_i} = \begin{cases} L_a, & \mathit{if} \quad a = \inf\{a_i : i \in \Delta\} \\ R, & \mathit{Otherwise}. \end{cases}$$

Hence $\bigcup_{i \in \Delta} L_{a_i} \in \tau$.

Example 2.1.10. (Left ray topology) Let R be a real number . Then

$$\tau = \{L_a = (-\infty, a) : a \in R\} \cup \{R, \varnothing\}$$

is the left ray topology on R .

Proof: O1) It is clear that $R, \varnothing \in \tau$.

O2) Let $L_a, L_b \in \tau$ for $a, b \in R$. Then,

$$L_a \cap L_b = \begin{cases} L_a, & if \quad a \le b \\ L_b, & if \quad a \ge b. \end{cases}$$

So, $L_a \cap L_b \in \tau$. Also, $L_a \cap \varnothing = \varnothing$ and $L_a \cap R = L_a$.

O3) Let $\left\{ L_{a_i} : a_i \in R \quad and \quad i \in I \right\} \subseteq \tau$. Then

$$\bigcup_{i \in \Delta} L_{a_i} = \begin{cases} L_a, & if \quad a = \sup\left\{ a_i : i \in \Delta \right\} \\ R, & Otherwise. \end{cases}$$

Hence $\bigcup_{i \in \Delta} L_{a_i} \in \tau$.

Example 2.1.11. Let N be a natural numbers and τ be the class

$$\tau = \left\{ A_n = \{1, 2, 3, \ldots\ldots., n\} : n \in N \right\} \bigcup \{N, \varnothing\}.$$

Then, prove that τ is a topology on N.

Proof: O1) It is clear that $N, \varnothing \in \tau$.

O2) Let $A_n, A_m \in \tau \; for \; n, m \in N$. Then,

$$A_n \cap A_m = A_k = \{1, 2, 3, \ldots\ldots., k\} \; where \quad k = \min\{n, m\}.$$

So, $A_n \cap A_m \in \tau$. Also, $A_n \cap \varnothing = \varnothing$ and $A_n \cap N = A_n$.

O3) Let $\left\{ A_{n_i} : n_i \in N \quad and \quad i \in \Delta \right\} \subseteq \tau$. Then

$$\bigcup_{i \in \Delta} A_{n_i} = A_k = \{1, 2, 3, \ldots\ldots., k\} \; where \quad k = \max\left\{ n_i : i \in \Delta \right\}.$$

Hence $\bigcup_{i \in \Delta} A_{n_i} \in \tau$.

Example 2.1.12. Let N be a natural numbers and τ be the class

$$\tau = \left\{ A_n = \{n, n+1, n+2, n+3, \ldots\ldots\ldots\} : n \in N \right\} \bigcup \{\varnothing\}.$$

Then, prove that τ is a topology on N.

Proof: O1) It is clear that $N, \varnothing \in \tau$.

O2) Let $A_n, A_m \in \tau$ *for* $n, m \in N$. Then,

$$A_n \cap A_m = A_k = \{k, k+1, k+2, ...\} \text{ where } k = \max\{n, m\}.$$

So, $A_n \cap A_m \in \tau$. Also, $A_n \cap \varnothing = \varnothing$ and $A_n \cap N = A_n$.

O3) Let $\left\{A_{n_i} : n_i \in N \text{ and } i \in \Delta\right\} \subseteq \tau$. Then

$$\bigcup_{i \in \Delta} A_{n_i} = A_k = \{k, k+1, k+2, ...\} \text{ where } k = \min\left\{n_i : i \in \Delta\right\}.$$

Hence $\bigcup_{i \in \Delta} A_{n_i} \in \tau$.

Example 2.1.13. Let X be a non empty set, $p \in X$ and $\tau = E_p \cup C = \{G \subset X : p \notin G \vee G^c$ is finite$\}$. Then,

(O1) $X, \varnothing \in E_p \cup C \Rightarrow X, \varnothing \in \tau$,

(O2) If $G_1, G_2 \in \tau - \{X\}$ then either (i) $G_1 \in E_p \vee G_2 \in E_p$ or (ii) $G_1, G_2 \in C$ from which we find that

$$(i) \Rightarrow p \notin G_1 \cap G_2 \Rightarrow G_1 \cap G_2 \in E_p \Rightarrow G_1 \cap G_2 \in \tau$$

$$(ii) \Rightarrow G_1 \cap G_2 \in C \Rightarrow G_1 \cap G_2 \in \tau$$

Clearly $G_1 = X$ implies that $G_1 \cap G_2 = G_1$ and $G_2 = X$ implies that $G_1 \cap G_2 = G_2$.

(O3) If $\{G_i : i \in \Delta\} \subset \tau$ then there are two cases:

(i) $\{G_i : i \in \Delta\} \subset E_p \Rightarrow \bigcup\{G_i : i \in \Delta\} \in E_p \subset \tau$.

(ii) There exists $i_o \in \Delta$ such that $G_{i_o} \in C$ in this case we find that

$$G_{i_o} \subset \bigcup\{G_i : i \in \Delta\} \Rightarrow \bigcup\{G_i : i \in \Delta\} \in C \subset \tau.$$

Therefore $\tau = E_p \cup C$ satisfies the axioms (O1), (O2) and (O3) and so a topology on X .

Example 2.1.14. Let X be a nonempty set and y and z be two distinct points of X . Then $\tau = P_y \cup E_z$ is a topology on X , for

(O1) $X, \varnothing \in P_y \cap E_z \Rightarrow X, \varnothing \in P_y \cup E_z$, (O2) If $G_1, G_2 \in \tau - \{X\} = (P_y \cup E_z) - \{X\}$ then either (i) $z \notin G_1$ or $z \notin G_2$ which implies that $z \notin G_1 \cap G_2$ which implies that $G_1 \cap G_2 \in E_z \subset \tau$ or (ii) $y \in G_1 \cap G_2$ which implies that $G_1 \cap G_2 \in P_y \subset \tau$.

(O3) If $\{G_i : i \in \Delta\} \subset \tau$ then either $\{G_i : i \in \Delta\} \subset E_z$ which implies that $\bigcup\{G_i : i \in \Delta\} \in E_z \subset \tau$ or there exists $i_o \in \Delta$ such that $G_{i_o} \in P_y - E_z$ in this case $p \in \bigcup\{G_i : i \in \Delta\}$ which implies that $\bigcup\{G_i : i \in \Delta\} \in P_p \subset \tau$

Therefore $\tau = E_z \cup P_y$ satisfies the axioms (O1), (O2) and (O3) , so is a topology on X . For example if $X = \{a,b,c\}$ then

$$\left.\begin{array}{l} E_a = \{X, \varnothing, \{b\}, \{c\}, \{b,c\}\}, \\ P_b = \{X, \varnothing, \{a,b\}, \{b\}, \{b,c\}\} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tau = E_a \cup P_b = \{X, \varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$$

Theorem 2.1.1. *Let X be a non empty set, τ be the family of all possible topologies on X and $\{\tau_i : i \in \Delta\} \subset \tau$. Then $\tau^* = \bigcap\{\tau_i : i \in \Delta\}$ is a topology on X .*

Proof: We are going to prove that τ^* satisfies the axioms (O1), (O2) and (O3) as follows:

Since τ_i is a topology on X for each $i \in \Delta$ then

(O1) $X, \varnothing \in \tau_i\ ; \forall i \in \Delta \Rightarrow X, \varnothing \in \tau^*$,

(O2)

$$G_1, G_2 \in \tau^* \Rightarrow G_1, G_2 \in \tau_i\ ; \forall i \in \Delta$$

$$\Rightarrow G_1 \cap G_2 \in \tau_i\ ; \forall i \in \Delta \Rightarrow (G_1 \cap G_2) \in \tau^*$$

(O3) Let $\{G_i : i \in \Delta\} \subset \tau^*$. Then $G_i \in \tau_i\ ; \forall i \in \Delta$ which implies that $(\bigcup_{i \in \Delta} G_i) \in \tau_i\ ; \forall i \in \Delta$ which implies that $\bigcup\{G_i : i \in \Delta\} \in \tau^*$.

In our last example, we show that the union of topologies need not be a topology.

Example 2.1.15. Each of the classes $\tau_1 = \{X, \varnothing, \{a\}\}$ and $\tau_2 = \{X, \varnothing, \{b\}\}$ is a topology on $X = \{a,b,c\}$. But the union

$\tau_1 \cup \tau_2 = \{X, \varnothing, \{a\}, \{b\}\}$ is not a topology on X since it violates (O3) . That is, $\{a\} \in \tau_1 \cup \tau_2$ and $\{b\} \in \tau_1 \cup \tau_2$ but $\{a\} \cup \{b\} = \{a,b\} \notin \tau_1 \cup \tau_2$.

2.2. Comparison between topologies.

Definition 2.2.1. Let τ and τ^* be two topologies defined on the same underlying set X .

- If $\tau \subseteq \tau^*$, then we say that τ^* is finer (stronger) than τ. Equivalently, we say that τ is coarser (weaker) than τ^*.
- If $\tau \subsetneq \tau^*$, then we say that τ^* is strictly finer than τ. Equivalently, we say that τ is strictly coarser than τ^*.
- If $\tau \subseteq \tau^*$ or $\tau^* \subseteq \tau$, then the two topologies are said to be comparable. Otherwise, the two topologies are incomparable.

Remark 2.2.1. *The indiscrete topology is the coarsest topology possible. While the discrete topology is the finest topology possible. So if* τ *is a topology on* X *, then* $I \subseteq \tau \subseteq D$ *.*

Remark 2.2.2. *If* $A \subseteq B$ *, then* $A \cap B = A$*. So if* $\tau \subseteq \tau^*$*, then* $\tau \cap \tau^* = \tau$*. Equivalently, if* $\tau \cap \tau^* \neq \tau$*, then* $\tau \not\subset \tau^*$*. Similarly, if* $\tau \cap \tau^* \neq \tau^*$*, then* $\tau^* \not\subset \tau$*. So if the intersection of the two topologies is not one of the two given topologies, then the two topologies are incomparable.*

Example 2.2.1. Consider the usual topology $\mho$ on R and let τ be the left ray topology on R . Then τ is weaker than $\mho$ since

$$G \in \tau - \{R, \varnothing\} \Rightarrow \exists\, a \in R \; : G = (-\infty, a) \Rightarrow G \in \mho \Rightarrow \tau \subset \mho$$

Example 2.2.2. Let $X = \{a,b,c,d\}$ and $\tau_1 = \{X, \varphi, \{a\}, \{b\}, \{c,d\}, \{a,b\}, \{a,c,d\}, \{b,c,d\}\}$ and $\tau_2 = \{X, \varphi, \{a,b\}, \{c,d\}\}$ be two topologies on X . Clearly $\tau_2 \subset \tau_1$ which means that τ_2 is weaker than τ_1 equivalently τ_1 is stronger than τ_2.

Example 2.2.3. Let P_p be the particular point topology on X and let E_p be the excluded point topology on X . Then,

$$\begin{aligned} P_p \cap E_p &= \{\{G \subseteq X : p \in G\} \cup \{\varnothing\}\} \cap \{\{G \subseteq X : p \notin G\} \cup \{X\}\} \\ &= \{\varnothing, X\}. \end{aligned}$$

So P_p and E_p are incomparable.

Example 2.2.4. Let $X=\{a,b,c\}$ and let $a\in X$. Then

- the indiscrete topology on X is $I=\{\varnothing,X\}$.
- the discrete topology on X is
$$D=\{\varnothing,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}.$$
- the particular point topology on X is
$$P_a=\{\varnothing,X,\{a\},\{a,b\},\{a,c\}\}.$$
- the excluded point topology on X is
$$E_a=\{\varnothing,X,\{b\},\{c\},\{b,c\}\}.$$

Notice $I\subsetneq P_a\subsetneq D$ and $I\subsetneq E_a\subsetneq D$.

Example 2.2.5. If $X=\{a\}$, then $I=\{X,\varnothing\}$ and $D=\{X,\varnothing\}$. Hence, $I=D$. So any topology defined on a singleton set is indistinguishable from the indiscrete topology.

Definition 2.2.2. *Suppose that τ_1 and τ_2 are two topologies defined on a nonempty set X then τ_2 is a strictly weaker topology than τ_1 if for each subfamily τ^* of $P(X)$ such that $\tau^*\neq\tau_2$ then $\tau_2\subset\tau^*\subset\tau_1$ implies that either $\tau^*=\tau_1$ or τ^* is not a topology on X .*

Example 2.2.6. Let $X=\{a,b,c,d\}$ and $\tau_1=\{X,\varnothing,\{a\},\{b\},\{c,d\},\{a,b\},\{a,c,d\},\{b,c,d\}\}$ and $\tau_2=\{X,\varnothing,\{a\},\{a,b\},\{c,d\},\{a,c,d\}\}$ be two topologies on X . Then τ_2 is a strictly weaker topology on X than τ_1 because if $\tau_2\neq\tau^*\subset P(X)$ such that $\tau_2\subset\tau^*\subset\tau_1$ then $G\in\tau^*-\tau_2$ implies that $G\in\tau_1-\tau_2=\{\{b\},\{b,c,d\}\}$ and there are two cases:

(1) $G=\{b\}$ Which implies that $\{b\}\cup\{c,d\}=\{b,c,d\}$ and either $\{b,c,d\}\in\tau^*$ which implies that $\tau^*=\tau_1$ or $\{b,c,d\}\notin\tau^*$ which implies that τ^* is not a topology on X .

(2) $G=\{b,c,d\}$ which implies that $\{b,c,d\}\cap\{a,b\}=\{b\}$ and either $\{b\}\in\tau^*$ which implies that $\tau^*=\tau_1$ or $\{b\}\notin\tau^*$ which implies that τ^* is not a topology on X . This means exactly that τ_2 is strictly weaker than τ_1.

Theorem 2.2.1 *Let* A *be a nonempty subset of a set* X, $z \in A$, $y \in A-\{z\}$, $P_{A-\{y\}}=\{G \subset X : A-\{y\} \subset G\} \cup \{\varnothing\}$ *and* $P_A=\{G \subset X : A \subset G\} \cup \{\varnothing\}$. *If* $\tau^+=E_z \cup P_{A-\{y\}}$ *then* $\tau=E_z \cup P_A$ *is a strictly weaker topology than* τ^+.

Proof: Suppose that $\tau^* \subset P(X)$ such that $\tau \subset \tau^* \subset \tau^+$ and $\tau^* \neq \tau$ then $G \in \tau^*-\tau$ implies that $G \in \tau^*$ and $G \notin \tau$ and so

(a) $G \notin \tau \Rightarrow G \notin E_z \wedge G \notin P_A$ and

(b) $G \in \tau^* \Rightarrow G \in \tau^+ \Rightarrow G \in P_{A-\{y\}}$ from which $G \notin E_z \Rightarrow z \in G$. Then

$$G \notin P_A \wedge G \in P_{A-\{y\}} \Rightarrow A \not\subset G, A-\{y\} \subset G \Rightarrow y \notin G.$$

Now if $H \in \tau^+$ such that $H \in P_{A-\{y\}}$ then $A \subset H^*=H \cup \{y\}$ from which we finds

$$H^* \in P_A \Rightarrow H^* \in \tau^* \Rightarrow H^* \cap G \in \tau^*$$

Also $y \notin G$ implies that $y \notin H^* \cap G$ which implies that $H^* \cap G=H \cap G \in \tau^*$ and so

$$z \in G \Rightarrow z \notin H-G \Rightarrow H-G \in E_z \Rightarrow H-G \in \tau^*$$

Therefore

$$(H-G) \cup (H \cap G)=H \in \tau^* \Rightarrow \tau^*=\tau^+$$

This means that τ is a strictly weaker topology than τ^+ on X.

Definition 2.2.3. If $X \neq \varnothing$ the topology on X which is strictly weaker than the discrete topology D is called an ultra topology on X.

Theorem 2.2.2. *Let* X *be a nonempty set and* $y, z \in X$ *such that* $y \neq z$. *Then* $D_{yz}=E_z \cup P_y=\{G \subset X : z \notin G \vee y \in G\}$ *is an ultra topology on* X.

Proof: It was proved that D_{yz} is a topology on X Example 2.1.14, and to prove that it is a strictly weaker topology than D, let $\tau^* \subset P(X)$ be such that $D_{yz} \subset \tau^* \subset D$. Then

$$G \in \tau^*-D_{yz} \Rightarrow G \in \tau^* \wedge G \notin D_{yz}$$

$$\Rightarrow G \in \tau^*-\{\varnothing\}, G \notin E_z, G \notin P_y \Rightarrow G \in \tau^*, z \in G, y \notin G$$

But $\{y,z\} \in D_{yz}$ implies that $\{y,z\} \in \tau^*$ and since $G \in \tau^*$ then $G \cap \{y,z\}=\{z\} \in \tau^*$ and since $\{\{x\} : x \in X-\{z\}\} \subset E_z \subset \tau^*$ then

$\{\{x\}:x\in X\}\subset\tau^*$ which implies that $\tau^*=D$. This means that D_{yz} is a strictly weaker topology than D on X. So D_{yz} is an ultra topology.

Remark 2.2.3. *In fact D_{yz} is a principal ultra topology, the ultra topologies are called principal or non principal which are defined by Frohlish in 1964. To define the non principal we need an idea about the filters which will be given at the following definitions, examples and Theorems.*

Definition 2.2.4. A nonempty family β of nonempty subsets of a set X is called a filter basis for X if it satisfies the following condition:

The intersection of two members of β is a member of β. This condition can be written in the form "the intersection of any finite number of members of β is a member of β".

Example 2.2.7. If A is a nonempty subset of a set X, then $\beta=\{A\}$ is a filter basis on X.

Example 2.2.8. The family of the tails of a sequence $<x_n>$ of points of a set X is a filter basis on X.

Example 2.2.9. Let X be an infinite set, then the family of the complements of all finite subsets of X i.e. the family $\beta=\{A^c$: A is a finite subset of X, $A\neq\varnothing\}$ is a filter basis on X.

Definition 2.2.5. A nonempty family $\Im$ of nonempty subsets of a set X is called a filter on X if it satisfies the following two conditions:

(i) A superset of a member of $\Im$ is a member of $\Im$ i. e. $A\in\Im, A\subset B\subset X \Rightarrow B\in\Im$.

(ii) The intersection of two or any finite number of members of $\Im$ is a member of $\Im$ i.e. $A_1,A_2\in\Im\Rightarrow A_1\cap A_2\in\Im$.

Clearly $\Im$ is a filter on X if it is a filter basis on X and satisfies the condition (i).

Example 2.2.10. If A is a nonempty subset of a set X, then the family of all super subsets of A is a filter on X determined by the filter basis $\{A\}$.

Example 2.2.11. Every filter basis on X determines a filter on X.

Example 2.2.12. If X is an infinite set then the family

$\Im=\{A^c:A$ is a finite subset of a nonempty subset of $X\}$ is a filter on X.

Example 2.2.13. The family of the tails of a sequence $<x_n>$ of points of a set X is a filter on X which is called an elementary filter.

Theorem 2.2.3. *If $\Im$ is a filter on a non empty set X and $z \in X$ is any point then $\tau = E_z \cup \Im$ is a topology on X where E_z is the excluding point topology on X .*

Proof: *Clearly* (i) $X, \varnothing \in E_z \Rightarrow X, \varnothing \in \tau$,

(ii) $G_1, G_2 \in \tau$ *implies that either*

(a) $G_1 \in E_z \vee G_2 \in E_z \Rightarrow G_1 \cap G_2 \in E_z \Rightarrow G_1 \cap G_2 \in \tau$ *or*

(b) $G_1, G_2 \in \Im \Rightarrow G_1 \cap G_2 \in \Im \Rightarrow G_1 \cap G_2 \in \tau$ *and* (iii) $\{G_i : i \in \Delta\} \subset \tau$ *implies that either*

(a) $\{G_i : i \in \Delta\} \subset E_z \Rightarrow (\bigcup_{i \in \Delta} G_i) \in E_z \Rightarrow (\bigcup_{i \in \Delta} G_i) \in \tau$ *or*

(b) There exists $i_o \in \Delta$ *such that* $G_{i_o} \in \Im$ *from which*

$G_{i_o} \subset (\bigcup_{i \in \Delta} G_i) \subset X \Rightarrow (\bigcup_{i \in \Delta} G_i) \in \Im \Rightarrow (\bigcup_{i \in \Delta} G_i) \in \tau$.

Definition 2.2.6. If $\Im_1$ and $\Im_2$ are two filters on a set X then $\Im_1$ is finer than $\Im_2$ if $\Im_2 \subset \Im_1$, if $\Im_1 \neq \Im_2$ we say that $\Im_1$ is genuinely finer than $\Im_2$.

Remark 2.2.4. *(1) Clearly* $\{X\}$ *is the weakest filter on* X .

(2) If $A \in P(X) - \{X, \varnothing\}$ *then* $\{A\}$ *and* $\{A^c\}$ *are filter bases determine the filters* $\Im_1$ *and* $\Im_2$. *If* $\Im$ *is a filter on* X *stronger than both* $\Im_1$ *and* $\Im_2$ *then* $A, A^c \in \Im \Rightarrow A \cap A^c = \varnothing \in \Im$ *a contradiction which means that such filter* $\Im$ *does not exist.*

Theorem 2.2.4. *Let* X *be a non empty set,* β_1 *and* β_2 *be two filter bases on* X *and* $\Im_1$ *and* $\Im_2$ *be the filters on* X *determined by* β_1 *and* β_2. *Then there exists a filter* $\Im$ *on* X *which is finer than both* $\Im_1$ *and* $\Im_2$ *iff* $A \cap B \neq \varnothing$ *for each* $A \in \beta_1$ *and* $B \in \beta_2$.

Proof: Let $A \cap B \neq \varnothing$ for each $A \in \beta_1$ and $B \in \beta_2$. Then the family $\{A \cap B : A \in \beta_1 \wedge B \in \beta_2\}$ is a filter basis on X determines a filter $\Im$ on X which is finer than both $\Im_1$ and $\Im_2$.

Clearly $\Im = \sup\{\Im_1, \Im_2\}$.

Conversely let $\Im$ be a filter on X which is finer than both $\Im_1$ and $\Im_2$ and $A \in \beta_1$ and $B \in \beta_2$ then $A, B \in \Im$ and

$$A \cap B = \varnothing \Rightarrow B \subset A^c \Rightarrow A^c \in \Im \Rightarrow A \cap A^c = \varnothing \in \Im$$

which contradicts the Definition 2.2.5, of the filters? Therefore $A \cap B \neq \varnothing$ for each $A \in \beta_1$ and $B \in \beta_2$.

Theorem 2.2.5. *Let A be a non empty subset of a set X and $\Im$ be a filter on X. If $\Im^*$ is a filter on X finer than $\Im$ then $A^c \in \Im \Rightarrow A \notin \Im^*$ and if $A^c \notin \Im$ then there exists a filter $\Im^*$ on X finer than $\Im$ such that $A \in \Im^*$.*

Proof: Let $\Im^*$ be a filter on X finer than $\Im$. Then $A^c \in \Im$ implies that $A^c \in \Im^*$ implies that $A \notin \Im^*$ since $A \in \Im^* \Rightarrow A \cap A^c = \varnothing \in \Im^*$ which contradicts Definition 2.2.5 .

Conversely let $A^c \notin \Im$ and $B \in \Im$. Then $B \cap A = \varnothing \Rightarrow B \subset A^c \Rightarrow A^c \in \Im$ which contradicts our assumption that $A^c \notin \Im$ and so $A \cap B \neq \varnothing$ for each $B \in \Im$. Hence by Theorem 2.2.4, the family $\{A \cap B \neq \varnothing : B \in \Im\}$ is a filter basis determines a filter $\Im^* = \sup\{\Im, \Im_A\}$ on X finer than $\Im$ where $\Im_A$ is the filter on X determined by the filter basis $\beta = \{A\}$ and clearly $A \in \Im^*$. Therefore $A^c \notin \Im \Rightarrow A \in \Im^*$ Equivalently $A \notin \Im^* \Rightarrow A^c \in \Im$.

Definition 2.2.7. If X is a nonempty set then $\Im$ is an ultra filter on X if for each filter $\Im^*$ on X, $\Im \subset \Im^*$ implies that $\Im^* = \Im$.

Example 2.2.14. If X is a nonempty set and $x \in X$ then the filter generated by the filter basis $\{\{x\}\}$ is an ultra filter denoted by $\Im_x$ and is called principal or elementary ultra filter.

Theorem 2.2.6. *If $X \neq \varnothing$ then a filter $\Im$ on X is an ultra filter iff for each nonempty subset A of X either $A \in \Im$ or $A^c \in \Im$.*

Proof: Suppose that $\Im$ is an ultra filter on X such that $A^c \notin \Im$ then by Theorem 2.2.5, there exists a filter $\Im^*$ on X finer than $\Im$ such that $A \in \Im^*$ but since $\Im$ is an ultra filter then $\Im = \Im^*$ and so $A \in \Im$.
Conversely suppose that $\Im$ is not ultra filter on X then there exists a filter $\Im^*$ on X such that $\Im \subset \Im^*$ and $\Im^* \neq \Im$ which implies that there is an element $A \in \Im^*$ such that $A \notin \Im$ which implies that $A^c \in \Im \Rightarrow A^c \in \Im^* \Rightarrow A \cap A^c = \emptyset \in \Im^*$ which contradicts Definition 2.2.5. Therefore if $\Im$ is not ultra filter on X then there exists a subset A of X such that $A \notin \Im$ and $A^c \notin \Im$ equivalently if $\Im$ is an ultra filter on X then for each nonempty subset A of X either $A \in \Im$ or $A^c \in \Im$.

Corollary 2.2.1. *If $\Im$ is a non principal ultra filter on X then $\{x\}^c \notin \Im$ for each point $x \in X$ since by Theorem 2.2.6, $\{x\} \in \Im$ implies that $\{x\}^c \notin \Im$ which implies in this case that $\Im = \Im_x$ is a principal ultra filter.*

Theorem 2.2.7. *If a filter $\Im$ on a nonempty set X has a finite element then it can not be ultra filter except when it is principal.*

Proof: the proof is omitted.

Remark 2.2.5. *As a direct consequence of Theorem 2.2.7, a non principal ultra filter has no finite elements.*

Now we arrive to the Theorem which because of it we gave the glance about the filters.

Theorem 2.2.8. *If $X \neq \emptyset$, $z \in X$ is any point and $\Im$ is an ultra filter on X then $\tau = E_z \cup \Im$ is an ultra topology on X.*

Proof: We proved in Theorem 2.2.3, that $\tau = E_z \cup \Im$ is a topology on X. To prove that it is an ultra topology, let $\tau^* \subset P(X)$ be such that $\tau \subset \tau^* \subset D$ and $\tau^* \neq \tau$. Then

$$G \in \tau^* - \tau \Rightarrow G \in \tau^*, G \notin \tau = E_z \cup \Im \Rightarrow G \notin E_z$$

and $G \notin \Im$. From which (1) $G \notin E_z \Rightarrow z \in G$ and (2) $G \notin \Im$ implies by Theorem 2.2.4, that $G^c \in \Im$ which implies by the Definition 2.2.5(i) that $G^c \cup \{z\} \in \Im$ and so $G^c \cup \{z\} \in \tau^*$. Since $G \in \tau^*$ then $G \cap (G^c \cup \{z\}) = \{z\} \in \tau^*$. Therefore

$$\{\{x\} : x \in X - \{z\}\} \subset E_z \subset \tau \subset \tau^*$$

$$\Rightarrow \{\{x\} : x \in X\} \subset \tau^* \Rightarrow \tau^* = D$$

Therefore $\tau = E_z \cup \Im$ is an ultra topology on X .

Remark 2.2.6. *If $\Im$ is a principal ultra filter on X that is if there exists a point $y \in X$ such that $\Im = \Im_y$ then $\Im_y = P_y - \{\varnothing\}$ and $\tau = E_z \cup \Im = E_z \cup \Im_y = E_z \cup P_y - \{\varnothing\} = E_z \cup P_y = D_{yz}$ which proved by Theorem 2.2.2, that it is an ultra topology on X which is called a principal ultra topology on X . If $\Im$ is a non principal ultra filter on X then $\tau = E_z \cup \Im$ is called a non principal ultra topology on X .*

2.3. Open and closed sets.

Definition 2.3.1. Let (X, τ) be a topological space. Then the elements of τ are called open sets i.e. $G \in \tau$ means that G is open and G is open means that $G \in \tau$.

Remark 2.3.1. *Clearly from Definition 2.3.1, the open sets in a topological space (X, τ) satisfy the following properties*

(1) X and $\varnothing$ are open sets,

(2) The intersection of any two and so of any finite number open sets is an open set and

(3) The union of any number (arbitrary union) of open sets is open.

In fact these properties are the axioms (O1), (O2) and (O3) of the Definition 2.1.1.

Example 2.3.1. In the discrete topological space (X, D) each subset of X is open for example the singleton subsets $\{\{x\} : x \in X\}$ are a family of open sets since $D = P(X)$.

Example 2.3.2. Consider the usual topological space (R, U) we defined the open subset $G \subset R$ i.e. $G \in U$ if for each point $x \in G$ there are two real numbers $a, b \in R$ such that $x \in (a,b) \subset G$ and clearly the family $\beta = \{(a,b) : a, b \in R\}$ is a family of open sets.

Example 2.3.3. Consider the set $X = \{a,b,c,d\}$ and let

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c,d\}\}.$$

Then the elements of τ is the family of all open sets like $\{a\}, \{a,b\}$ and $\{a,c,d\}$ we find also $\{b,d\}, \{d\}, \{b\}, \{b,c\} \notin \tau$ which gives a family of subsets of X which are not open.

Definition 2.3.2. Let (X,τ) be a topological space and $F \subset X$. Then F is called a closed subset of X If F^c is an open set equivalently if there exists $G \in \tau$ such that $F^c = G$ or equivalently $G^c = F$.

Remark 2.3.2. *From Definition 2.3.2, if $G \subset X$ is an open set i.e. $G \in \tau$ then $(G^c)^c = G$ which implies that G^c is a closed set. Hence the open and closed sets are complements of one another. Therefore in any topological space (X,τ) the family of the closed sets i.e. the family of the complements of the members of the topology τ, through this book it will be denoted by τ_c. Hence $\tau_c = \{G^c : G \in \tau\}$. For example if $X = \{a,b,c,d\}$ and $\tau = \{X, \varnothing, \{a,b\}, \{c\}, \{c,d\}, \{a,b,c\}\}$ is a topology on X then $\tau_c = \{\varnothing, X, \{c,d\}, \{a,b,d\}, \{a,b\}, \{d\}\}$ is the family of all closed subsets of X and we remarks that $X, \varnothing, \{a,b\}, \{c,d\}$ are both open and closed sets at the same time and the sets $\{b,c,d\}, \{b,d\}, \{b,c\}\}$ are neither open nor closed sets.*

Example 2.3.4. Consider the right ray topological space (R,τ) where $\tau = \{(a,\infty) : a \in R\} \cup \{R, \varnothing\}$ the family of the closed sets is $\tau_c = \{(-\infty, a] : a \in R\} \cup \{\varnothing, R\}$.

Example 2.3.5. In any topological space (X,τ), $X, \varnothing \in \tau$ implies that $\varnothing, X \in \tau_c$ which means that X and $\varnothing$ are both open and closed sets.

Example 2.3.6. Let $X = \{a,b\}$ and let $a \in X$. Then

- in the particular point topology $P_a = \{X, \varnothing, \{a\}\}$ the subset $\{a\}$ is open but is not closed.
- in the excluded point topology $E_a = \{X, \varnothing, \{b\}\}$ the subset $\{b\}$ is open but is not closed.

Theorem 2.3.1. *In any topological space (X,τ) the family τ_c of the closed sets satisfy the following conditions:*
(C1) X and $\varnothing$ are closed sets.,
(C2) Arbitrary intersections of closed sets is a closed set and

(C3) The union of two and so of a finite number of closed sets is a closed set.

Proof: (C1) From Example 2.3.5, X and $\varnothing$ are two closed sets.

(C2) Let $\{F_i : i \in \Delta\} \subset \tau_c$ be a family of closed sets. Then $\{F_i^c : i \in \Delta\} \subset \tau$ is a family of open sets then

$$(\bigcap_{i \in \Delta} F_i)^c = (\bigcup_{i \in \Delta} F_i^c) \in \tau \Rightarrow (\bigcap_{i \in \Delta} F_i) \in \tau_c .$$

(C3) Let $F_1, F_2 \in \tau_c$. Then $F_1^c, F_2^c \in \tau$ and so

$$(F_1 \cup F_2)^c = (F_1^c \cap F_2^c) \in \tau \Rightarrow (F_1 \cup F_2) \in \tau_c .$$

Which complete the proof.

Remark 2.3.3. *Through this book in any topological space (X, τ) the subsets of X which are both open and closed are called clopen sets i.e. if $G \in (\tau \cap \tau_c)$ then G is called clopen set. For example*

(1) In any topological space (X, τ), X and $\varnothing$ are two clopen sets.

(2) In the discrete topological space (X, D), each subset of X is a clopen set since $D = P(X) = D_c$.

2.4. Convergence of sequences in topological spaces:

Definition 2.4.1. If (X, τ) is a topological space and $< x_n >$ is a sequence of points of X then the convergence of the real sequence one can defined the convergence of the sequence $< x_n >$ to a point $x \in X$ written $x_n \to x$ as $n \to \infty$ if each open set G containing x i.e. $x \in G$ contains all points of the sequence $< x_n >$ except a finite number of them i.e. there is a positive number $n_o \in N$ such that $n \geq n_o \Rightarrow x_n \in G$.

Example 2.4.1. In any topological space (X, τ) if a sequence $< x_n >$ is in the form $< x_n > = \{x_1, x_2, \dots\dots, x_n, p, p, p, \dots\dots\}$. Then it is convergent to the point $p \in X$ i.e. $x_n \to p$.

Example 2.4.2. Consider the co-finite topological space (X, C) where X is an infinite set and let $< x_n >$ be an infinite sequence of distinct points of X. If $x \in X$ and $G \in C$ such that $x \in G$ then G contains all elements except a finite number of $<x_n>$ which may be in G^c since G^c is finite. Therefore $<x_n>$ is convergent to each point $x \in X$. If X is finite then $C = D$ and the infinite sequence $<x_n>$ does not convergent to any point of X .

Example 2.4.3. Let X be an infinite set and τ be the co-countable topology on X i.e. $G \in \tau - \{\varnothing\}$ iff G^c is countable. Then the sequence $<x_n>$ of points of X converges to a point $p \in X$ iff $<x_n> = \{x_1, x_2, \ldots\ldots, x_n, p, p, p, \ldots\ldots\}$. Since first if $<x_n> = \{x_1, x_2, \ldots, x_n, p, p, p, \ldots\ldots\}$ then by Example 2.4.2, $x_n \to p$. Secondly If $G \subset X$ such that G^c contains only the elements of the sequence $<x_n>$ except the point p i.e. $G^c = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\} - \{p\}$ then G^c is countable and so $G \in \tau$ and $p \in G$. $x_n \to p$ implies by the definition that G contains all elements of $<x_n>$ except a finite number of elements and accordingly G^c is finite which implies that there exists a natural number $n_o \in N$ such that $G^c = \{x_1, x_2, \ldots\ldots, x_{n_o}\}$ and so $<x_n>$ will be of the form

$$<x_n> = \{x_1, x_2, \ldots, x_n, p, p, p, \ldots\ldots\}.$$

Exercise

1 – Construct all possible topologies on the set
(i) $X = \{a, b\}$ (There are four topologies on X).
(ii) $X = \{a, b, c\}$ (There are 29 topologies on X).

2 – Construct the largest number of topologies as possible on the given set
(1) $X = \{a, b, c, d\}$ (There are 355 topologies on X).
(2) $X = \{a, b, c, d, e\}$ (There are 6942 topologies on X).
(3) $X = \{a, b, c, d, e, g\}$ (There are 207527 topologies on X).

3 – Suppose that $\{a, b, c, d\} \subset X$ write a topology on X such that the union of all members of $\tau - \{X\}$ equals $\{a, b, c, d\}$.

4 – If $X \neq \emptyset$ and τ is a topology on X, prove that

(i) $\{\{x\} : x \in X\} \subset \tau$ iff $\tau = D$.

(ii) $Y \in \tau;\ \forall\ Y \subset X : |Y| = 2 \wedge |X| \geq 3 \Rightarrow \tau = D$

(iii) $Y \in \tau;\ \forall\ Y \subset X : |Y| = 3 \wedge |X| \geq 5 \Rightarrow \tau = D$

5 – Prove that if X is infinite and τ is a topology on X such that the infinite subsets of X are members of τ then $\tau = D$.

6 – If X is a finite set, prove that each of the co-finite and the co-countable topology coincides with the discrete topology D on X and show that for any set X,

$$G \subset H \subset X : G \in C\ (G \in \tau) \Rightarrow H \in C\ (H \in \tau).$$

7 – If $X = \{a, b, c, d\}$, write the topologies D, E_a, E_b, E_c, E_d, P_a, P_b, P_c and P_d.

8 – If $A \subset X$ such that $A \neq \emptyset$ and $\tau = \{G \subset X : G \cap A = \emptyset\} \cup \{X\}$ then prove that τ is a topology on X. If $A = \{p\}$ then what is the topology τ in this case?.

9 – If $A \subset X$ such that $A \neq \emptyset$ and $\tau = \{G \subset X : A \subset G\} \cup \{\emptyset\}$ then prove that τ is a topology on X. If $A = \{p\}$, what is the topology τ in this case ?.

10 – If X is an infinite set and $p, q \in X$, prove that

(i) $\tau = E_p \cup (P_q \cap C)$ is a topology on X,

(ii) $\tau = P_p \cup C$ is not a topology on X and

(iii) $\tau = P_p \cup (E_q \cap C)$ is not topology on X.

11 – Consider the set R of the real numbers and let $\tau \subset P(R)$ be such that each subset $G \subset R$ is an element of τ iff $x \in G \Rightarrow \exists\ a, b \in R : x \in [a, b) \subset G$, prove that τ is a topology on R. This topology is called the lower limit point topology on R.

12 – Consider the set R of the real numbers and let $\tau \subset P(R)$ be such that each subset $G \subset R$ is an element of τ iff $x \in G \Rightarrow \exists\ a, b \in R : x \in (a, b] \subset G$, prove that τ is a topology on R. This topology is called the upper limit point topology on R.

13 – Let (X, τ) be a topological space, Y be a nonempty set and $f : X \to Y$ be any function. Then show that $f^*(\tau) = \{V \subset Y : f^{-1}(V) \in \tau\}$ is a topology on Y.

14 – Let (Y, τ^*) be a topological space, X be a nonempty set and $f : X \to Y$ be any function, prove that $f^{-1}(\tau^*) = \{f^{-1}(V) : V \in \tau^*\}$ is topology on X.

15 – If $X \neq \varnothing$, compare the following topologies on X

(I) The co-finite topology C and the co-countable topology τ if

(a) X is an infinite set. (b) X is a finite set.

(ii) The excluding point topology E_p and the particular point topology P_q where $p, q \in X$ (a) $p = q$ (b) $p \neq q$.

(iii) The right rays topology τ on R and the usual topology U.

(iv) The co-finite topology C on R and the usual topology U.

16 – Let $X = \{a, b, c, d\}$, construct three distinct topologies τ_1, τ_2 and τ_3 on X such that

(a) $I, D \notin \{\tau_1, \tau_2, \tau_3\}$,

(b) $|\tau_i| \geq 4$ for each $i \in \{1, 2, 3\}$ where $\left|\tau_i\right|$ is the cardinal number of τ_i.

(c) $\tau_1 \subset \tau_2 \subset \tau_3$ such that the topologies τ_1 and τ_2 are not are not comparable .

(d) $\tau_1 \supset \tau_2 \cup \tau_3$ such that the topologies τ_2 and τ_3 are not comparable .

17 – Let (X, τ) be a topological space and $A \subset X$, prove that A is not open iff A^c is not closed.

18 – Let (X, τ) be a topological space, $U \in \tau$ and $F \in \tau_c$. Then prove that $U - F \in \tau$ and $F - U \in \tau_c$.

19 – Consider the co-finite topological space (X, C) and show that $\{\{x\} : x \in X\}$ is family of (a) closed sets if X is an infinite set and (b) clopen sets if X is a finite set.

20– Consider the usual topological space (R, U) and show that $\{\{x\} : x \in R\}$ is a family of closed sets and so each finite subset of R is a closed set.

21 – Describe the closed sets in the following topological spaces:

(a) (X, E_p) where $p \in X$. (b) (X, P_p) where $p \in X$.

(c) (R, τ) where τ is the left ray's topology on R.

22 – Let $X \neq \varnothing$ and $\Im \subset P(X)$ be such that

(i) $X, \emptyset \in \Im$, (ii) Arbitrary intersection of members of $\Im$ is a member of $\Im$ and (iii) The union of any two members of $\Im$ is a member of $\Im$. Prove that there exists a unique topology τ on X such that $\Im$ is the family of all closed sets with respect to the topology τ on X .

Chapter III

Some Concepts in Topological Spaces

Topics:

- The Closure of a set and neighborhood system.
- Limit points and some properties of the closure of sets.
- The minimal sets.
- Interior, Exterior and boundary points.

3.1. The Closure of a set and neighborhood system.

Definition 3.1.1. Let (X,τ) be a topological space and $A \subset X$. Then the closure of A denoted $\overline{A}$ and defined to be the intersection of the closed sets which contains A i.e.

$$\overline{A} = \bigcap\{F \in \tau_C : A \subset F\}.$$

Remark 3.1.1. *Directly from this definition*

(i) $\overline{A}$ *is a closed set since it is an intersection of closed sets,*

(ii) $A \subset \overline{A}$*,*

(iii) If $F \in \tau_C$ *i.e.* F *is a closed set then* $A \subset F \Rightarrow \overline{A} \subset F$ *i.e.* $\overline{A}$ *is the smallest closed set which contains* A *and*

$$\overline{A} = \bigcap\{F \in \tau_C : A \subset F\} = \bigcap\{G^c : G \in \tau \ \ and \ \ A \subset G^c\}$$

$$= \bigcap\{G^c : G \in \tau \ \ and \ \ G \cap A = \varnothing\}.$$

Example 3.1.1. Let $X = \{a,b,c\}$, $\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{c\}, \{a,c\}\}$. Then, $\tau_C = \{\varnothing, X, \{\mathrm{b,c}\}, \{c\}, \{\mathrm{a,b}\}, \{b\}\}$ and by using Remark 3.1.1, or the Definition 3.1.1, to gets

(i) $A = \{a\} \Rightarrow \overline{A} = \{a,b\}$ and

(ii) $A = \{b,c\} \Rightarrow \overline{A} = \{b,c\}\} = A$.

Example 3.1.2. Consider the discrete topological space (X, D) then by using Definition 3.1.1, or Remark 3.1.1, we find that $\overline{A} = A$ for any $A \subset X$ since A is the smallest closed set which contains A is A itself.

Theorem 3.1.1. *Let* (X,τ) *be a topological space and* $A \subset X$. *Then* A *is a closed set iff* $\overline{A} = A$ *i.e.* $A \in \tau_C \Leftrightarrow \overline{A} = A$.

Proof: If $\overline{A} = A$ then by the Definition 3.1.1, of $\overline{A}$, A is a closed set.

Conversely by Remark 3.1.1, (i) $A \subset \overline{A}$ and if $A \in \tau_c$ and since $A \subset A$ then by Remark 3.1.1 (iii) $\overline{A} \subset A$ and so $\overline{A} = A$.

Remark 3.1.2. *(a) In the part (ii) of Example 3.1.1, clearly by Theorem 3.1.1,*

$$\{b,c\} = A \in \tau_c \Rightarrow \overline{A} = A .$$

(b) In Example 3.1.2,

$$A \in D_c \Rightarrow \overline{A} = A ; \forall A \subset X .$$

Definition 3.1.2. Let (X,τ) be a topological space and $x \in X$. Then $N \subset X$ is called a neighborhood of the point x if there exists an open set G such that $x \in G \subset N$.

Remark 3.1.3. *Directly from this definition if $G \in \tau$ and $x \in G$ then G is a neighborhood of x called the open neighborhood of x .*

Remark 3.1.4. *From Remark 3.1.3, if $A \in \tau$ then it is a neighborhood of each of its points and conversely if $A \subset X$ is a neighborhood of each of its points therefore, for each point $x \in A$ there exists $G_x \in \tau$ such that $x \in G_x \subset A$ hence $A = \bigcup_{x \in A} G_x$ which implies that A is open. Therefore, $A \subset X$ is open iff it is a neighborhood of each of its points.*

Definition 3.1.3. Let (X,τ) be a topological space and $x \in X$. Then the family of all possible neighborhood of the point x is called the neighborhood system of the point x and is denoted by N_x .

$$N_x = \{N \subseteq X : \exists G \in \tau \text{ such that } x \in G \subseteq N\}.$$

Example 3.1.3. Consider the following topology on $X = \{a,b,c,d,e\}$,

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}.$$

Then,

$$\begin{aligned} N_c &= \{N \subseteq X : \exists G \in \tau \text{ such that } c \in G \subseteq N\} \\ &= \{N \subseteq X : \ \{a,c,d\} \subseteq N\} \\ &= \{X, \{a,c,d\}, \{a,c,d,e\}, \{a,b,c,d\}\}, \\ N_e &= \{N \subseteq X : \exists G \in \tau \text{ such that } e \in G \subseteq N\} \\ &= \{N \subseteq X : \ \{a,b,e\} \subseteq N\} \\ &= \{X, \{a,b,e\}, \{a,b,c,e\}, \{a,b,d,e\}\}. \end{aligned}$$

Remark 3.1.5. *In a topological space* (X,τ) *the neighborhood system* N_x *of the point* $x\in X$ *satisfies the following conditions:*

(1) $N_x\neq\varnothing$ *and* $x\in N$ *for each* $N\in N_x$ *.*

(2) The intersection of two members of N_x *is a member of* N_x *.*

(3) Any subset of X *contains a member of* N_x *is a member of* N_x *.*

In the following Theorem we employ the concept of the neighborhood system to define a topology on a nonempty set.

Theorem 3.1.2. *Let* $X\neq\varnothing$ *and for each point* $x\in X$ **,** N_x^* *be a family of subsets of* X *satisfies the following conditions:*

(a) $N_x^*\neq\varnothing$ *and* $x\in N$ *for each* $N\in N_x^*$*,*

(b) $N_1,N_2\in N_x^*\Rightarrow N_1\cap N_2\in N_x^*$*,*

(c) $N\subset G\subset X\wedge N\in N_x^*\Rightarrow G\in N_x^*$ *,*

(d) For each $N\in N_x^*$ *there exists* $G\in N_x^*$ *such that* $G\subset N$ *and* $G\in N_y^*$ *for each* $y\in G$ *.*

If $\tau=\{G\subset X:G\in N_x^*;\forall x\in G\}\cup\{\varnothing\}$ *then* τ *is a topology on* X *such that* N_x^* *is the neighborhood system of the point* x *with respect to the topology* τ *on* X *.*

Proof: (O1) From (a) and (c) $X\in N_x^*$ for each $x\in X$ and so $X\in\tau$ and by the definition $\varnothing\in\tau$.

(O2) Let $G_1,G_2\in\tau$. Then from (b)

$$x\in(G_1\cap G_2)\Rightarrow x\in G_1\in, x\in G_2\Rightarrow G_1,G_2\in N_x^*$$

$$\Rightarrow(G_1\cap G_2)\in N_x^*$$

Hence $(G_1\cap G_2)\in\tau$.

(O3) Let $\{G_i:i\in\Delta\}\subset\tau$. Then

$$x\in\bigcup_{i\in\Delta}G_i\Rightarrow\exists i_o\in\Delta:x\in G_{i_o}\wedge G_{i_o}\in\tau\Rightarrow G_{i_o}\in N_x^* .$$

But form (c) we find

$$G_{i_o}\subset\bigcup_{i\in\Delta}G_i\Rightarrow(\bigcup_{i\in\Delta}G_i)\in\tau .$$

Therefore from (O1), (O2) and (O3), τ is a topology on X ,

Now if N_x is a neighborhood system of the arbitrary point $x\in X$ with respect to the topology τ on X we gets

Firstly $N \in N_x \Rightarrow \exists G \in \tau : x \in G \subset N$ and according to (c) $G \in \tau \Rightarrow G \in N_x^* \Rightarrow N \in N_x^*$ which implies that

$$N_x \subset N_x^* \text{ ----}(I).$$

Secondly From (d)

$$N \in N_x^* \Rightarrow \exists G \in N_x^* : G \subset N, G \in N_y^*; \forall y \in G$$

and hence $G \in \tau$ and from the definition of τ and Remark 3.1.4, we find that $G \in N_x$. Hence from Remark 3.1.5 (3) $G \subset N \Rightarrow N \in N_x$ which implies that

$$N_x^* \subset N_x \text{ ----}(II).$$

Therefore $(I) \wedge (II) \Rightarrow N_x^* = N_x$.

3.2. limit points and some properties of the closure of sets.

Definition 3.2.1. Let (X, τ) be a topological space and $A \subset X$. Then $x \in X$ is a limit or accumulation point of A if each neighborhood O of x contains at least a point of A different from x i.e. $(O - \{x\}) \cap A \neq \varnothing$.

We can rewrite this definition equivalently in terms of the open sets as follows: The point x is a limit point of A if for each open set $G \in \tau$ containing x ,$(O - \{x\}) \cap A \neq \varnothing$.

One can easily prove that the two definitions are equivalent.

Remake 3.2.1. *If (X, τ) is a topological space and $A \subset X$ then the point x can not be a limit point of A if there exists an open set $G \in \tau$ such that $x \in G$ and $(O - \{x\}) \cap A = \varnothing$ equivalently $x \in G$ and either $G \cap A = \varnothing$ or $G \cap A = \{x\}$. From this remark clearly if $\{x\} \in \tau$ then x can not be a limit point of any subset A of X .*

Definition 3.2.2. If (X, τ) is a topological space and $A \subset X$, the set of all limit point of A is called the derived set of A denoted by A' that is if $x \in X$ then $x \in A'$ means that x is a limit point of A . From this definition

(1) $x \in A' \Leftrightarrow (G - \{x\}) \cap A \neq \varnothing; \forall G \in \tau : x \in G$ equivalently

(2) $x \notin A' \Leftrightarrow \exists G \in \tau : x \in G, (G - \{x\}) \cap A = \varnothing$ equivalently

$x\notin A'\Leftrightarrow\exists G\in\tau : x\in G$ and either $G\cap A=\varnothing$ or $G\cap A=\{x\}$.

Remark 3.2.2. *If $A\subset X$ and $G\in\tau$ then $G\cap A=\varnothing\Rightarrow G\cap A'=\varnothing$. For suppose that $G\cap A=\varnothing$ then*

$$x\in G\ \Rightarrow\ (G-\{x\})\cap A=\varnothing\Rightarrow x\notin A'\Rightarrow G\cap A'=\varnothing .$$

Example 3.2.1. Consider the topology

$$\tau=\{X,\varnothing,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$$

on $X=\{a,b,c,d,e\}$ and $A=\{a,b,d\}$, $B=\{a,b\}$, $C=\{c,e\}$. Then,

First, the derived set of A.

Members of X		Cluster points
$a\in X$	$X\in\tau$ and $a\in X$, $(X-\{a\})\bigcap A\neq\varnothing$	$a\notin A'$
	$\{a\}\in\tau$ and $a\in\{a\}$, $(\{a\}-\{a\})\bigcap A=\varnothing$	
	$\{a,c,d\}\in\tau$ and $a\in\{a,c,d\}$, $(\{a,c,d\}-\{a\})\bigcap A\neq\varnothing$	
$b\in X$	$X\in\tau$ and $b\in X$, $(X-\{b\})\bigcap A\neq\varnothing$	$b\in A'$
	$\{b,c,d,e\}\in\tau$ and $b\in\{b,c,d,e\}$, $(\{b,c,d,e\}-\{b\})\bigcap A\neq\varnothing$	
$c\in X$	$X\in\tau$ and $c\in X$, $(X-\{c\})\bigcap A\neq\varnothing$	$c\in A'$
	$\{c,d\}\in\tau$ and $c\in\{c,d\}$, $(\{c,d\}-\{c\})\bigcap A\neq\varnothing$	
	$\{a,c,d\}\in\tau$ and $c\in\{a,c,d\}$, $(\{a,c,d\}-\{c\})\bigcap A\neq\varnothing$	
	$\{b,c,d,e\}\in\tau$ and $c\in\{b,c,d,e\}$, $(\{b,c,d,e\}-\{c\})\bigcap A\neq\varnothing$	
$d\in X$	$X\in\tau$ and $d\in X$, $(X-\{d\})\bigcap A\neq\varnothing$	$d\notin A'$
	$\{c,d\}\in\tau$ and $d\in\{c,d\}$, $(\{c,d\}-\{d\})\bigcap A=\varnothing$	
	$\{a,c,d\}\in\tau$ and $d\in\{a,c,d\}$, $(\{a,c,d\}-\{d\})\bigcap A\neq\varnothing$	
	$\{b,c,d,e\}\in\tau$ and $d\in\{b,c,d,e\}$, $(\{b,c,d,e\}-\{d\})\bigcap A\neq\varnothing$	
$e\in X$	$X\in\tau$ and $e\in X$, $(X-\{e\})\bigcap A\neq\varnothing$	$e\in A'$
	$\{b,c,d,e\}\in\tau$ and $e\in\{b,c,d,e\}$, $(\{b,c,d,e\}-\{e\})\bigcap A\neq\varnothing$	
$\therefore A'=\{b,c,e\}$		

Second, the derived set of B.

Members of X		Cluster points
$a \in X$	$X \in \tau$ and $a \in X$, $(X - \{a\}) \cap B \neq \varnothing$	$a \notin B'$
	$\{a\} \in \tau$ and $a \in \{a\}$, $(\{a\} - \{a\}) \cap B = \varnothing$	
	$\{a,c,d\} \in \tau$ and $a \in \{a,c,d\}$, $(\{a,c,d\} - \{a\}) \cap B = \varnothing$	
$b \in X$	$X \in \tau$ and $b \in X$, $(X - \{b\}) \cap B \neq \varnothing$	$b \notin B'$
	$\{b,c,d,e\} \in \tau$ and $b \in \{b,c,d,e\}$, $(\{b,c,d,e\} - \{b\}) \cap B = \varnothing$	
$c \in X$	$X \in \tau$ and $c \in X$, $(X - \{c\}) \cap B \neq \varnothing$	$c \notin B'$
	$\{c,d\} \in \tau$ and $c \in \{c,d\}$, $(\{c,d\} - \{c\}) \cap B = \varnothing$	
	$\{a,c,d\} \in \tau$ and $c \in \{a,c,d\}$, $(\{a,c,d\} - \{c\}) \cap B \neq \varnothing$	
	$\{b,c,d,e\} \in \tau$ and $c \in \{b,c,d,e\}$, $(\{b,c,d,e\} - \{c\}) \cap B \neq \varnothing$	
$d \in X$	$X \in \tau$ and $d \in X$, $(X - \{d\}) \cap B \neq \varnothing$	$d \notin B'$
	$\{c,d\} \in \tau$ and $d \in \{c,d\}$, $(\{c,d\} - \{d\}) \cap B = \varnothing$	
	$\{a,c,d\} \in \tau$ and $d \in \{a,c,d\}$, $(\{a,c,d\} - \{d\}) \cap B \neq \varnothing$	
	$\{b,c,d,e\} \in \tau$ and $d \in \{b,c,d,e\}$, $(\{b,c,d,e\} - \{d\}) \cap A \neq \varnothing$	
$e \in X$	$X \in \tau$ and $e \in X$, $(X - \{e\}) \cap B \neq \varnothing$	$e \in B'$
	$\{b,c,d,e\} \in \tau$ and $e \in \{b,c,d,e\}$, $(\{b,c,d,e\} - \{e\}) \cap B \neq \varnothing$	
$\therefore B' = \{e\}$		

Third, the derived set of C.

Members of X		Cluster points
$a \in X$	$X \in \tau$ and $a \in X$, $(X - \{a\}) \cap C \neq \varnothing$	$a \notin C'$
	$\{a\} \in \tau$ and $a \in \{a\}$, $(\{a\} - \{a\}) \cap C = \varnothing$	
	$\{a,c,d\} \in \tau$ and $a \in \{a,c,d\}$, $(\{a,c,d\} - \{a\}) \cap C \neq \varnothing$	
$b \in X$	$X \in \tau$ and $b \in X$, $(X - \{b\}) \cap C \neq \varnothing$	$b \in C'$
	$\{b,c,d,e\} \in \tau$ and $b \in \{b,c,d,e\}$, $(\{b,c,d,e\} - \{b\}) \cap C \neq \varnothing$	
	$X \in \tau$ and $c \in X$, $(X - \{c\}) \cap C \neq \varnothing$	
	$\{c,d\} \in \tau$ and $c \in \{c,d\}$, $(\{c,d\} - \{c\}) \cap C = \varnothing$	

$c \in X$	$\{a,c,d\} \in \tau$ and $c \in \{a,c,d\}$, $(\{a,c,d\}-\{c\}) \cap C = \varnothing$	$c \notin C'$
	$\{b,c,d,e\} \in \tau$ and $c \in \{b,c,d,e\}$, $(\{b,c,d,e\}-\{c\}) \cap C \neq \varnothing$	
$d \in X$	$X \in \tau$ and $d \in X$, $(X-\{d\}) \cap C \neq \varnothing$	$d \in C'$
	$\{c,d\} \in \tau$ and $d \in \{c,d\}$, $(\{c,d\}-\{d\}) \cap C \neq \varnothing$	
	$\{a,c,d\} \in \tau$ and $d \in \{a,c,d\}$, $(\{a,c,d\}-\{d\}) \cap C \neq \varnothing$	
	$\{b,c,d,e\} \in \tau$ and $d \in \{b,c,d,e\}$, $(\{b,c,d,e\}-\{d\}) \cap C \neq \varnothing$	
$e \in X$	$X \in \tau$ and $e \in X$, $(X-\{e\}) \cap C \neq \varnothing$	$e \in C'$
	$\{b,c,d,e\} \in \tau$ and $e \in \{b,c,d,e\}$, $(\{b,c,d,e\}-\{e\}) \cap C \neq \varnothing$	
$\therefore C' = \{b,d,e\}$		

Example 3.2.2. Let X be a countable set and let $D = P(X)$. Then

$$\forall A \subseteq X,\ A' = \varnothing$$

Because, in the discrete topology on X every singleton set is open. So $\{p\}$ is an open neighborhood of p where $(\{p\}-\{p\}) \cap A = \varnothing$ for any subset $A \subseteq X$. Since this is true $\forall p \in X$, A has no cluster points. That is $A' = \varnothing$.

Example 3.2.3. Let $I = \{X, \varnothing\}$. Then,

subset	Derived set
$\varnothing$	$\varnothing$
$\{p\}$	$X-\{p\}$
$\{p,q\}$ or more points	X

Example 3.2.4. Let $P_p = \{G \subseteq X : p \in G\} \cup \{\varnothing\}$ and let $A \subseteq X$. Then,

$p \in A$	$p \notin A$
$p \in X\ \exists \{p\} \in P_p$ *such that* $(\{p\}-\{p\}) \cap A = \varnothing$ $\therefore p \notin A'$	$\forall x \in X\ \exists \{x,p\} \in P_p$ *such that* $(\{x,p\}-\{x\}) \cap A = \varnothing$ $\therefore x \notin A'$
$q \in X$ *and* $p \neq q$, then any open set containing q contains also p $\therefore q \in A'$	
So, $A' = X-\{p\}$	So, $A' = \varnothing$

Example 3.2.5. Let $E_p = \{G \subseteq X : p \notin G\} \bigcup \{X\}$ and let $\{p\} \neq A \subseteq X$. Then,

- for $p \in X$, then the only open neighborhood of p is the set X and $(X - \{p\}) \bigcap A \neq \varnothing$. So, $p \in A'$.
- for $q \in X$ *and* $p \neq q$, then $\exists \{q\} \in E_p$ such that $(\{q\} - \{q\}) \bigcap A = \varnothing$. So, $q \notin A'$.

$$\therefore A' = \{p\}.$$

Example 3.2.6. Let $\mho$ be the usual topology on R and let $A = [a,b)$, $B = (a,b)$, $C = (a,b]$, $D = [a,b]$, $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ where $a,b \in R$. Then,

subset	Derived set
$\varnothing$	$\varnothing$
A	D
B	D
C	D
D	D
E	$\{0\}$
Q	R
Z	$\varnothing$

Example 3.2.7. Let $\tau = \{E_a = (a,\infty) : a \in R\} \bigcup \{R, \varnothing\}$ be a topology on R and let $A = [a,b)$, $B = (a,b)$, $C = (a,b]$, $D = [a,b]$, where $a,b \in R$ Then,

subset	Derived set
A	$(-\infty, b]$
B	$(-\infty, b]$
C	$(-\infty, b]$
D	$(-\infty, b]$

Example 3.2.8. Let $\tau = \{A_n = \{n, n+1, n+2, n+3, \ldots\ldots\ldots\} : n \in N\} \bigcup \{\varnothing\}$ be a topology on N and let $A = \{1,2,4,6\}$, $B = \{4,13,28,37\}$. Then,

subset	Derived set
A	$\{1,2,3,4,5\}$
B	$\{1,2,3,......,36\}$
N	N

Theorem 3.2.1. *Let (X,τ) be a topological space and $A\subset X$. Then $A\cup A'$ is a closed set.*

Proof: we shall prove that $(A\cup A')^c$ is an open set, for

$$x\in(A\cup A')^c\Rightarrow x\notin(A\cup A')\Rightarrow x\notin A'\wedge x\notin A$$

Then $x\notin A'\Rightarrow\exists G\in\tau: x\in G\wedge(G-\{x\})\cap A=\varnothing$ and $x\notin A$ implies that $G\cap A=\varnothing$ which implies by Remark 3.2.2, that $G\cap A'=\varnothing$. Hence

$$\varnothing=(G\cap A)\cup(G\cap A')=G\cap(A\cup A')$$
$$\Rightarrow x\in G\subset(A\cup A')^c$$

This means that $(A\cup A')^c$ is a neighborhood of each point $x\in(A\cup A')^c$ and since x is arbitrary then $(A\cup A')^c$ is a neighborhood of each of its points and so it is an open set. Then $A\cup A'$ is a closed set.

Theorem 3.2.2. *Let (X,τ) be a topological space and let $A,B\subseteq X$. Then*

1) *if $A\subseteq B$, then $A'\subseteq B'$.*
2) $(A\bigcup B)'=A'\bigcup B'$.
3) $(A\bigcap B)'\subsetneq A'\bigcap B'$.
4) $\overline{A}=A\bigcup A'$.
5) *A is closed iff $A'\subseteq A$.*

Proof: 1) For each $p\in A'$, then $\forall$ open neighborhood G of p, $(G-\{p\})\bigcap A\neq\varnothing$. But $A\subseteq B$, then $(G-\{p\})\bigcap B\neq\varnothing$. So, $p\in B'$. Hence $A'\subseteq B'$.

2) It is clear that $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$. Then by using the property (1) we have

$$A' \subseteq (A \bigcup B)' \textit{ and } B' \subseteq (A \bigcup B)'.$$

Then, $A' \bigcup B' \subseteq (A \bigcup B)'$.

Conversely, let $p \in (A \bigcup B)'$. Then $\forall$ open neighborhood G of p, $(G-\{p\}) \bigcap (A \bigcup B) \neq \varnothing$. That is $(G-\{p\}) \bigcap A \neq \varnothing$ or $(G-\{p\}) \bigcap B \neq \varnothing$. Then $p \in A'$ or $p \in B'$. So $p \in A' \bigcup B'$. Hence $(A \bigcup B)' \subseteq A' \bigcup B'$.

$$\therefore (A \bigcup B)' = A' \bigcup B'.$$

3) It is clear that $A \bigcap B \subset A$ and $A \bigcap B \subset B$. Then by using the property (1) we have

$$(A \bigcap B)' \subset A' \textit{ and } (A \bigcap B)' \subset B'.$$

Then, $(A \bigcap B)' \subset A' \bigcap B'$.

Now to prove $(A \bigcap B)' \neq A' \bigcap B'$ we give the following example:

Example: Let $\tau = \{X, \varnothing, \{b\}, \{a,b\}\}$ be a topology on $X = \{a,b,c\}$, and let $A = \{a,c\}$, $B = \{b,c\}$. Then, $A' = \{c\}$, $B' = \{a,c\}$ and $(A \bigcap B)' = \{c\}' = \varnothing$. Hence

$$\varnothing = (A \bigcap B)' \neq A' \bigcap B' = \{c\}.$$

4) Let $p \in \overline{A}$. Since $A \subseteq \overline{A}$, we consider two cases: $p \in A$ or $p \notin A$

Suppose $p \in A$. So $p \in A \bigcup A'$. Now suppose $p \notin A$, then as $p \in \overline{A}$, $\forall$ open neighborhood G of $p, G \bigcap A \neq \varnothing$. Further, as $p \notin A$, G intersects A in some points different from p. So p. is a cluster point of A Thus, $p \in A'$. So $p \in A \bigcup A'$. Hence, in either case $\overline{A} \subseteq A \bigcup A'$.

Conversely, let $p \in A \bigcup A'$. Then $p \in A$ or $p \in A'$. Suppose $p \in A$. Then as $A \subseteq \overline{A}$, $p \in \overline{A}$. Now suppose $p \in A'$. Then p. is a cluster point of A. So $\forall$ open neighborhood G of p, $(G - \{p\}) \bigcap A \neq \varnothing$. That is $G \bigcap A \neq \varnothing$. So $p \in \overline{A}$. Hence, in either case, $A \bigcup A' \subseteq \overline{A}$.

$$\therefore \overline{A} = A \bigcup A'.$$

5) Suppose that A is closed, then $A = \overline{A} = A \bigcup A'$. So $A' \subseteq A$.

Conversely, suppose $A' \subseteq A$, then $A \bigcup A' \subseteq A \bigcup A = A$. That is, $\overline{A} \subseteq A$. Since $A \subseteq \overline{A}$, $A = \overline{A}$. Hence A. is closed.

Proposition 3.2.1. *Let* (X, τ) *be a topological space and* $A \subset X$. *Then*
(i) $x \in \overline{A} \Leftrightarrow G \cap A \neq \varnothing ; \forall G \in \tau : x \in G$. *Equivalently*

$$x \notin \overline{A} \Leftrightarrow \exists G \in \tau : x \in G \ \wedge \ G \cap A = \varnothing$$

(ii) If $G \cap A = \varnothing$ *then* $G \cap \overline{A} = \varnothing$.

Proof: (i) $x \in \overline{A} \Rightarrow x \in A \vee x \in A'$ and firstly, if $G \in \tau$ such that $x \in G$ then

(1) $x \in A \Rightarrow x \in G \cap A \Rightarrow G \cap A \neq \varnothing$.

(2) $x \in A' \Rightarrow (G - \{x\}) \cap A \neq \varnothing \Rightarrow G \cap A \neq \varnothing$.

Secondly, if $G \cap A \neq \varnothing$ for each $G \in \tau$ such that $x \in G$ then either $x \in A$ or $x \notin A$, $x \in A$ implies that $x \in \overline{A}$ and

$$x \notin A \Rightarrow G - \{x\} \cap A \neq \varnothing \Rightarrow x \in A' \Rightarrow x \in \overline{A}.$$

(ii) If $G \in \tau$ then by using Remark 3.2.2, and Theorem 3.2.2(4),

$$G \cap A = \varnothing \Rightarrow G \cap A' = \varnothing \Rightarrow (G \cap A) \cup (G \cap A') = \varnothing$$
$$\Rightarrow G \cap (A \cup A') = \varnothing \Rightarrow G \cap \overline{A} = \varnothing$$

Lemma 3.2.1. *Let* (X, τ) *be a topological space and* $A, B \subset X$. *Then*

$$A \subset B \Rightarrow \overline{A} \subset \overline{B}.$$

Proof: By using Theorem 3.2.2, we gets

$$A \subset B \Rightarrow A' \subset B' \Rightarrow A \cup A' \subset B \cup B' \Rightarrow \overline{A} \subset \overline{B}.$$

Theorem 3.2.3. *Let (X, τ) be a topological space and $A, B \subset X$. Then*

(1) $\overline{X} = X$, $\overline{\varnothing} = \varnothing$ and $\overline{\overline{A}} = \overline{A}$

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and so the closure of a finite number of subsets of X equals the union of their closures i.e. if $\{A_1, A_2, ..., A_n\} \subset P(X)$ then $\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ and

(3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and may not be equals i.e. may be $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Proof: (1) Since $X, \varnothing, \overline{A} \in \tau_c$ implies that $\overline{X} = X$, $\overline{\varnothing} = \varnothing$ and $\overline{\overline{A}} = \overline{A}$.

(2) By Proposition 3.2.1, firstly

$$\begin{aligned} x \notin \overline{A} \cup \overline{B} &\Rightarrow x \notin \overline{A} \wedge x \notin \overline{B} \Rightarrow \exists G_1, G_2 \in \tau : x \in G_1, x \in G_2, \\ &G_1 \cap A = \varnothing \wedge G_2 \cap B = \varnothing \Rightarrow (G_1 \cap G_2) \in \tau, \\ &x \in (G_1 \cap G_2) \wedge (G_1 \cap G_2) \cap (A \cup B) = \varnothing \\ &\qquad \Rightarrow x \notin \overline{A \cup B} \Rightarrow \overline{A \cup B} \subset \overline{A} \cup \overline{B} \text{ --}(I) \end{aligned}$$

Equivalently in a direct way

$$\left.\begin{array}{l} A \subset \overline{A} \\ B \subset \overline{B} \end{array}\right\} \Rightarrow A \cup B \subset \overline{A} \cup \overline{B} \Rightarrow \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}}$$

But $\overline{A}, \overline{B} \in \tau_c \Rightarrow \overline{A} \cup \overline{B} \in \tau_c$ and so by Theorem 3.1.1, $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ which implies that $\overline{A \cup B} \subset \overline{A} \cup \overline{B} \text{ --}(I)$.

Conversely,

$$x \notin \overline{A \cup B} \Rightarrow \exists G \in \tau : x \in G, \varnothing = G \cap (A \cup B) =$$

$$(G \cap A) \cup (G \cap B) \Rightarrow G \cap A = \varnothing \wedge G \cap B = \varnothing$$

$$\Rightarrow x \notin \overline{A} \wedge x \notin \overline{B} \Rightarrow x \notin \overline{A} \cup \overline{B}$$

$$\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B} \quad --(II)$$

Equivalently

$$\left.\begin{array}{l} A \subset A \cup B \\ B \subset A \cup B \end{array}\right\} \Rightarrow \left.\begin{array}{l} \overline{A} \subset \overline{A \cup B} \\ \overline{B \subset A \cup B} \end{array}\right\} \Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B} \quad --(II)$$

Therefore $(I) \wedge (II) \Rightarrow \overline{A \cup B} = \overline{A} \cup \overline{B}$.

(3)

$$x \in \overline{A \cap B} \Rightarrow G \cap (A \cap B) \neq \varnothing; \forall G \in \tau : x \in G$$

$$\Rightarrow G \cap (A \cap B) = (G \cap A) \cap (G \cap B) \neq \varnothing; \forall G \in \tau : x \in G$$

$$\Rightarrow (G \cap A) \neq \varnothing \wedge (G \cap B) \neq \varnothing; \forall G \in \tau : x \in G$$

$$\Rightarrow x \in \overline{A} \wedge x \in \overline{B} \Rightarrow x \in \overline{A} \cap \overline{B}$$

$$\Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B} \quad ----(I)$$

Equivalently

$$\left.\begin{array}{l} A \cap B \subset A \\ A \cap B \subset B \end{array}\right\} \Rightarrow \left.\begin{array}{l} \overline{A \cap B} \subset \overline{A} \\ \overline{A \cap B} \subset \overline{B} \end{array}\right\} \Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B}$$

Another proof:

$x \notin \overline{A} \cap \overline{B} \Rightarrow x \notin \overline{A} \vee x \notin \overline{B}$ from which

$$x \notin \overline{A} \Rightarrow \exists G \in \tau : x \in G \wedge G \cap A = \varnothing \Rightarrow G \cap (A \cap B) = \varnothing$$

$$\Rightarrow x \notin \overline{A \cap B} \quad ----(I)$$

and

$$x \notin \overline{B} \Rightarrow \exists H \in \tau : x \in H \wedge H \cap B = \varnothing \Rightarrow H \cap (A \cap B) = \varnothing$$

$$\Rightarrow x \notin \overline{A \cap B} \quad ----(II)$$

From (I) and (II), $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$

To prove that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ let $X = \{1,2,3,4,5\}$, $A = \{1,2,4\}$. $B = \{3\}$ and

$\tau = \{X, \varnothing, \{1,2\}, \{3\}, \{3,4,5\}, \{1,2,3\}\}$ then

$\tau_C = \{\varnothing, X, \{3,4,5\}, \{1,2,4,5\}, \{1,2\}, \{4,5\}\}$ and so

(1) $A\cap B=\emptyset\Rightarrow\overline{A\cap B}=\emptyset$

(2) $\overline{A}=\{1,2,4,5\}$ and $\overline{B}=\{3,4,5\}$ which implies that $\overline{A}\cap\overline{B}=\{4,5\}$.

Then from (1) and (2), $\overline{A\cap B}\neq\overline{A}\cap\overline{B}$ **.**

Remark 3.2.3. *If (X,τ) is a topological space and $\{A_i:i\in\Delta\}$ is a family of subsets of X then*

(1) $\bigcap\{\overline{A_i}:i\in\Delta\}\in\tau_c\Rightarrow\overline{\bigcap\{\overline{A_i}:i\in\Delta\}}=\bigcap\{\overline{A_i}:i\in\Delta\}$.

(2) $\bigcup\{\overline{A_i}:i\in\Delta\}\subset\overline{\bigcup\{A_i:i\in\Delta\}}$ *but not equal, they are equal if* Δ *is finite since for example in the co-finite topological space* (X,C), X *is infinite and* $A\subset X$ *is infinite such that* $A\neq X$ *then* $\bigcup\{\overline{\{x\}}:x\in A\}=A$ *while* $\overline{\bigcup\{\overline{\{x\}}:x\in A\}}=\overline{A}=X$ *.*

Definition 3.2.4. If (X,τ) is a topological space and $y,z\in X$ are two distinct points then, an interesting topology on X in terms of τ and the principal ultra topology D_{yz} will be defined as follows:

$$\tau_{yz}=\tau\cap D_{yz}=\tau\cap(E_z\cup P_y)=\{G\in\tau:z\notin G\vee y\in G\}.$$

From the definition we remarks that for each $G\in\tau$,

- $G\in\tau_{yz}$ iff $y\in G$ or $z\notin G$.
- $G\in\tau_{yz}:z\in G\Rightarrow y\in G$.

But

$$D_{yzc}=(E_z\cup P_y)_c=E_{zc}\cup P_{yc}=P_z\cup E_y=D_{zy}$$

$$\Rightarrow\tau_{yzc}=(\tau\cap D_{yz})_c=\tau_c\cap D_{yzc}=\tau_c\cap D_{zy}=\tau_{czy}$$

$$=\{F\in\tau_c:y\notin F\vee z\in F\}.$$

And we remarks that for each $F\in\tau_c$,

- $F\in\tau_{yzc}$ iff $z\in F$ or $y\notin F$.
- $F\in\tau_{yzc}:y\in F\Rightarrow z\in F$ **.**

If τ_1 and τ_2 are two topologies on a nonempty set X such that $(\tau_1\cup\tau_2)$ is a topology on X then

- $(\tau_1\cup\tau_2)_{yz}=(\tau_1\cup\tau_2)\cap D_{yz}=\tau_{1yz}\cup\tau_{2yz}$.
- $(\tau_1\cap\tau_2)_{yz}=\tau_{1yz}\cap\tau_{2yz}$.

Remark 3.2.4. *If τ and τ^* are two topologies on a nonempty set X and $A\subset X$ then $\tau^*\subset\tau\Rightarrow\overline{A}\subset\overline{A}^*$ where $\overline{A}^*$ is the closure of A w.r.t τ^**

accordingly $\overline{A} \subset \overline{A}yz$ *where* $\overline{A}yz$ *is the closure of* A *w.r.t. the topology* τ_{yz} .

Theorem 3.2.4. *Let* (X,τ) *be a topological space,* $y,z \in X$ *be two distinct points and* $A \subset X$ *. Then*

(a) $\overline{A}yz \subset \overline{A} \cup \overline{\{z\}}$

(b) $\overline{A}yz = \begin{cases} \overline{A}; & y \notin \overline{A} \vee z \in \overline{A} \\ \overline{A} \cup \overline{\{z\}}; & y \in \overline{A}. \end{cases}$

Proof: (a) From the definition of τ_{yz} and since $A \subset \overline{A} \cup \overline{\{z\}}$ then

$$z \in \overline{A} \cup \overline{\{z\}} \in \tau_c \Rightarrow \overline{A} \cup \overline{\{z\}} \in \tau_{yzc} \Rightarrow \overline{A}yz \subset \overline{A} \cup \overline{\{z\}}.$$

(b) From the definition of τ_{yzc} and since $\overline{A} \in \tau_c$ then

$$y \notin \overline{A} \vee z \in \overline{A} \Rightarrow \overline{A} \in \tau_{yzc} \Rightarrow \overline{A}yz \subset \overline{A}$$

And from Remark 3.2.4, $\overline{A}yz = \overline{A}$.

Secondly from Remark 3.2.4, $\overline{A} \subset \overline{A}yz$ and from the definition of τ_{yzc} ,

$$y \in \overline{A} \Rightarrow y \in \overline{A}yz \Rightarrow z \in \overline{A}yz \Rightarrow \overline{\{z\}} \subset \overline{\{z\}}_{yz} \subset \overline{A}yz$$

and so $\overline{A} \cup \overline{\{z\}} \subset \overline{A}yz \quad (I)$. From (a) and (I), $\overline{A}yz = \overline{A} \cup \overline{\{z\}}$.

Corollary 3.2.1. *From Theorem 3.2.4, directly we gets*

(i) $\overline{\{y\}}_{yz} = \overline{\{y\}} \cup \overline{\{z\}}$,

(ii) $y \notin \overline{\{x\}} \vee z \in \overline{\{x\}} \Rightarrow \overline{\{x\}}_{yz} = \overline{\{x\}}$

(iii) $y \in A \Rightarrow y \in \overline{A} \Rightarrow \overline{A}yz = \overline{A} \cup \overline{\{z\}}$.

3.3. The minimal sets:

Definition 3.3.1. If (X,τ) is a topological space and $x \in X$ then the set $\{x\}^\wedge = \bigcap\{G \in \tau : x \in G\}$ is called the minimal set at the point x with respect to the topology τ and it may and may not be an open set.

If $\{x\}^\wedge \in \tau$ then it is called the minimal open set at x and is denoted by M_x i.e. in such case $\{x\}^\wedge = M_x$.

Clearly $\{x\}^\wedge \subset G$ for each $G \in \tau$ such that $x \in G$. So $\{x\}^\wedge$ is defined to be such that $G \in \tau : x \in G \Rightarrow \{x\}^\wedge \subset G$.

Remark 3.3.1. *If* (X,τ) *is a topological space and* $A \subset X$ *then*

(1) $\left(\{x\}^\wedge - \{x\}\right) \cap A \neq \varnothing \Rightarrow x \in A'$ *but not conversely since for example in the co-finite topological space if* X *is infinite and* A *is infinite then* $\left(\{x\}^\wedge - \{x\}\right) \cap A = \left(\{x\} - \{x\}\right) \cap A = \varnothing$ *while* $x \in A'$ *for each* $x \in X$ **.**

(2) $\bar{A} = \bigcap\limits_{x \in A} \{U_x^c : U_x \cap A = \varnothing\} = [\bigcup\limits_{x \in A} \{U_x : U_x \cap A = \varnothing\}]^c$.

Definition 3.3.2. If (X, τ) is a topological space and $A \subset X$ then the set $A^\wedge = \bigcup\{\{x\}^\wedge : x \in A\}$ is called the minimal set at A with respect to the topology τ on X and it may and may not be open. Clearly $A \subset A^\wedge$, the following theorem explains why $A^\wedge$ is called the minimal set about A with respect to τ on X.

Theorem 3.3.1. *If* (X, τ) *is a topological space and* $A \subset X$ *then*

$$A^\wedge = \bigcap\{G \in \tau : A \subset G\}.$$

Proof: If (X, τ) is a topological space and $A \subset X$ **,** clearly if $G \in \tau$ such that $A \subset G$ then $\{x\}^\wedge \subset G$ for each $x \in A$ and therefore $A^\wedge = \bigcup\{\{x\}^\wedge : x \in A\} \subset G$ for each $G \in \tau$ such that $A \subset G$ .Hence

$$A^\wedge \subset \bigcap\{G \in \tau : A \subset G\} -- (I)$$

Conversely $t \notin A^\wedge$ implies that $t \notin \{x\}^\wedge$ for each $x \in A$ and so for each point $x \in A$ there exists $G_x \in \tau$ such that $x \in G_x$ and $t \notin G_x$ **.** Hence $G = \bigcup\{G_x : x \in A\} \Rightarrow G \in \tau, A \subset G$ and $t \notin G$ which implies that $t \notin \bigcap\{G \in \tau : A \subset G\}$ and therefore

$$\bigcap\{G \in \tau : A \subset G\} \subset A^\wedge -- (II)$$

From (I) and (II), $A^\wedge = \bigcap\{G \in \tau : A \subset G\}$.

According to this theorem we can say that $A^\wedge$ is quasi-open set regarding that $A^\wedge$ may and may not be open.

Remark 3.3.2. *If* (X, τ) *is a topological space and* $A \subset X$ *then*

(1) $A \subset A^\wedge$ *and* $A \in \tau \Rightarrow A^\wedge = A$ *and so* $\tau \subset P^\wedge(X)$ *where* $P^\wedge(X) = \{A^\wedge : A \subset X\}$.

(2) $A \subset G \Leftrightarrow A^\wedge \subset G$ *for each* $G \in \tau$.

(3) One can find a subset B *of* X *such that* $A \neq B$ *while* $A^\wedge = B^\wedge$*. For example let* $X = \{1,2,3,4,5,6,7\}$ *and*

$$\tau = \{X, \varnothing, \{1\}, \{1,2,3\}, \{1,2,3,4\}, \{5,6,7\}, \{1,5,6,7\}, \{1,2,3,5,6,7\}\}.$$

We finds that $\{1,2,3\} = \{2,3\}^\wedge = \{1,2\}^\wedge = \{1,3\}^\wedge$ *while* $\{1,2\}$*,* $\{1,3\}$ *and* $\{2,3\}$ *are distinct.*

(4) $x \in \{y\}^\wedge \Rightarrow \{x\}^\wedge \subset \{y\}^\wedge$ *for each* $x, y \in X$ *, since* $x \in \{y\}^\wedge$ *implies that if* $G \in \tau$ *such that* $y \in G$ *then* $x \in G$ *which implies that* $\{x\}^\wedge \subset G$ *which implies that* $\{x\}^\wedge \subset \{y\}^\wedge$.

(5) $y \in \{x\}^\wedge \Leftrightarrow x \in \overline{\{y\}}$ *for each* $x, y \in X$ *since,*

$$y \in X - \{x\} : y \in \{x\}^{\wedge} \Leftrightarrow G \cap \{y\} = \{y\}; \forall G \in \tau : x \in G$$
$$\Leftrightarrow G - \{x\} \cap \{y\} = \{y\}; \forall G \in \tau : x \in G \Leftrightarrow x \in \{y\}'$$
$$\Leftrightarrow x \in \overline{\{y\}} - \{y\}$$

As a direct consequence of (5),

(i) $\{x\}^{\wedge} = \{y \in X : x \in \overline{\{y\}}\}$.

(ii) $\overline{\{x\}} = \{y \in X : x \in \{y\}^{\wedge}\}$.

(iii) $\overline{\{x\}} = \overline{\{y\}}$ *iff* $\{x\}^{\wedge} = \{y\}^{\wedge}$ *for each* $x, y \in X$.

Theorem 3.3.2. *Let* (X, τ) *be a topological space and* $A, B \subset X$ *. Then*

(1) $X^{\wedge} = X$, $\varnothing^{\wedge} = \varnothing$ *and* $A^{\wedge\wedge} = A^{\wedge}$,

(2) $A \subset B \Rightarrow A^{\wedge} \subset B^{\wedge}$,

(3) $(\bigcup_{i\in\Delta} A_i)^{\wedge} = \bigcup_{i\in\Delta} A_i^{\wedge}$ *and*

(4) $(\bigcap_{i\in\Delta} A_i)^{\wedge} \subset \bigcap_{i\in\Delta} A_i^{\wedge}$ *but may be* $(\bigcap_{i\in\Delta} A_i)^{\wedge} \neq \bigcap_{i\in\Delta} A_i^{\wedge}$.

Proof: (1) Clearly $X, \varnothing \in \tau \Rightarrow X^{\wedge} = X \wedge \varnothing^{\wedge} = \varnothing$ and

$$A^{\wedge} \subset A^{\wedge\wedge} --(I)$$

Also, $x \notin A^{\wedge}$ implies that there exists $G \in \tau$ such that $A \subset G$ and $x \notin G$ but by Remark 3.3.2, $A \subset G \Rightarrow A^{\wedge} \subset G$ according to which $x \notin A^{\wedge\wedge}$ so

$$A^{\wedge\wedge} \subset A^{\wedge} --(II).$$

From (I) and (II), $A^{\wedge\wedge} = A^{\wedge}$.

(2) Clearly $A \subset B \Rightarrow \{G \in \tau : B \subset G\} \subset \{G \in \tau : A \subset G\}$ which implies that $A^{\wedge} \subset B^{\wedge}$ in another way if $x \notin B^{\wedge}$ then there exists $G \in \tau$ such that $B \subset G$ and $x \notin G$ but $A \subset B \Rightarrow A \subset G$ which implies that $x \notin A^{\wedge}$ and hence $A^{\wedge} \subset B^{\wedge}$.

(3) Firstly $A_i \subset \bigcup_{i\in\Delta} A_i \Rightarrow A_i^{\wedge} \subset (\bigcup_{i\in\Delta} A_i)^{\wedge}$ for each $i \in \Delta$ and so

$$\bigcup_{i\in\Delta} A_i^{\wedge} \subset (\bigcup_{i\in\Delta} A_i)^{\wedge} --(I).$$

Conversely if $x \notin \bigcup_{i\in\Delta} A_i^{\wedge}$ then $x \notin A_i^{\wedge}$ for each $i \in \Delta$ accordingly for each $i \in \Delta$ there exists $G_i \in \tau$ such that $A_i \subset G_i$ and $x \notin G_i$ and then $\bigcup_{i\in\Delta} G_i = G \in \tau$, $\bigcup_{i\in\Delta} A_i \subset G$ and $x \notin G$ which implies that $x \notin (\bigcup_{i\in\Delta} A_i)^{\wedge}$ and hence

$$(\bigcup_{i\in\Delta} A_i)^{\wedge} \subset \bigcup_{i\in\Delta} A_i^{\wedge} \text{ --}(II).$$

From (I) and (II), $(\bigcup_{i\in\Delta} A_i)^{\wedge} = \bigcup_{i\in\Delta} A_i^{\wedge}$.

(4) $\bigcap_{i\in\Delta} A_i \subset A_i \Rightarrow (\bigcap_{i\in\Delta} A_i)^{\wedge} \subset A_i^{\wedge}$ for each $\alpha \in \Delta$ then

$$(\bigcap_{i\in\Delta} A_i)^{\wedge} \subset \bigcap_{i\in\Delta} A_i^{\wedge}.$$

To prove the inequality we give two examples

Example: Consider the left rays topological space (R,τ) where R is the set of the real numbers and $\tau = \{(a,\infty): a \in R\} \cup \{R, \varnothing\}$ and let $A = Q$ and $B = Q^c \cup \{1,2\}$ where Q is the set of the rational numbers and Q^c is the set of the irrational numbers then

(1) $A^{\wedge} = B^{\wedge} = R$.

(2) $A \cap B = \{1,2\} \Rightarrow (A \cap B)^{\wedge} = \{1,2\}^{\wedge} = \{1\}^{\wedge} \cup \{2\}^{\wedge}$

$$= \{1\}^{\wedge} = \bigcap_{x\in R} \{(x,\infty): x < 1\} = [1,\infty)$$

From (1) and (2), $(A \cap B)^{\wedge} \neq A^{\wedge} \cap B^{\wedge}$.

Example: Let $X = \{1,2,3,4,5\}$, $A = \{1,2,5\}$, $B = \{1,2,4\}$ and

$$\tau = \{X, \varnothing, \{1\}, \{1,2\}, \{3\}, \{3,4\}, \{3,5\}, \{1,3\}, \{1,3,4\}, \{1,3,5\},$$
$$\{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, \{3,4,5\}, \{1,3,4,5\}\}$$

Then,

(a) $A^{\wedge} = \{1,2,3,5\}, B^{\wedge} = \{1,2,3,4\} \Rightarrow A^{\wedge} \cap B^{\wedge} = \{1,2,3\}$.

(b) $A \cap B = \{1,2\} \Rightarrow (A \cap B)^{\wedge} = \{1,2\}$.

From (a) and (b), $(A \cap B)^{\wedge} \neq A^{\wedge} \cap B^{\wedge}$.

Theorem 3.3.3. *If τ and τ^* are two topologies on a nonempty set X and $x \in X$ such that $\{x\}^{\wedge} \neq \{x\}^{*\wedge}$ then $\tau \neq \tau^*$ where $\{x\}^{*\wedge}$ is the minimal set at the point x with respect to τ^* but not conversely.*

Proof: If $t \in \{x\}^{*\wedge} - \{x\}^{\wedge}$ then there exists $G \in \tau$ such that $x \in G$ and $t \notin G$ which implies that $G \notin \tau^*$ since each open set containing x contains t since $t \in \{x\}^{*\wedge}$, Hence $\tau \neq \tau^*$.

To show that the converse need not be true let X be an infinite set and $p \in X$. Then consider the topological space (X,τ) where $\tau = E_p \cup C$ clearly $\tau \neq C$ while $\{x\}^{\wedge 1} = \{x\}^{\wedge 2} = \{x\}$ for each $x \in X$ where $\{x\}^{\wedge 1}$ and $\{x\}^{\wedge 2}$ are the minimal sets about the point x with respect to τ and C respectively.

Remark 3.3.3. *If* τ *and* τ^* *are two topologies on a nonempty set* X *and* $A \subset X$ *then* $\tau^* \subset \tau \Rightarrow A^\wedge \subset A^{\wedge^*}$ *where* $A^{\wedge^*}$ *is the minimal set about* A *with respect to* τ^* *specifically* $A^\wedge \subset A_{yz}^\wedge$ *where* $A_{yz}^\wedge$ *is the minimal set with respect to* $\tau_{yz} = \tau \cap D_{yz}$.

Theorem 3.3.4. *Let* (X, τ) *be a topological space,* $y, z \in X$ *be two distinct points and* $A \subset X$ *. Then,*

(a) $A_{yz}^\wedge \subset A^\wedge \cup \{y\}^\wedge$

(b) $A_{yz}^\wedge = \begin{cases} A^\wedge; & z \notin A^\wedge \vee y \in A^\wedge \\ A^\wedge \cup \{y\}^\wedge; & z \in A^\wedge \end{cases}$.

Proof: By Theorem 3.3.2, $A^\wedge \cup \{y\}^\wedge = (A \cup \{y\})^\wedge$ and from the definition of τ_{yz}

$$t \notin A^\wedge \cup \{y\}^\wedge \Rightarrow \exists H \in \tau : (A \cup \{y\}) \subset H, t \notin H$$

form which $y \in H \Rightarrow H \in \tau_{yz}$, $A \subset H$ and $t \notin H$ which implies that $t \notin A_{yz}^\wedge$ and so $A_{yz}^\wedge \subset A^\wedge \cup \{y\}^\wedge$.

Firstly (1) $z \notin A^\wedge$ implies that there is $U \in \tau$ such that $A \subset U$ and $z \notin U$ and $t \notin A^\wedge$ implies that there is $V \in \tau$ such that $A \subset V$ and $t \notin V$. From which $A \subset U \cap V$, $t \notin U \cap V$ and $z \notin U \cap V$. But from the definition of τ_{yz} , $z \notin U \cap V$ implies that $U \cap V \in \tau_{yz}$ which implies that $t \notin A_{yz}^\wedge$ i.e. $t \notin A^\wedge \Rightarrow t \notin A_{yz}^\wedge$ under the condition $z \notin A^\wedge$ which implies that $A_{yz}^\wedge \subset A^\wedge$. Then, $A_{yz}^\wedge = A^\wedge$.

(2) Let $y \in A^\wedge$. Then from the definition of τ_{yz} ,

$$G \in \tau : A \subset G \Rightarrow y \in G \Rightarrow G \in \tau_{yz}$$

$$\Rightarrow \{G \in \tau : A \subset G\} \subset \{G \in \tau_{yz} : A \subset G\} \Rightarrow A_{yz}^\wedge \subset A^\wedge$$

$$\Rightarrow A_{yz}^\wedge = A^\wedge$$

Secondly $z \in A^\wedge$ from which and from the definition of τ_{yz} , if $z \in G$ then $y \in G$ and we gets

$$G \in \tau_{yz} : A \subset G \Rightarrow G \in \tau : A \subset G \Rightarrow z \in G \Rightarrow y \in G$$

$$\Rightarrow \{y\}^\wedge \subset G \Rightarrow \{y\}^\wedge \subset A_{yz}^\wedge \Rightarrow A^\wedge \cup \{y\}^\wedge \subset A_{yz}^\wedge \quad (I)$$

From (I) and (a), $A_{yz}^\wedge = A^\wedge \cup \{y\}^\wedge$.

Remark 3.3.4. *If we remark precisely the definitions of* τ_c, τ_{yz}, τ_{yzc}, $\overline{A}$, $\overline{A}yz$, $A^{\wedge}$ *and* $A^{\wedge}_{yz}$ *one can introduced any of the Theorems 3.2.4, from one another simply by replace* y *by* z, $\overline{A}$ *by* $A^{\wedge}$ *and* $\overline{A}yz$ *by* $A^{\wedge}_{yz}$ *or conversely.*

Corollary 3.3.1. *Directly from Theorem 3.3.2,*

(1) $\{z\}^{\wedge}_{yz} = \{y\}^{\wedge} \cup \{z\}^{\wedge}$.

(2) $y \in \{x\}^{\wedge} \vee z \notin \{x\}^{\wedge} \Rightarrow \{x\}^{\wedge}_{yz} = \{x\}^{\wedge}$ *for each* $x \in X$.

(3) If $z \in A$ *then* $z \in A^{\wedge}$ *which implies that* $A^{\wedge}_{yz} = A^{\wedge} \cup \{y\}^{\wedge}$ *and* $y \in A$ *implies that* $y \in A^{\wedge}$ *implies by Theorem 3.3.2, that* $\{y\}^{\wedge}_{yz} = \{y\}^{\wedge}$.

3.4. Interior, Exterior and boundary.

Definition 3.4.1. Let (X,τ) be a topological space and let $A \subseteq X$. Then the interior of A is the set

$$\operatorname{int}(A) = \bigcup\{G : G \in \tau \ and \ G \subseteq A\},$$

that is, $\operatorname{int}(A)$ is the union of all open sets contained in A .

Remark 3.4.1.

- *As* $\operatorname{int}(A)$ *is an arbitrary union of open sets, it is open.*
- *As* $\operatorname{int}(A)$ *is the union of all open sets contained in* A *, it is the largest open set contained in* A *. So* $\operatorname{int}(A) \subseteq A$.
- *If* G *is an open set such that* $G \subseteq A$, *then* $G \subseteq \operatorname{int}(A)$.
- *The interior of* A *will be denoted* A^{o} *.*
- A *is an open set iff each point* $x \in A$ *is an interior point of* A *iff* A *is neighborhood of* x *for each point* $x \in A$ *.*

Definition 3.4.2. Let (X,τ) be a topological space and let $A \subseteq X$. Then the exterior of A is the set $ext(A) = X - \overline{A}$.

Notice $ext(A)$ is open as it is the complement of a closed set. Further

$$ext(A)=X-\overline{A} \Leftrightarrow X-ext(A)=X-\left(X-\overline{A}\right) \Leftrightarrow \overline{A}=X-ext(A).$$

Definition 3.4.3. Let (X,τ) be a topological space and let $A\subseteq X$. Then the boundary of A is the set $b(A)=X-\left[A^O \bigcup ext(A)\right]$.

Notice $A^O \bigcup ext(A)$ is a union of open sets, it is open. So $b(A)=X-\left[A^O \bigcup ext(A)\right]$ is closed set.

Theorem 3.4.1. *Let (X,τ) be a topological space and let $A\subseteq X$. Then, A is open iff $A=A^O$.*

Proof: Suppose A is open. Then, since $A\subseteq A$, A is open set contained in itself. so, $A\subseteq A^O$. Since $A^O\subseteq A$, $A^O=A$.

Conversely, suppose $A=A^O$. Then since A^O is an open set, A is open.

Theorem 3.4.2. *Let (X,τ) be a topological space and let $A\subseteq X$. Then*

$$b(A)=\overline{A}-A^O.$$

Proof:

$$\begin{aligned} b(A)&=X-\left[A^O \bigcup ext(A)\right]\\ &=\left[X-A^O\right]\bigcap\left[X-ext(A)\right]\\ &=\left[X-A^O\right]\bigcap\overline{A}.\\ &=\left[X\bigcap\overline{A}\right]-\left[A^O\bigcap\overline{A}\right]\\ &=\overline{A}-A^O. \end{aligned}$$

Example 3.4.1. Let $\tau=\{X,\varnothing,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$ be a topology on $X=\{a,b,c,d,e\}$, and let $A=\{b\}, B=\{a,c\}, C=\{b,d\}$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	X
A	$\varnothing$	$\{b,e\}$	$\{b,e\}$	$\{a,c,d\}$
B	$\{a\}$	X	$\{b,c,d,e\}$	$\varnothing$

C	$\varnothing$	$\{b,c,d,e\}$	$\{b,c,d,e\}$	$\{a\}$
X	X	X	$\varnothing$	$\varnothing$

Example 3.4.2. Let $I=\{R,\varnothing\}$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
$A\subset R$	$\varnothing$	R	R	$\varnothing$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.3. Let $D=P(R)$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
$A\subset R$	A	A	$\varnothing$	$R-A$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.4. Let $E_p=\{G\subseteq R : p\notin G\}\bigcup\{R\}$. Then,

- Each $G_1\subset R,\, p\in G_1\Rightarrow G_1$ is not open.
- Each $G_1\subset R,\, p\in G_1\Rightarrow G_1$ is closed, because $p\notin R-G_1$ and so $R-G_1\in E_p$.
- Each $G_2\subset R,\, p\notin G_2\Rightarrow G_2$ is open.
- Each $G_2\subset R,\, p\notin G_2\Rightarrow G_2$ is not closed, because $p\in R-G_2$ and so $R-G_2\notin E_p$.

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
G_1	$G_1-\{p\}$	G_1	$\{p\}$	$R-G_1$
G_2	G_2	$G_2\bigcup\{p\}$	$\{p\}$	$R-\left[G_2\bigcup\{p\}\right]$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.5. Let $P_p=\{G\subseteq R : p\in G\}\bigcup\{\varnothing\}$. Then,

- Each $G_1\subset R,\, p\in G_1\Rightarrow G_1$ is open.
- Each $G_1\subset R,\, p\in G_1\Rightarrow G_1$ is not closed, because $p\notin R-G_1$ and so $R-G_1\notin P_p$.
- Each $G_2\subset R,\, p\notin G_2\Rightarrow G_2$ is not open.

- Each $G_2 \subset R,\ p \notin G_2 \Rightarrow G_2$ is closed, because $p \in R - G_2$ and so $R - G_2 \in P_p$.

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
G_1	G_1	R	$R - G_1$	$\varnothing$
G_2	$\varnothing$	G_2	G_2	$R - G_2$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.6. Let (X, C) be a co-finite topological space where $|X| \geq \aleph$. Then the only finite set which is open is $\varnothing$ and the only infinite set which is closed is X. For $A \subseteq X$ be finite and $B \subsetneq X$ be infinite

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	X
A	$\varnothing$	A	A	A^c
B	B	X	B^c	$\varnothing$
X	X	X	$\varnothing$	$\varnothing$

Example 3.4.7. Let $\mho$ be the usual topology on R and let $A = [a,b),\ B = (a,b),\ C = (a,b],\ D = [a,b],\ where\ \ a,b \in R$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
A	B	D	$\{a,b\}$	$(-\infty,a) \cup (b,\infty)$
B	B	D	$\{a,b\}$	$(-\infty,a) \cup (b,\infty)$
C	B	D	$\{a,b\}$	$(-\infty,a) \cup (b,\infty)$
D	B	D	$\{a,b\}$	$(-\infty,a) \cup (b,\infty)$
Q	$\varnothing$	R	R	$\varnothing$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.8. Let $\tau = \{E_a = (a,\infty) : a \in R\} \cup \{R, \varnothing\}$ be a topology on R and let $A = [a,b),\ B = (a,b),\ C = (a,b],\ D = [a,b],$ where $a,b \in R$ and $E = \{7,24,47,85\},\ F = \{3,6,9,....\}$ Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
A	$\varnothing$	$(-\infty,b]$	$(-\infty,b]$	(b,∞)
B	$\varnothing$	$(-\infty,b]$	$(-\infty,b]$	(b,∞)
C	$\varnothing$	$(-\infty,b]$	$(-\infty,b]$	(b,∞)

D	$\varnothing$	$(-\infty,b]$	$(-\infty,b]$	(b,∞)
E	$\varnothing$	$(-\infty,85]$	$(-\infty,85]$	$(85,\infty)$
F	$\varnothing$	R	R	$\varnothing$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.9. Let $\tau=\{E_a=(-\infty,a):a\in R\}\cup\{R,\varnothing\}$ be a topology on R and let $A=[a,b), B=(a,b), C=(a,b], D=[a,b],$ where $a,b\in R$ and $E=\{7,24,47,85\}, F=\{3,6,9,....\}$ Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	R
A	$\varnothing$	$[a,\infty)$	$[a,\infty)$	$(-\infty,a)$
B	$\varnothing$	$[a,\infty)$	$[a,\infty)$	$(-\infty,a)$
C	$\varnothing$	$[a,\infty)$	$[a,\infty)$	$(-\infty,a)$
D	$\varnothing$	$[a,\infty)$	$[a,\infty)$	$(-\infty,a)$
E	$\varnothing$	$[7,\infty)$	$[7,\infty)$	$(-\infty,7)$
F	$\varnothing$	$[3,\infty)$	$[3,\infty)$	$(-\infty,3)$
R	R	R	$\varnothing$	$\varnothing$

Example 3.4.10. Let $\tau=\{A_n=\{1,2,3,.......,n\}:n\in N\}\cup\{N,\varnothing\}$ be a topology on N, and let $A=\{1,2,4,6\}, B=\{5,7,9,20\}$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	N
A	$\{1,2\}$	$\mathbb{N}$	$\{3,4,5,......\}$	$\varnothing$
B	$\varnothing$	$\{5,6,7,......\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$
N	N	N	$\varnothing$	$\varnothing$

Example 3.4.11. Let

$$\tau=\{A_n=\{n,n+1,n+2,n+3,.........\}:n\in N\}\cup\{\varnothing\}$$

be a topology on N, and let $A=\{7,24,47,85\}, B=\{3,6,9,......\}$. Then,

subset	interior	closure	boundary	exterior
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	N
A	$\varnothing$	$\{1,2,3,......,85\}$	$\{1,2,3,......,85\}$	$\{86,87,88,........\}$
B	$\varnothing$	N	N	$\varnothing$

N	N	N	$\varnothing$	$\varnothing$

Theorem 3.4.3. *If* (X,τ) *is a topological space and* $A\subset X$ *then*

(1) $(A^{O})^{O}=A^{O}$, $X^{O}=X$, $\varnothing^{O}=\varnothing$.

(2) $A\subset B\Rightarrow A^{O}\subset B^{O}$.

(3) $(A\cap B)^{O}=A^{O}\cap B^{O}$.

(4) $(A\cup B)^{O}\subset A^{O}\cup B^{O}$ *and* $(A\cup B)^{O}\neq A^{O}\cup B^{O}$.

Proof: We prove only (3) and show in (4) that $(A\cup B)^{O}$ may be not equals to $A^{O}\cup B^{O}$.

(3)

$$\begin{aligned} x\in(A\cap B)^{O} &\Rightarrow \exists G\in\tau : x\in G\subset A\cap B \\ &\Rightarrow x\in G\subset A \;\wedge\; x\in G\subset B \\ &\Rightarrow x\in A^{O} \;\wedge\; x\in B^{O}\Rightarrow x\in(A^{O}\cap B^{O}) \\ &\Rightarrow (A\cap B)^{O}\subset A^{O}\cap B^{O} ----(I) \end{aligned}$$

Another proof: (Contra positive way)

$$x\notin A^{O}\cap B^{O}\Rightarrow x\notin A^{O}\vee x\notin B^{O}$$

But

$$x\notin A^{O}\Rightarrow G\not\subset A;\forall G\in\tau : x\in G\Rightarrow x\in G\not\subset A\cap B$$

$$\Rightarrow x\notin(A\cap B)^{O}$$

Similarly $x\notin B^{O}\Rightarrow x\notin(A\cap B)^{O}$. Hence

$$x\notin A^{O}\cap B^{O}\Rightarrow x\notin(A\cap B)^{O}$$

which implies that $(A\cap B)^{O}\subset A^{O}\cap B^{O}$

Third way to proof: From (2) of this Theorem,

$$\left.\begin{array}{l} A\cap B\subset A \\ A\cap B\subset B \end{array}\right\}\Rightarrow \left.\begin{array}{l} (A\cap B)^{O}\subset A^{O} \\ (A\cap B)^{O}\subset B^{0} \end{array}\right\}\Rightarrow (A\cap B)^{O}\subset A^{O}\cap B^{O}.$$

Conversely

$$x \in A^o \cap B^o \Rightarrow x \in A^o \wedge x \in B^o$$
$$\Rightarrow \exists G_1, G_2 \in \tau : x \in G_1 \subset A \wedge x \in G_2 \subset B$$
$$\Rightarrow G_1 \cap G_2 = G \in \tau \wedge x \in G \subset (A \cap B)$$
$$\Rightarrow x \in (A \cap B)^o \Rightarrow A^o \cap B^o \subset (A \cap B)^o - (II)$$

Another proof: (contra positive way)

$$x \notin (A \cap B)^o \Rightarrow \exists G \in \tau : x \in G \wedge G \not\subset A \cap B$$
$$\Rightarrow G \cap (A \cap B)^c \neq \varnothing \Rightarrow G \cap (A^c \cup B^c) \neq \varnothing$$
$$\Rightarrow (G \cap A^c) \cup (G \cap B^c) \neq \varnothing$$
$$\Rightarrow (G \cap A^c) \neq \varnothing \vee (G \cap B^c) \neq \varnothing \Rightarrow G \not\subset A \vee G \not\subset B$$
$$\Rightarrow x \notin A^o \vee x \notin B^o \Rightarrow x \notin (A^o \cap B^o)$$
$$\Rightarrow (A^o \cap B^o) \subset (A \cap B)^o$$

Third way of the proof:

$$\left.\begin{array}{l} A^o \subset A \\ B^o \subset B \end{array}\right\} \Rightarrow A^o \cap B^o \subset A \cap B \Rightarrow (A^o \cap B^o)^o \subset (A \cap B)^o$$

But,

$$A^o, B^o \in \tau \Rightarrow (A^o \cap B^o) \in \tau \Rightarrow (A^o \cap B^o)^o = A^o \cap B^o .$$

Then $(A^o \cap B^o) \subset (A \cap B)^o$.

To prove that $(A \cup B)^o \neq A^o \cup B^o$, let $X = \{1,2,3,4,5\}$, $A = \{1,4\}$, $B = \{2,3\}$ and

$$\tau = \{X, \varnothing, \{1\}, \{1,2\}, \{3\}, \{3,4,5\}, \{1,3\}, \{1,3,4,5\}, \{1,2,3\}\}.$$

Then,

(1) $A^o = \{1\} \wedge B^o = \{3\} \Rightarrow A^o \cup B^o = \{1,3\}$.

(2) $A \cup B = \{1,2,3,4\} \Rightarrow (A \cup B)^o = \{1,2,3\}$.

From (1) and (2), $(A \cup B)^o \neq A^o \cup B^o$.

Another example: Consider the usual topological space $(R, \mho)$ and let $A = [0,1]$ and $B = [1,2]$. Then,

(1) $A^o = (0,1) \wedge B^o = (1,2) \Rightarrow A^o \cup B^o = (0,1) \cup (1,2)$

$\Rightarrow A^o \cup B^o = (0,2) - \{1\}$.

(2) $A \cup B = [0,2] \Rightarrow (A \cup B)^o = (0,2)$.

From (1) and (2), $(A \cup B)^o \neq A^o \cup B^o$.

Theorem 3.4.4. *If* (X,τ) *is a topological space and* $A,B\subset X$ *then,*

(1) $ext(X)=\varnothing$ *and* $ext(\varnothing)=X$ *,*

(2) $ext((ext(A))^C)=ext(A)$*,*

(3) $A\subset B\Rightarrow ext(B)\subset ext(A)$*,*

(4) $ext(A\cup B)=ext(A)\cap ext(B)$ *and*

(5) $ext(A)\cup ext(B)\subset ext(A\cap B)$**.**

Proof: We shall prove (2) and (5) and we left the others to the reader

(2) $ext((ext(A))^C)=ext((\overline{A}^C)^C)=ext(\overline{A})=(\overline{\overline{A}})^C=\overline{A}^C=ext(A)$

equivalently

$$ext((ext(A))^C)=ext((A^{CO})^C)=((A^{CO})^C)^{CO}=(A^{CO})^O=A^{CO}=ext(A)$$

(5) By (3) $A\cap B\subset A\Rightarrow\overline{A\cap B}\subset\overline{A}\Rightarrow\overline{A}^C\subset(\overline{A\cap B})^C$ and $A\cap B\subset B\Rightarrow\overline{A\cap B}\subset\overline{B}\Rightarrow\overline{B}^C\subset(\overline{A\cap B})^C$ which implies that $\overline{A}^C\cup\overline{B}^C\subset(\overline{A\cap B})^C\Rightarrow ext(A)\cup ext(B)\subset ext(A\cap B)$. The inequality will be proved by the following example.

Example. Consider the topology in example given in Theorem 3.4.3, where $X=\{1,2,3,4,5\}$ and let $A=\{1,4\}$ and $B=\{2,4\}$ then

(1) $A^C=\{2,3,5\}\Rightarrow ext(A)=A^{CO}=\{3\}$,

(2) $B^C=\{1,3,5\}\Rightarrow ext(B)=B^{CO}=\{1,3\}$,

(3) $A\cup B=\{1,2,4\}\Rightarrow(A\cup B)^C=\{3,5\}\Rightarrow ext(A\cup B)=(A\cup B)^{CO}=\{3\}$
$\Rightarrow ext(A)\cap ext(B)=ext(A\cup B)$

and

(4) $A\cap B=\{4\}\Rightarrow(A\cap B)^C=\{1,2,3,5\}\Rightarrow$

$$\Rightarrow ext(A\cap B)=(A\cap B)^{CO}=\{1,2,3\}$$

from which $ext(A\cap B)\neq ext(A)\cup ext(B)=\{1,3\}$.

Example. Consider the usual topology $(R,\mho)$ then $ext(Z)=Z^C$ and $ext(Q)=\varnothing$ where

$$Z^C=\bigcup\{(n,n+1):n\in Z\}\in U\Rightarrow Z^{CO}=Z^C$$

and

$$Q\cap(a,b)\neq\varnothing;\forall a,b\in R\Rightarrow\overline{Q}=R\Rightarrow ext(Q)=\overline{Q}^C=\varnothing.$$

Remark 3.4.2. *If* (X,τ) *is a topological space and* $A \subset X$ *then* $x \in X$ *is a boundary point of* A *if*

$$G \in \tau : x \in G \Rightarrow G \cap A \neq \varnothing \wedge G \cap A^{c} \neq \varnothing$$

The set of all boundary points of a set A *denoted by* $b(A)$ *i.e.*

$$x \in b(A) \Leftrightarrow G \cap A \neq \varnothing \wedge G \cap A^{c} \neq \varnothing ; \forall G \in \tau : x \in G$$

Equivalently

$$x \notin b(A) \Leftrightarrow \exists G \in \tau : x \in G,\ G \cap A = \varnothing \vee G \cap A^{c} = \varnothing$$

Theorem 3.4.5. *(i) If* $x \notin A$ *then* $x \in b(A) \Leftrightarrow x \in A'$

(ii) If $x \in A$ *then* $x \notin b(A) \Leftrightarrow x \in A^{o}$*.*

(iii) $A' - A = b(A)$ *and* $A^{o} = A - b(A)$*.*

Proof: Suppose that (X,τ) is a topological space and $A \subset X$ then,

(i) If $x \notin A$ then

$$x \in b(A) \Rightarrow G \cap A \neq \varnothing ; \forall G \in \tau : x \in G,$$

which implies that $(G - \{x\}) \cap A \neq \varnothing$ for each $G \in \tau$ such that $x \in G$ because $x \notin A$ and so $x \in A'$.

Conversely $x \in A' \Rightarrow \left(G - \{x\}\right) \cap A \neq \varnothing ; \forall G \in \tau : x \in G$

$$\Rightarrow G \cap A \neq \varnothing ; \forall G \in \tau : x \in G \text{ --}(I)$$

Also

$$x \notin A \Rightarrow x \in G \cap A^{c} \neq \varnothing ; \forall G \in \tau : x \in G$$

$$\Rightarrow G \cap A^{c} \neq \varnothing ; \forall G \in \tau : x \in G \text{ ---}(II)$$

Therefore from (I) and (II), $x \in b(A)$.

(ii) If $x \in A$ then,

$$x \notin b(A) \Leftrightarrow \exists G \in \tau : x \in G,\ G \cap A = \varnothing \vee G \cap A^{c} = \varnothing$$

and

$$x \in A \Rightarrow G \cap A \neq \varnothing \Rightarrow G \cap A^{c} = \varnothing \Rightarrow x \in G \subset A \Rightarrow x \in A^{o}$$

Conversely

$$x \in A^{o} \Rightarrow \exists G \in \tau : x \in G \subset A \Rightarrow G \cap A^{c} = \varnothing \Rightarrow x \notin b(A).$$

Theorem 3.4.6. *If* (X,τ) *is a topological space and* $A \subset X$ *then,*

$$b(A) = \overline{A} \cap \overline{A^{c}} = (ext(A) \cup A^{o})^{c}.$$

Proof: From the definition of $b(A)$ and Proposition 3.2.1,

$$x \in b(A) \Leftrightarrow G \cap A \neq \emptyset \wedge G \cap A^c \neq \emptyset ; \forall G \in \tau : x \in G$$
$$\Leftrightarrow x \in \overline{A} \wedge x \in \overline{A^c} \Leftrightarrow x \in \overline{A} \cap \overline{A^c}$$
$$\Leftrightarrow b(A) = \overline{A} \cap \overline{A^c}$$

Secondly,

$$b(A) = \overline{A} \cap \overline{A^c} = (\overline{A}^c \cup \overline{A^c}^c)^c = (ext(A) \cup ext(A^c))^c$$
$$= (ext(A) \cup A^o)^c$$

Remark 3.4.4. *If* $G \in \tau$ *and* $A \subset X$ *then*

(1) $G \cap A = \emptyset \Rightarrow G \cap A' = \emptyset \Rightarrow (G \cap A') \cup (G \cap A) = \emptyset \Rightarrow$
$\Rightarrow G \cap \overline{A} = \emptyset$.

(2) $G \cap A = \emptyset \Rightarrow x \notin b(A); \forall x \in G \Rightarrow G \cap b(A) = \emptyset$.

also from (1)

$$G \cap A = \emptyset \Rightarrow G \cap \overline{A} = \emptyset \Rightarrow G \cap (\overline{A} \cap \overline{A^c}) = \emptyset$$
$$\Rightarrow G \cap b(A) = \emptyset.$$

Theorem 3.4.7. *If* (X, τ) *is a topological space and* $A \subset X$ *then,*

$$\overline{A} = A \cup b(A) = A^o \cup b(A).$$

Proof: From Theorem 3.4.5(3),

$$\overline{A} = A \cup A' = A \cup (A' - A) = A \cup (b(A) - A) = A \cup b(A).$$

Secondly also from Theorem 3.4.5(3),

$$\overline{A} = A \cup A' = A \cup (A' - A) = A \cup (b(A) - A)$$
$$= (A - b(A)) \cup (A \cap b(A)) \cup (b(A) - A)$$
$$= A^o \cup [(A \cap b(A) \cup (b(A) - A)]$$
$$= A^o \cup b(A)$$

Remarking that $b(A)) = (A \cap b(A)) \cup (b(A) - A)$.

Another way:

$$A^o \cup b(A) = A^o \cup (\overline{A} \cap \overline{A^c}) = (A^o \cup \overline{A}) \cap (A^o \cup \overline{A^c})$$
$$= \overline{A} \cap (A^o \cup A^{oc}) = \overline{A} \cap X = \overline{A}$$

Remarking that $A^o \subset \overline{A}$ and $A^o = \overline{(A^c)}^c$.

Definition 3.4.4. If (X, τ) is a topological space and $A \subset X$ then,

(1) $x \in X$ is called an isolated point of the topological space X if $\{x\} \in \tau$.

(2) $x \in A$ is called an isolated point of A if there is an open set $G \in \tau$ such that $G \cap A = \{x\}$.

The set of all isolated points of A will be denoted by $isd(A)$ i.e.

$$x \in isd(A) \Leftrightarrow \exists G \in \tau : G \cap A = \{x\}.$$

Example 3.4.12. Consider the discrete topological space (X, D) we finds that each point $x \in X$ is an isolated point that is $isd(X) = X$ since $\{\{x\} : x \in X\} \subset D$.

Example 3.4.13. Consider the usual topological space (R, U) then, $isd(N) = N$ and $isd(R) = isd((a,b)) = \varnothing; \forall a, b \in R$.

Theorem 3.4.8. *If (X, τ) is a topological space and $A, B \subset X$ then,*

(1) $isd(\varnothing) = \varnothing$

(2) $isd(isd(A)) = isd(A)$.

(3) $isd(A) \cap isd(B) \subset isd(A \cap B)$ but not equal.

(4) $isd(A \cup B) \subset isd(A) \cup isd(B)$ but not equal.

Proof: (1) Directly from (2) of Definition 3.4.4.

(2)

$$x \in isd(isd(A)) \Rightarrow \exists G \in \tau : G \cap isd(A) = \{x\} \Rightarrow x \in isd(A)$$

$$\Rightarrow isd(isd(A)) \subset isd(A) \quad ----(I)$$

Now

$$x \in isd(A) \Rightarrow \exists H \in \tau : H \cap A = \{x\} \Rightarrow x \in H$$

But,

$$isd(A) \subset A \Rightarrow x \in H \cap isd(A) \subset H \cap A = \{x\}$$

$$\Rightarrow H \cap isd(A) = \{x\} \Rightarrow x \in isd(isd(A))$$

$$\Rightarrow isd(A) \subset isd(isd(A)) \quad ----(II)$$

Then, from (I) and (II) $isd(isd(A)) = isd(A)$.

(3)

$$x \in isd(A) \cap isd(B) \Rightarrow x \in isd(A) \wedge x \in isd(B)$$

$$\Rightarrow \exists G, H \in \tau : G \cap A = \{x\} \wedge H \cap B = \{x\}$$

$$\Rightarrow x \in G \cap H \in \tau \wedge (G \cap H) \cap (A \cap B) =$$
$$= ((G \cap H) \cap A) \cap ((G \cap H) \cap B) = \{x\}$$

$$\Rightarrow x \in isd(A \cap B) \Rightarrow isd(A) \cap isd(B) \subset isd(A \cap B)$$

(4)

$$x \in isd(A \cup B) \Rightarrow \exists G \in \tau : (G \cap A) \cup (G \cap B) =$$

$$= G \cap (A \cup B) = \{x\}$$

$$\Rightarrow G \cap A = \{x\} \vee G \cap B = \{x\}$$

$$\Rightarrow x \in isd(A) \vee x \in isd(B) \Rightarrow x \in isd(A) \cup isd(B)$$

$$\Rightarrow isd(A \cup B) \subset isd(A) \cup isd(B)$$

We shall prove the inequality in both (3) and (4) by the following example.

Example. Let $X = \{1,2,3,4,5\}$, $A = \{4,5\}$, $B = \{1,3,5\}$ and

$$\tau = \{X, \varnothing, \{1\}, \{1,2\}, \{3\}, \{3,4\}, \{1,2,5\}, \{1,3\}\{1,3,4\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}\}$$

Then,

$$isd(A) = \{4,5\}, isd(B) = \{1,3\} \Rightarrow isd(A) \cap isd(B) = \varnothing \wedge$$

$$isd(A) \cup isd(B) = \{1,3,4,5\} - - - - - (I)$$,

$isd(A \cap B) = \{5\}$ and $isd(A \cup B) = \{1,3\}$. Accordingly

$$isd(A \cap B) \neq isd(A) \cap isd(B) \wedge isd(A \cup B) \neq isd(A) \cup isd(B).$$

Definition 3.4.5. If (X, τ) is a topological space and $A \subset X$ then

(1) A is called dense in X if $\overline{A} = X$.

(2) A is called nowhere dense in X if $\overline{A}^{o} = \varnothing$.

(3) A is called an isolated set if $isd(A) = A$.

Example 3.4.14. Consider the usual topological space $(R, \mho)$ then

(1) $\overline{Q} = R$ that is Q is a dense subset of R.

(2) $Z' = \varnothing \Rightarrow \overline{Z} = Z \Rightarrow \overline{Z}^{o} = \varnothing$ that is Z is a nowhere dense in R. Also $isd(Z) = Z$ that is Z is an isolated subset of R. Remark that

$$n \in Z \Rightarrow (n-1, n+1) \cap Z = \{n\} \Rightarrow n \in isd(Z).$$

Example 3.4.15. Let $X = \{1,2,3,4,5\}$ and $\tau = \{X, \varnothing, \{1,2\}, \{1,2,3,4\}, \{5\}, \{1,2,5\}\}$. Then, $A = \{1,5\} \Rightarrow \overline{A} = X$ Also $isd(A) = A$ which means that A is dense in X and isolated subset of X.

Example 3.4.16. Let $I = \{X, \varnothing\}$ and let A be a nonempty proper subset of X Then,

subset	interior	closure	Interior (closure
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

A	$\varnothing$	X	X
X	X	X	X

So $\forall$ nonempty proper subset $A \subsetneq X$ is dense in X. Also, $\nexists$ nonempty subset $A \subseteq X$ such that $\left(\overline{A}\right)^{o} = \varnothing$. Hence there are no nowhere dense set in an indiscrete space.

Example 3.4.17. Let $|X| \leq \aleph$, let $D = P(X)$ and let A be a nonempty proper subset of X. Then,

subset	interior	closure	Interior (closure
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
A	A	A	A
X	X	X	X

So $\nexists$ a proper subset $A \subsetneq X$. Hence there are no dense subset in a discrete space. Also, $\nexists$ nonempty subset $A \subseteq X$ such that $\left(\overline{A}\right)^{o} = \varnothing$. Hence there are no nowhere dense set in a discrete space.

Example 3.4.18. Let $P_p = \{G \subseteq X : p \in G\} \cup \{\varnothing\}$ and let A and B be nonempty proper subsets of X such that $A, X - B \in P_p$. Then,

subset	interior	closure	Interior (closure
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
A	A	X	X
B	$\varnothing$	B	$\varnothing$
X	X	X	X

So $\forall$ nonempty proper subset $A \subsetneq X$ is dense in X. Also, So $\forall$ nonempty proper closed subset $B \subsetneq X$ is nowhere dense in X.

Example 3.4.19. Let $E_p = \{G \subseteq X : p \notin G\} \cup \{X\}$ and let A and B be nonempty proper subsets of X such that $A, X - B \in E_p$. Then,

subset	interior	closure	Interior (closure
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
A	A	$A \cup \{p\}$	A
B	$B - \{p\}$	B	$B \cup \{p\}$
$X - \{p\}$	$X - \{p\}$	X	$\varnothing$

X	X	X	X

The only dense subset of X is $X-\{p\}$. Also, So the only nonempty nowhere dense subset of X is the closed set $\{p\}$.

Theorem 3.4.9. *If (X,τ) is a topological space and $A\subset X$ then A is an isolated set iff $A\cap A'=\varnothing$.*

Proof: Suppose that A is an isolated set i.e. $isd(A)=A$ then

$$x\in A\Rightarrow x\in isd(A)\Rightarrow \exists G\in\tau: G\cap A=\{x\}\Rightarrow x\notin A'$$

$$\Rightarrow A\cap A'=\varnothing$$

Conversely suppose that $A\cap A'=\varnothing$ then

$$x\in A\Rightarrow x\notin A'\Rightarrow \exists G\in\tau: x\in G, G-\{x\}\cap A=\varnothing$$

$$\Rightarrow G\cap A=\{x\}\Rightarrow x\in isd(A)\Rightarrow isd(A)=A$$

Definition 3.4.6. The topological space (X,τ) is said to be separable if there is a countable subset A of X which is dense in X that is A is countable and $\overline{A}=X$. The set A is dense in itself if $A\subset A'$.

Example 3.4.20. Clearly from Example 3.4.14, that (R,U) is a separable space since the set of the rational numbers Q is dense in X and as we know Q is countable.

Theorem 3.4.7. *If (X,τ) is a topological space and $A\subset X$ then A is dense in X iff $G\cap A\neq\varnothing$ for each nonempty open set $G\in\tau-\{\varnothing\}$ i.e. $\overline{A}=X\Leftrightarrow G\cap A\neq\varnothing;\forall G\in\tau-\{\varnothing\}$.*

Proof: Clearly $\overline{A}=X\Leftrightarrow \overline{A}^{c}=\varnothing$ so we shall proves that

$\overline{A}^{c}=\varnothing\Leftrightarrow G\cap A\neq\varnothing;\forall G\in\tau-\{\varnothing\}$ which is valid since

$$\overline{A}^{c}=\varnothing\Leftrightarrow \overline{A}^{c}=\bigcup\{G\in\tau: G\cap A=\varnothing\}=\varnothing$$

$$\Leftrightarrow G\cap A\neq\varnothing;\forall G\in\tau-\{\varnothing\}$$

Exercise

1 – (a) Let $X=\{a,b,c,d,e\}$ and

$$\tau=\{X,\varnothing,\{a\},\{b,c\},\{b,c,d\},\{b,c,e\},\{a,b,c\},$$
$$\{a,b,c\,d\},\{a,b,c,e\},\{b,c,d,e\}\}.$$

Find, $A',\overline{A},A^{o},ext(A),b(A)$ and $isd(A)$ if

$(i)\ A=\{a,b\}$, $(ii)\ A=\{b,d,e\}$ and $(iii)\ A=\{a,e\}$.

(b) Consider the topological spaces (X,E_p) and (X,P_p) where $p\in X$. If $A\subset X$ find $A',\overline{A},A^o,ext(A),b(A)$ and $isd(A)$ if $(i)\ p\in A$ and $(ii)\ p\notin A$.

(c) Consider the topological spaces (X,D) and (X,I). If $A\subset X$ find $A',\overline{A},A^o,ext(A),b(A)$ and $isd(A)$.

2 – If (X,τ) is a topological space, $A\subset X$ and $p\in A$, show that

$(i)\ p\in A' \Leftrightarrow p\in(A-\{p\})' \Leftrightarrow p\in\overline{(A-\{p\})}$

$(ii)\ \{p\}\in\tau \Rightarrow p\notin A';\forall A\subset X \quad (iii)\quad p\notin\{p\}'$

3 – Consider the usual topological space (R,U), prove that

$(i)\ A'\in U_c;\forall A\subset R$ (Hint: Prove that $A''\subset A'$).

(ii) If $A\subset R$ is a finite set then, $A\in U_c$.

(iii) If $A\subset R$ and $G\in U$ such that $x\in G$ then $x\in A'$ implies that $G\cap A$ is an infinite set.

4 – Considers the left and the right rays topological spaces, and find $N',\overline{N},N^o,ext(N),b(N)$ and $isd(N)$. Is $N\in\tau_c$?.

5 – If (X,τ) is a topological space, $A\subset X$ then A is said to be complete if $A'=A$, prove that

(i) If A is complete then $isd(A)=\varnothing$.

$(ii)\ isd(A)=\varnothing\Rightarrow A\subset A'$.

$(iii)\ A$ is complete iff $A\in\tau_c$ and $isd(A)=\varnothing$.

6 – If (X,τ) is a topological space and $A,B\subset X$, prove that

$(i)\ (A\cup B)'=A'\cup B'$.

$(ii)\ (A\cap B)'\subset A'\cap B'$ but need not be equal.

7 – Prove all theorems in this section in different ways and give examples to verifying these theorems.

8 – Give an example of a topological space X and a subset $A\subset X$ such that $isd(A)\neq\phi$ and $isd(X)=\varnothing$.

9 – If (X,τ) is a topological space and $A,B\subset X$ prove that

$(i)\ A'\subset B\subset A\Rightarrow B\in\tau_c$.

$(ii)\ b(\overline{A})\subset b(A)$ and $b(A^o)\subset b(A)$

$(iii)\ b(A)\cap(A^o\cup ext(A))=\varnothing$. $(iv)\ isd(A)\cap(A'\cup ext(A))=\varnothing$.

$(v)\ ext(A)\cap(A^{o}\cup A')=\varnothing$. $(vi)\ ext(A)\cap\overline{A}'=\varnothing$

$(vii)\ ext(\overline{A})=ext(A)$. $(viii)\ b(A)=\overline{A}-A^{o}$.

$(ix)\ A^{oc}=\overline{A^{c}}$. $(x)\ b(A^{o})=b(\overline{A^{c}})$.

$(xi)\ b(ext(A))=b(\overline{A})$. $(xii)\ x\in A-isd(A)\Rightarrow x\in A'$.

$(xiii)\ \overline{\{x\}}\neq\overline{\{y\}}\Leftrightarrow x\notin\overline{\{y\}}\vee y\notin\overline{\{x\}};\forall x,y\in X$.

$(ixv)\ \overline{A}^{o}=\varnothing\Leftrightarrow\overline{\overline{A}^{c}}=X$.

$(xv)\ \{x\}\notin\tau\Leftrightarrow x\in X';\forall x\in X$.

$(xvi)\ \{x\}^{\wedge}-\{x\}\cap A\neq\varnothing\Rightarrow x\in A'$ but not conversely. Also, prove that the converse is valid if $\{x\}^{\wedge}\in\tau$

$(xvii)\ A\cap b(A)\neq\varnothing\Leftrightarrow A\notin\tau\Leftrightarrow A\cap A^{c\prime}\neq\varnothing$.

$(xviii)\ A^{c}\cap b(A)\neq\varnothing\Leftrightarrow A\notin\tau_{c}\Leftrightarrow A'\cap A^{c}\neq\varnothing$.

$(xvvi)\ A\in\tau\Leftrightarrow b(A)\subset A^{c}$. $(xvv)\ A\in\tau_{c}\Leftrightarrow b(A)\subset A$.

10 – If (X,τ) is a topological space and $A\subset X$, prove that A is a clopen set iff $b(A)=\varnothing$.

11 – If (X,τ) is a topological space, prove that

$(i)\ isd(X)=\varnothing\Leftrightarrow isd(G)=\varnothing;\forall G\in\tau$. $(ii)\ G\in\tau\Rightarrow isd(G)\in\tau$.

$(ii)\ isd(X)=X\Leftrightarrow\tau=D$.

12 – Prove that if (X,τ_{2}) is separable and $\tau_{1}\subset\tau_{2}$ where τ_{1} is a topology on X then (X,τ_{1}) is separable.

13 – Prove that the right rays topology τ on R is a separable space where $\tau=\{R,\varnothing,(a,\infty):a\in R\}$.and left rays topology τ on R is a separable space where $\tau=\{R,\varnothing,(-\infty,a):a\in R\}$

14 – Prove that (X,E_{p}) is not separable where X is an uncountable set otherwise it is separable.

15 – Prove that (X,P_{p}) where $p\in X$ is a separable for any set X.

Chapter IV

Bases and subbases

Topics:

- Bases.
- Subbasis and local bases.
- Classification of topological spaces.
- Topological subspaces.

In the definition of the usual topological space $(R,\mho)$ a subset G of R is a member of $\mho$ i.e. $G\in\mho$ if for each point $x\in G$ there is an open interval $(a,b)\subset R$ such that $x\in(a,b)\subset G$ equivalently $G\in\mho$ if it can be written as a union of open intervals. This fact is valid for all topological spaces that is in any topological space (X,τ), there is a subfamily β of τ such that any non empty member of τ can be written as a union of members of β, the family β is called a basis for the topology τ this means that the topology τ can be constructed if we know any of its bases. Many properties of the topological spaces can be described in terms of the members of the basis instead of the members of the topology which makes the basis of the topology more interesting concept. In this chapter we discuss and study the bases and subbasis of the topologies and their properties.

4.1. Bases.

Definition 4.1.1. Consider the topological space (X,τ), a subfamily β of τ i.e. $\beta\subset\tau$ is said to be a basis for the topology τ if it satisfies the condition

$$G\in\tau,\ x\in G\Rightarrow\exists B\in\beta: x\in B\subset G$$

Equivalently $\beta\subset\tau$ is a basis for the topology τ if can be express any non empty member of τ as a union of members of β that is if for each $G\in\tau-\{\varnothing\}$ there exists $\{B_i: i\in\Delta\}\subset\beta$ such that $G=\bigcup_{i\in\Delta}B_i$.

Remark 4.1.1. *If β is a basis for a topology τ on X then τ is the family of all possible unions of the members of β.*

Remark 4.1.2. *If (X,τ) is a topological space then $\beta_{\tau} = \tau$ is a basis for τ, called the trivial basis for τ. Then always there exists a basis for any topology τ on any nonempty set X.*

Example 4.1.1. Let $X=\{1,2,3,4,5\}$ and

$$\tau=\{X,\varnothing,\{1\},\{2,3\},\{1,4\},\{1,2,3,5\},\{1,2,3\},\{1,2,3,4\}\}$$

And consider the families

$\beta_1=\{\{1\},\{4,5\},\{2,3,5\},\{1,4\},\{1,2,3,5\}\}$,

$\beta_2=\{\{1\},\{2,3\},\{1,4\},\{1,2,3,5\}\}$,

$\beta_3=\{\{1\},\{2,3\},\{1,4\},\{1,2,3,5\},X\}$ and

$\beta_4=\{\{1\},\{2,3\},\{1,2,3\},X\}$.

Clearly β_2 and β_3 are bases for the given topology τ. The family β_1 is not a basis for τ since β_1 is not a subfamily of τ. The family β_4 is not a basis for the topology τ since $\{1,2,3,4\}\in\tau$ but it can not be written as a union of members of β_4 since $4\in\{1,2,3,4\}$ while

$$B\in\beta_4 : B\subset\{1,2,3,4\}\Rightarrow 4\notin B.$$

Example 4.1.2. Consider the topological space (X,τ), where $X=\{a,b,c,d\}$ and $\tau=\{X,\varnothing,\{a,b\},\{c,d\}\}$. Then,

(1) $\beta_1=\{\{a,b\},\{c,d\}\}$ is a basis for the topology τ.

(2) $\beta_2=\{X,\{a,b\},\{c,d\}\}$ is a basis for the topology τ.

(3) $\beta_3=\{X,\{a,b\}\}$ is not a basis for the topology τ. because $\{c,d\}\in\tau$ and $\{c,d\}$ can't be written as a union of members of β_3.

Example 4.1.3. If X is a nonempty set and D is the discrete topology on X then $\beta_o=\{\{x\}: x\in X\}$ is a basis for the topology D which is the weakest basis for D. For example If $X=\{a,b,c\}$ then $\beta_o=\{\{a\},\{b\},\{c\}\}$ is a basis for the topology D while $\beta_o-\{\{c\}\}=\{\{a\},\{b\}\}$ can not be a basis for D since for example $\{b,c\}\in D$ can not be written as a union of members of $\beta_o-\{\{c\}\}$ since $c\in\{b,c\}$ while $B\in\beta_o-\{\{c\}\}\Rightarrow c\notin B$. In such case β_o is called the minimal basis for the topology D on X , we remarks that if β is a basis for D then $\beta_O \subset \beta$.

Definition 4.1.2. If (X,τ) is a topological space then $\beta_O \subset \tau$ is said to be the minimal basis for τ if we delete just a member B of β_O the members $\beta_O - \{B\}$ can not be a basis for τ. Equivalently $\beta_O \subset \tau$ is called the minimal basis for the topology τ if $\beta_O \subset \beta$ for each basis β for τ.

Remark 4.1.3. *If* (X,τ) *is a topological space and* $\beta \subset \tau$ *is a basis for* τ *then* $\beta \subset \beta^* \subset \tau$ *implies that* β^* *is also a basis for* τ.

Remark 4.1.4. *Consider the usual topological space* $(R,\mho)$ *then the family* $\beta = \{(a,b); a,b \in R\}$ *is a basis for* $\mho$.

Theorem 4.1.1. *Let* (X,τ) *be a topological space and* β *be a basis for* τ *then the set* $G \subset X$ *is open iff*

$$x \in G \Rightarrow \exists B \in \beta : x \in B \subset G.$$

Proof: If $G \in \tau$ and β is a basis for τ then the condition given in the theorem is satisfied by the definition of the basis.

Conversely suppose that the condition is satisfied then

$$x \in G \Rightarrow \exists B_x \in \beta : x \in B_x \subset G \Rightarrow G = \bigcup\{B_x : x \in G\}$$

Since $\beta \subset \tau$ then G written as a union of members of τ and so $G \in \tau$.

A direct result of this theorem is, if β is a basis for a topology τ on a non empty set X then τ is the family of all possible unions of the members of β which will be denoted by $\tau(\beta)$ i.e. $\tau = \tau(\beta)$.

Theorem 4.1.2. *Let* β_1 *and* β_2 *be two bases for the topologies* τ_1 *and* τ_2 *on a non empty set* X *then* $\tau_1 \subset \tau_2$ *iff*

$$B_1 \in \beta_1 \wedge x \in B_1 \Rightarrow \exists B_2 \in \beta_2 : x \in B_2 \subset B_1.$$

Proof: Firstly, $\tau_1 \subset \tau_2 \Rightarrow \beta_1 \subset \tau_2$ from which if $B_1 \in \beta_1$ then $B_1 \in \tau_2$ and since β_2 is a basis for τ_2 then

$$x \in B_1 \Rightarrow \exists B_2 \in \beta_2 : x \in B_2 \subset B_1.$$

Hence the condition in the theorem is satisfied.

Conversely suppose that the condition is given, then β_1 is a basis for τ_1 and $G \in \tau_1$ implies that

$$x \in G \Rightarrow \exists B_1 \in \beta_1 : x \in B_1 \subset G$$

And from the given condition in the theorem we gets

$$x \in B_1 \Rightarrow \exists B_2 \in \beta_2 : x \in B_2 \subset B_1 \subset G$$

Therefore

$$G \in \tau_1 \wedge x \in G \Rightarrow \exists B_2 \in \beta_2 : x \in B_2 \subset G$$

And by Theorem 4.1.1, $G \in \tau_2$ that is $G \in \tau_1 \Rightarrow G \in \tau_2$ which implies that $\tau_1 \subset \tau_2$.

Remark 4.1.4. *If X is a nonempty set and $\beta \subset P(X)$, is β a basis for a topology τ on X ?. To answer this question we give the following example.*

Example 4.1.4. Let $X = \{a,b,c,d\}$, $\beta_1 = \{\{a\},\{b\},\{a,c\}\}$ and $\beta_2 = \{\{a,b\},\{b,c\},\{a,d\}\}$. If β_1 and β_2 are bases for two topologies on X then by Theorem 4.1.1, these topologies must be

$\tau_1 = \tau(\beta_1) = \{\{a\},\{b\},\{a,c\},\{a,b\},\{a,b,c\},\varnothing\}$ and

$\tau_2 = \tau(\beta_2) = \{\{a,b\},\{b,c\},\{a,d\},\{a,b,c\},\{a,b,d\},X,\varnothing\}$ Respectively but clearly neither τ_1 nor τ_2 is a topology on X .

Firstly, $X \notin \tau_1$ and so τ_1 fail to satisfies (O1) of the topology where the union of the members of β_1 not equals to X .

Secondly τ_2 fail to satisfy (O2) where $\{a,b\},\{b,c\} \in \tau_2$ while

$$\{a,b\} \cap \{b,c\} = \{b\} \notin \tau_2.$$

From this example clearly the answer of the question given in Remark 4.1.4, is negative and a family $\beta \subset P(X)$ must be satisfied some conditions to be a basis for a topology on X as it is given in the following theorem.

Theorem 4.1.3. *Let X be a non empty set and $\beta \subset P(X)$ then β form a basis for a unique topology on X if it satisfies the following two conditions:*

(i) *for each $x \in X$ there exists $B \in \beta$ such that $x \in B$ equivalently, the union of the members of β equals to X that is $X = \bigcup\{B : B \in \beta\}$*

(ii) $B_1, B_2 \in \beta, x \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \beta : x \in B_3 \subset B_1 \cap B_2$ *equivalently, for each two members $B_1, B_2 \in \beta$, $B_1 \cap B_2$ can be written as a union of members of β i.e.*

$$B_1, B_2 \in \beta \Rightarrow \exists \{B_i : i \in \Delta\} \subset \beta : B_1 \cap B_2 = \bigcup\{B_i : i \in \Delta\}$$

where Δ is an indexed set.

Proof: Firstly if β is a basis for the topology τ and $X \in \tau$ then X can be written as a union of members of β that is

$\bigcup\{B : B\in\beta\} = X$ i.e. β satisfies the condition (i) and if $B_1, B_2\in\beta$ then $B_1\cap B_2\in\tau$ which implies that

$$x\in B_1\cap B_2 \Rightarrow \exists\, B_3\in\beta : x\in B_3\subset B_1\cap B_2$$

Which means that β satisfies the condition (ii).

Conversely, suppose that β satisfies the conditions (i) and (ii) and let $\tau(\beta)$ be the family of all possible unions of the members of β i.e.

$$G\in\tau(\beta) \Leftrightarrow \exists\{B_i : i\in\Delta\}\subset\beta : G = \bigcup_{i\in\Delta} B_i$$

Where Δ is an indexed set. At the following we shall prove that $\tau(\beta)$ is a topology on X by proving that $\tau(\beta)$ satisfies (O1), (O2) and (O3) of the topology.

(O1) From the condition (i), $X = \bigcup\{B : B\in\beta\}$ from which $X\in\tau(\beta)$ while $\varnothing$ is the union of the members of the empty subfamily of β and so $\varnothing\in\tau(\beta)$. Hence $\tau(\beta)$ satisfies (O1).

(O2) suppose that $G_1, G_2\in\tau(\beta)$ then there are two subfamilies $\{B_i : i\in\Delta\}\subset\beta$ and $\{B_j : j\in\Lambda\}\subset\beta$ where Δ and Λ are two indexed sets, $G_1 = \bigcup_{i\in\Delta} B_i$ and $G_2 = \bigcup_{j\in\Lambda} B_j$ from which

$$G_1\cap G_2 = (\bigcup_{i\in\Delta} B_i)\cap(\bigcup_{j\in\Lambda} B_j) = \bigcup_{i\in\Delta,\, j\in\Lambda} (B_i\cap B_j)$$

From which and from the condition (i)

$$\begin{aligned} x\in G_1\cap G_2 &\Rightarrow \exists i\in\Delta,\, j\in\Lambda : x\in B_i\cap B_j\subset G_1\cap G_2 \\ &\Rightarrow \exists B_x\in\beta : x\in B_x\subset B_i\cap B_j\subset G_1\cap G_2 \\ &\Rightarrow G_1\cap G_2 = \bigcup\{B_x : x\in G_1\cap G_2\}\Rightarrow G_1\cap G_2\in\tau(\beta) \end{aligned}$$

Hence $\tau(\beta)$ satisfies (O2).

(O3) If $\{G_i : i\in\Delta\}\subset\tau(\beta)$ where Δ is an indexed set, then for each $i\in\Delta$ there exists $\Lambda(i)$ such that $\{B_j : j\in\Lambda(i)\}\subset\beta$ and $G_i = \bigcup_{j\in\Lambda(i)} B_j$ from which

$$\bigcup_{i\in\Delta} G_i = \bigcup_{i\in\Delta} (\bigcup_{j\in\Lambda(i)} B_j)$$

Which implies that $\bigcup_{i\in\Delta} G_i\in\tau(\beta)$. Hence $\tau(\beta)$ satisfies (O3). Therefore $\tau(\beta)$ is a topology on X and β is its basis. To prove that $\tau(\beta)$ is the unique topology on X which has the basis β, let τ be a topology on

X and β be its basis then the family of all possible unions of the members of β is a subfamily of τ i.e. $\tau(\beta) \subset \tau ---(I)$.

Conversely since β is a basis for τ so by the definition of the basis each member $G \in \tau$ can be written as a union of members of β that is

$$G \in \tau \Rightarrow \exists \{B_i : i \in \Delta\} \subset \beta : G = \bigcup \{B_i : i \in \Delta\} \Rightarrow G \in \tau(\beta)$$
$$\Rightarrow \tau \subset \tau(\beta) ---(II)$$

Therefore $\tau(\beta) = \tau$.

Example 4.1.5. If X is a nonempty set, $p \in X$, τ is a topology on X, β is a basis for τ and β_p is a basis for the topology E_p on X then $\tau^* = E_p \cup \tau$ may be not a topology on X while $\beta^* = \beta_p \cup \beta$ is a basis for a topology τ^* on X stronger than both τ and E_p.

Proof: Consider the set $X = \{1,2,3,4,5\}$ and let

$$\tau = \{X, \emptyset, \{1,2\}, \{3\}, \{3,4\}, \{1,2,5,\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}\}.$$

Then $\{4\} \in E_1$ and $\{1,2\} \in \tau$ while $\{1,2\} \cup \{4\} = \{1,2,4\} \notin E_1 \cup \tau$ which implies that $\tau^* = E_p \cup \tau$ is not a topology on X.

Clearly $\bigcup\{B : B \in \beta^*\} = X$ i.e. β^* satisfies the condition (i) Theorem 4.1.3, and if $B_1, B_2 \in \beta^*$ then either

(1) $p \notin B_1$ or $p \notin B_2$ which implies that $p \notin B_1 \cap B_2$ which implies that $x \in \{x\} \subset B_1 \cap B_2$ for each $x \in B_1 \cap B_2$ where $\{x\} \in \beta_p$ for each $x \in X - \{p\}$ or

(2) $p \in B_1 \cap B_2$ which means that $B_1, B_2 \in \beta$ or may be $X \in \{B_1, B_2\}$ and since β is a basis for τ, then there exists $B_3 \in \beta$ such that $p \in B_3 \subset B_1 \cap B_2$ and $x \in \{x\} \subset B_1 \cap B_2$ for each $x \in B_1 \cap B_2$ such that $x \neq p$.

(1) and (2) means that β^* satisfies the condition (ii) of Theorem 4.1.3, and so it is a basis for a topology $\tau(\beta^*)$ on X. Clearly $\tau(\beta^*)$ is stronger than both τ and E_p. Prove that $\tau^* = E_p \cup \tau$ is a basis for a topology $\tau(\tau^*)$ on X such that $\tau(\tau^*) = \tau(\beta^*)$.

Example 4.1.6. Consider the set R^2 of all points in the plane i.e. $R^2 = \{(x, y) : x, y \in R\}$ and let $S(p; r)$ be the set of all points of the plane inside the circle with center at $p \in R^2$ and radius r where

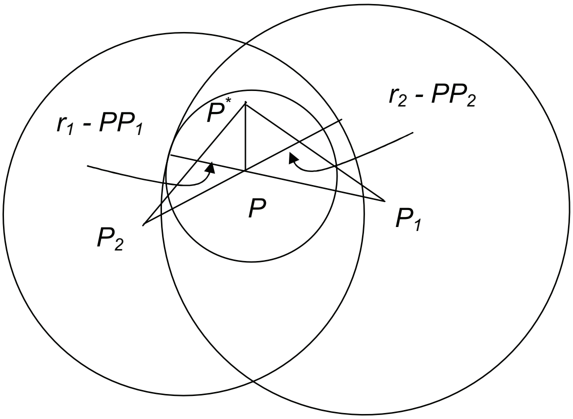
$r \in R^+ = (0,\infty)$, if $p_o \in R^2$ then $S(p_o;r)=\{p \in R^2 : \overline{pp_o} < r\}$ where $\overline{pp_o}$ is the distance between P and P_o then $\beta=\{S(p;r): p \in R^2, r \in R^+\}$ form a basis for a topology $\mho^2$ on R^2 called the usual topology on R^2 to prove this we shall prove that β satisfies the conditions (i) and (ii) of Theorem 4.1.3.

Firstly for each $r \in R^+$ and each $p \in R^2$, $p \in S(p;r) \in \beta$ i.e. β satisfies (i).

Secondly consider the circles $S(p_1;r_1)$ and $S(p_2;r_2)$ where $r_1, r_2 \in R^+$ and $p_1, p_2 \in R^2$ and let $r = \inf.\{r_1 - \overline{pp_1}, r_2 - \overline{pp_2}\}$ then $p \in S(p_1;r_1) \cap S(p_2,r_2) \Rightarrow S(p;r) \subset S(p_1;r_1) \cap S(p_2,r_2)$
as illustrated in the figure since

$$p^* \in S(p;r) \Rightarrow \overline{p^*p_1} \le \overline{pp^*} + \overline{pp_1} < r + \overline{pp_1} \le r_1 - \overline{pp_1} + \overline{pp_1}$$
$$= r_1 \Rightarrow p^* \in S(p_1;r_1) \Rightarrow S(p;r) \subset S(p_1;r_1)$$

Similarly one can show that $S(p;r) \subset S(p_2;r_2)$



Example 4.1.7. Let β^* be the family of all subsets of R^2 containing the points inside the rectangles in the plane with ribs parallel to the coordinate axes i.e. the sets of the form $(a,b)\times(c,d)$ where $a,b,c,d \in R$

. Then β^* form a basis for a topology τ^* on R^2, the proof left to the reader.

Theorem 4.1.4. *If X is a non empty set and $\sigma \subset P(X)$ then the family of all possible finite intersections of the members of σ denoted by $\beta(\sigma)$ form a basis for a topology on X denoted by $\tau(\beta(\sigma))$ and it is the weakest topology on X which contains σ.*

Proof: To prove that $\beta(\sigma)$ is a basis for a topology on X we shall prove that it satisfies the two conditions (i) and (ii) of Theorem 4.1.3.

Firstly X can be considered as the intersection of the members of the empty subfamily $\varnothing$ of σ and so $X \in \beta(\sigma)$ where

$$\bigcup\{A : A \in \varnothing\} = \varnothing \Rightarrow \bigcap\{X - A : A \in \varnothing\} = X \Rightarrow \bigcap\{A : A \in \varnothing\} = X$$

which implies that $\beta(\sigma)$ satisfies the condition (i).

Secondly If $B_1, B_2 \in \beta(\sigma)$ then from the definition of $\beta(\sigma)$ there are two families $\{U_1, U_2, \ldots\ldots, U_r\} \subset \sigma$ and $\{V_1, V_2, \ldots\ldots, V_k\} \subset \sigma$ Such that $B_1 = \bigcup_{i=1}^{r} U_i$ and $B_2 = \bigcup_{j=1}^{k} V_j$ and so there is a positive integer $n \in N$ such that $n \le r + k$ and

$$\{U_1, U_2, \ldots, U_r\} \cup \{V_1, V_2, \ldots, V_k\} = \{W_1, W_2, \ldots, W_n\}$$

for each W_m there exists either U_i such that $U_i = W_m$ or V_j such that $V_j = W_m$. Hence $B_1 \cap B_2 = \bigcap_{m=1}^{n} W_m$ this means that $B_1 \cap B_2 \in \beta(\sigma)$ and so $\beta(\sigma)$ satisfies the condition (ii). Therefore $\beta(\sigma)$ form a basis for a topology on X denoted by $\tau(\beta(\sigma))$ and is called the topology on X generated by the family σ i.e.

$$\tau(\beta(\sigma)) = \{G \subset X : G \text{ is a union of members of } \beta(\sigma)\}.$$

From the definition of $\beta(\sigma)$ and the definition of the topology if τ is a topology on X then

$$\sigma \subset \tau \Rightarrow \beta(\sigma) \subset \tau \Rightarrow \tau(\beta(\sigma)) \subset \tau$$

That is $\tau(\beta(\sigma))$ is the weakest topology on X which contains σ.

Example 4.1.8. Let $X = \{1,2,3,4,5\}$. Then

$$\sigma = \{\{1,2,\},\{2,3\},\{3,4,5\},\{2,4,5\}\}$$
$$\Rightarrow \beta(\sigma) = \{\{1,2\},\{2,3\},\{3,4,5\},\{2,4,5\},\{2\},\varnothing,\{3\},\{4,5\},X\}$$

$$\Rightarrow \tau(\beta(\sigma)) = \{X,\varnothing,\{1,2\},\{2\},\{3\},\{4,5\}\{1,2,3\}\{1,2,4,5\},$$
$$\{2,3\},\{2,4,5\},\{3,4,5\},\{2,3,4,5\}\}$$

Example 4.1.9. Let $X = \{1,2,3,4\}$. Then,

$$\sigma = \{\{1,2,\},\{2,3\},\{3,4,\},\{1,4,\}\}$$
$$\Rightarrow \beta(\sigma) = \{\{1,2\},\{2,3\},\{3,4,\},\{1,4,\},\{2\},\varnothing,\{1\},\{3\},\{4\},X\}$$

From which

$$\{\{1\},\{2\},\{3\},\{4\}\} \subset \beta(\sigma) \subset \tau(\beta(\sigma)) \Rightarrow D \subset \tau(\beta(\sigma)) \Rightarrow \tau(\beta(\sigma)) = D..$$

Hence the topology $\tau(\beta(\sigma))$ generated on X by σ is the discrete topology D.

4.2. Subbases and local bases:

Definition 4.2.1. Let (X,τ) be a topological space, $\sigma \subset \tau$ and $\beta(\sigma)$ be the family of all possible finite intersections of the members of σ. Then σ is called a subbasis for τ if $\beta(\sigma)$ form a basis for τ.

Example 4.2.1. Let $X = \{a,b,c,d\}$ and

$$\tau = \{X,\varnothing,\{a,b\},\{b\},\{b,c\},\{d\},\{a,b,c,\},\{a,b,d\},\{b,d\},\{b,c,d\}\}$$

Then $\sigma = \{\{a,b\},\{b,c\},\{d\}\}$ is a subbasis for the topology τ on X since $\beta(\sigma) = \{\{a,b\},\{b,c\},\{d\},\{b\},\varnothing,X\}$ is a basis for the topology τ.

Example 4.2.2. Let X be a nonempty set then the family $\sigma = \{\{x\}^C : x \in X\} \subset C$ is a subbasis for the co-finite topology on X since

$\beta(\sigma)=\{G\subset X : G^{c}$ is finite $\}\cup\{X,\varnothing\}$ form a basis for the topology C on X in fact $\beta(\sigma)=C$.

Example 4.2.3. Consider the usual topology (R,U) and let

$$\sigma=\{(-\infty,b),(a,\infty):a,b\in R,a<b\}.$$

Then

$$\begin{aligned} a,b\in R : a<b &\Rightarrow(-\infty,b)\cap(a,\infty)=(a,b)\\ &\Rightarrow \beta(\sigma)\supset\beta=\{(a,b):a,b\in R\}\end{aligned}$$

And since $\sigma\subset U$, β is a basis for U and from the definition of $\beta(\sigma)$ then $\beta(\sigma)\subset U$ then by Remark 4.1.3, $\beta(\sigma)$ is a basis for U. Accordingly σ is a subbasis for U.

Example 4.2.4. Consider the set R^{2} and let σ be the family of the subsets of R^{2} which consists all infinite tapes in the plane which are parallel to the coordinate's axes i.e. $\sigma=\{(a,b)\times R,\ R\times(c,d):a,b,c,d\in R\}$ as illustrated in the figure. Then

$$\begin{aligned} a,b,c,d\in R : a<b,c<d &\Rightarrow[(a,b)\times R]\cap[R\times(c,d)]\\ &=[(a,b)\cap R]\times[R\cap(c,d)]=(a,b)\times(c,d)\end{aligned}$$

That is the intersection of any two different tapes, $[(a,b)\times R]$ and $[R\times(c,d)]$ of the members of σ is an open rectangle with parallel sides to the coordinates axes which is a basis for $\mho$ by Example 4.1.7. Hence $\beta(\sigma)\supset\beta$ where β is the family of all open rectangles which is a basis for $\mho$. Then $\sigma\subset\mho\Rightarrow\beta(\sigma)\subset\mho$ which implies that $\beta(\sigma)$ is a basis for $\mho$ and so σ is a subbasis for $\mho$.

Theorem 4.2.1. *If X is a nonempty set then any non empty family σ of subsets of X form a subbasis of a unique topology on X weakest than any topology on X having σ as a subbasis.*

Proof: In Theorem 4.1.4, we proved that $\beta(\sigma)$ is a basis for a topology on X denoted by $\tau(\beta(\sigma))$ which is weakest than any topology contains

σ on X then σ is its subbasis. Clearly if σ is a subbasis for a topology τ on X then

$$\sigma \subset \tau \Rightarrow \tau(\beta(\sigma)) \subset \tau \;---(I)$$

and

$$G \in \tau \Rightarrow \exists \{B_i : i \in \Delta\} \subset \beta(\sigma) : G = \bigcup_{i \in \Delta} B_i \Rightarrow G \in \tau(\beta(\sigma))$$
$$\Rightarrow \tau \subset \tau(\beta(\sigma)) \;----(II)$$

From (I) and (II) $\tau = \tau(\beta(\sigma))$.

Definition 4.2.2. Let (X, τ) be a topological space and $x \in X$ be any point. Then $\beta_x \subset \tau$ is called a local basis for the point x if

(1) $x \in B$ for each $B \in \beta_x$.

(2) $G \in \tau : x \in G \Rightarrow \exists B \in \beta_x : B \subset G$.

Example 4.2.5. If (X, τ) is a topological space, β is a basis for τ and $x \in X$ then any of the families

(1) $\beta_x = \{G \in \tau : x \in G\}$ and (2) $\beta_x = \{B \in \beta : x \in B\}$ forms a local basis for the point x .

Example 4.2.6. Consider the excluding point topological space (X, E_p) then

(1) $\beta_x = \{\{x\}\}$ is a local basis for the point x for each point $x \in X - \{p\}$ and (2) $\beta_p = \{X\}$ is the local basis for p .

Example 4.2.7. Consider the usual topological space $(R, \mho)$, if $x \in R$ then the family $\beta_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n \in N\}$ is a local basis for the point x .

Proof: Clearly $\beta_x \subset \mho$ and if $G \in \mho$ and $x \in G$ then there are $a, b \in R$ such that $x \in (a, b) \subset G$ which implies that $a < x < b$ from which by Archimedes's axiom

$$a < x \Rightarrow x - a > 0 \Rightarrow \exists\, n_1 \in N : x - a > \frac{1}{n_1} \Rightarrow a < x - \frac{1}{n_1} \qquad (I)$$

and

$$x < b \Rightarrow b - x > 0 \Rightarrow \exists\, n_2 \in N : b - x > \frac{1}{n_2} \Rightarrow x + \frac{1}{n_2} < b \qquad (II)$$

Now let $n_o = \max\{n_1, n_2\}$. Then

$$n_o \geq n_1 \Rightarrow \frac{1}{n_0} \leq \frac{1}{n_1} \Rightarrow -\frac{1}{n_1} \leq -\frac{1}{n_0} \Rightarrow x - \frac{1}{n_1} \leq x - \frac{1}{n_0} \qquad (III)$$

and

$$n_o \geq n_2 \Rightarrow \frac{1}{n_0} \leq \frac{1}{n_2} \Rightarrow \Rightarrow x + \frac{1}{n_0} \leq x + \frac{1}{n_2} \qquad (IV)$$

From (I), (II), (III) and (IV),

$$a < x - \frac{1}{n_1} \leq x - \frac{1}{n_0} < x < x + \frac{1}{n_0} \leq x + \frac{1}{n_2} < b \Rightarrow$$

$$\Rightarrow a < x - \frac{1}{n_0} < x < x + \frac{1}{n_0} < b \Rightarrow x \in (x - \frac{1}{n_0}, x + \frac{1}{n_0}) \subset (a,b)$$

Then $x \in (x - \frac{1}{n_0}, x + \frac{1}{n_0}) \subset (a,b) \subset G$ that is there is a member $B \in \beta_x$ such that $x \in B \subset G$ where $B = (x - \frac{1}{n_0}, x + \frac{1}{n_0})$. Hence β_x is a subbasis for $\mho$.

Example 4.2.8. Consider the usual topological space $(R^2, \mho^2)$, if $x \in R^2$ then the family $\beta_x = \{S(x;r) : r \in R^+\}$ of all sets each of which is the set of the interior points of a circle with radius r and centered at the point x is a local basis for x.

Remark 4.2.1. *Let (X,τ) be a topological space, σ be a subbasis for τ, $x \in X$ and $\sigma_x = \{U \in \sigma : x \in U\}$. Then $\beta(\sigma_x)$ the family of all finite* *intersections of the members of* σ_x *forms a local basis for the point* x *, generally if* $\sigma_x \subset \tau$ *and* $\beta(\sigma_x)$ *is a local basis for* x *then* σ_x *is called a local subbasis for the point* x *.*

Remark 4.2.2. *If* (X,τ) *is a topological space,* $A \subset X$ *,* $x \in X$ *and* β_x *is a local basis for the point* x *then*

$$x \in A' \Leftrightarrow \left(B-\{x\}\right)\cap A \neq \varnothing ; \forall B \in \beta_x$$

Since firstly if $x \in A'$ *then*

$$B \in \beta_x \Rightarrow B \in \tau \Rightarrow \left(B-\{x\}\right)\cap A \neq \varnothing$$

Secondly if $\left(B-\{x\}\right)\cap A \neq \varnothing$ *for each* $B \in \beta_x$ *then*

$$\begin{aligned} G \in \tau : x \in G &\Rightarrow \exists B \in \beta_x : B \subset G \\ &\Rightarrow \varnothing \neq \left(B-\{x\}\right)\cap A \subset \left(G-\{x\}\right)\cap A \\ &\Rightarrow \left(G-\{x\}\right)\cap A \neq \varnothing \Rightarrow x \in A' \end{aligned}$$

4.3. Classification of topological spaces:

Definition 4.3.1. If (X,τ) is a topological space and $x \in X$, then M_x is called the minimal open set about or at the point $x \in X$ if

(1) $M_x \in \tau$,

(2) $x \in M_x$ and

(3) $G \in \tau : x \in G \Rightarrow M_x \subset G$

i.e, M_x is an open set contains the point x and contained in each open set containing x .

Remark 4.3.1. $M_x = \bigcap\{G \in \tau : x \in G\} = \{x\}^{\wedge}$ *for each point* $x \in X$ *.*

Remark 4.3.2. *If* (X,τ) *is a topological space,* $x \in X$ *and the family* $\{G \in \tau : x \in G\} \subset \tau$ *is infinite then* $\{x\}^{\wedge} = \bigcap\{G \in \tau : x \in G\}$ *may be not open in such case* M_x *does not exist. For example consider the co-finite topological space* (X,C) *where* X *is an infinite set then for each point* $x \in X$ *,*

$$x \in \{x\}^{\wedge} = \bigcap\{G \in \tau : x \in G\} \subset \bigcap\{\{y\}^{c} : y \in \{x\}^{c}\} = \{x\}$$

$$\Rightarrow \{x\}^{\wedge} = \{x\} \notin C$$

which means that there is no minimal open set at any point of X .

Theorem 4.3.1. *If (X,τ) is a topological space and $x \in X$ such that $\{x\}^{\wedge} \notin \tau$ then each open set containing x is infinite.*

Proof: Suppose that $G \in \tau$ such that G is finite and $x \in G$ then either

(1) $G - \{x\}^{\wedge} = \varnothing$ which implies that $G = \{x\}^{\wedge}$ or

(2) $G - \{x\}^{\wedge} = \{x_1, x_2, \ldots, x_n\}$ in this case there exists $G_i \in \tau$ for each $i \in \{1,2,\ldots,n\}$ such that $x \in G_i$ and $x_i \notin G_i$ which implies that $\{x\}^{\wedge} = G \bigcap (\bigcap_{i=1}^{n} G_i)$.

In both cases (1) and (2) $\{x\}^{\wedge} \in \tau$ and the contra positive of this result is the required aim.

Corollary 4.3.1. *In any topological space (X,τ) if $G \in \tau$ is finite then $M_x = \{x\}^{\wedge} \in \tau$ for each point $x \in G$ and if X or τ is finite then $\{x\}^{\wedge} \in \tau$ which means that $U_x = \{x\}^{\wedge}$ for each point $x \in X$.*

Example 4.3.1. Let $X = \{1,2,3,4,5,6\}$ and

$$\tau = \{X, \varnothing, \{1,2\}, \{3\}, \{3,4,5\}, \{1,2,6\}, \{1,2,3\}, \{1,2,3,4,5\}, \{1,2,3,6\}\}$$

Then $M_1 = M_2 = \{1,2\}$, $M_3 = \{3\}$, $M_4 = M_5 = \{3,4,5\}$ and $M_6 = \{1,2,6\}$

Remark 4.3.3. *Let (X,τ) be a topological space such that each point $x \in X$ has a minimal open set M_x and $A \subset X$. Then the minimal open set M_x at a point $x \in A$ denoted by M_A if $y \in A \Leftrightarrow M_x = M_y$ which means that $M_x = M_A$ for each $x \in A$. According of this definition of M_A we have the following remarks and propositions.*

(1) If $A \subset X$ such that $M_A \in \tau$ and $x \in A$ then

$$A = \{y \in X : M_y = M_x\}$$

(2) $M_A = A^{\wedge}$ if $M_A \in \tau$ since in this case

$$x \in A \Rightarrow M_A = M_x \Rightarrow M_x = M_A = \bigcup\{M_x : x \in A\}$$

where $M_x = \bigcup\{M_x : x \in A\} = A^{\wedge}$ since M_x is the same for each $x \in A$ and then

$$M_A = \bigcup\{M_x : x \in A\} = \bigcup\{\{x\}^{\wedge} : x \in A\} = A^{\wedge}$$

The converse of this result generally is not valid since there is $A^{\wedge}$ *for each subset* A *of* X *and it may and may not be in* τ *while* M_A *may be does not exist. Also we may have a subset* B *of* X *such that* $M_A = B^{\wedge}$ *while* $A \neq B$ *to explain this fact we give the following example where* $X = \{1,2,3,4,5,6,7,8\}$ *and*

$$\tau = \{X, \emptyset, \{1\}, \{1,2,3\}, \{1,2,3,4\}, \{5,6,7\}, \{1,5,6,7\},$$

$$\{1,2,3,5,6,7\}, \{1,2,3,4,5,6,7\}\}$$

from which

$$U_3 = U_2 = U_{\{2,3\}} = \{1,2,3\} = \{2,3\}^{\wedge} = \{1,2\}^{\wedge} = \{1,3\}^{\wedge}$$

$$= \{2\}^{\wedge} = \{3\}^{\wedge} = \{1,2,3\}^{\wedge}$$

where $\{1,2\} \neq \{2,3\}$, $\{1,3\} \neq \{2,3\}$ *and* $M_{\{1,2\}}$ *does not exist.*

(3) Let (X, τ) *be a topological space and* $A \subset X$ *be such that* $M_A \in \tau$ *then* $M_A \in \beta_o$ *where* $\beta_o = \{M_A : A \subset X, M_A \in \tau\} = \{M_x \in \tau : x \in X\}$ *is the minimal basis for the topology* τ *its proof will be given later in this section. Accordingly* $M_A \in \beta_o$ *iff* $M_A \in \tau$.

(4) $M_A - A \in \tau$ *Since* $x \in M_A - A$ *implies that* $x \in M_A$ *and* $x \notin A$ *implies that* $M_x \subset M_A$ *and* $M_x \neq M_A$. *Now if* $M_x \cap A \neq \emptyset$ *then* $M_A = M_x$ *which contradicts that* $x \notin A$. *Hence* $M_x \cap A = \emptyset$. *So* $x \in M_A - A$ *implies that* $M_x \subset M_A - A$ *which implies that* $M_A - A \in \tau$.

(5) Let $M_A, M_B \in \beta_o$. *Then*

(a) $A \cap B \neq \emptyset$ *implies that there is a point* $x \in A \cap B$ *and so*

$$x \in A \cap B \Rightarrow x \in A, x \in B \Rightarrow M_A = M_x = M_B \Rightarrow A = B.$$

(b) If $A \cap B = \emptyset$ *then*

$$x \in A \Rightarrow x \notin B \Rightarrow M_A = M_x \neq M_B \Rightarrow A \neq B.$$

Therefore

$$M_A \neq M_B \Leftrightarrow A \cap B = \emptyset \Leftrightarrow A \neq B$$

and $\beta_o^* = \{A \subset X : M_A \in \beta_o\}$ *is a partition of* X.

(6) If $M_A \neq M_B$ *then* $M_A \subset M_B \Rightarrow M_A - B = M_A$

since $M_A \cap B \neq \emptyset \Rightarrow M_B \subset M_A \Rightarrow M_A = M_B$, *a contradiction*

which implies that $M_A \cap B = \emptyset$ *which implies that* $M_A - B = M_A$ *Accordingly if* $M_A \subset M_B \subset M_C \subset \ldots \subset M_H \subset \ldots \subset M_E$
then

$$\begin{aligned} & M_A - (A \cup B \cup C \cup \ldots \cup H) \\ & = M_A \cap (A^c \cap B^c \cap C^c \cap \ldots \cap H^c) \\ & = (M_A - A) \cap [(M_A - B) \cap (M_A - C) \cap \ldots \ldots \cap (M_A - H)] \\ & = (M_A - A) \cap M_A \cap M_A \cap \ldots \ldots \cap M_A = M_A - A \in \tau \end{aligned}$$

Proposition 4.3.1. *Let* (X, τ) *be a topological space and* $A \subset X$ *such that* $U_A \in \beta_o$ *where* β_o *is the minimal basis for the topology* τ. *Then*

(I) $\bigcup_{U_M \in \beta_o} \{M : U_M \cap A = \emptyset\} = \bigcup_{U_M \in \beta_o} \{U_M : U_M \cap A = \emptyset\}$

(II) $\bigcup_{U_M \in \beta_o} \{M : U_M \cap A \neq \emptyset\}$ *is a closed set.*

(III) $\bigcup_{U_M \in \beta_o} \{M : U_M \cap A \neq \emptyset\} \neq \bigcup_{U_M \in \beta_o} \{U_M : U_M \cap A \neq \emptyset\}$

Proof: (I) Since $M \subset U_M$ for each $U_M \in \beta_o$ then

$$\bigcup_{U_M \in \beta_o} \{M : U_M \cap A = \emptyset\} \subset \bigcup_{U_M \in \beta_o} \{U_M : U_M \cap A = \emptyset\}$$

Conversely if $x \in \bigcup_{U_M \in \beta_o} \{U_M : U_M \cap A = \emptyset\}$ then there exists a subset M of X such that $U_M \in \beta_o$, $x \in U_M$ and $U_M \cap A = \emptyset$. Now from Remark 4.3.3, there exists $B \subset X$ such that $U_B \in \beta_o$ and $x \in B$ which implies that $U_B = U_x \subset U_M$ because $x \in B$ and $x \in U_M$ which implies that $U_B \cap A = \emptyset$. Therefore there exists $B \subset X$, $U_B \in \beta_o$ and $x \in B$ such that $U_B \cap A = \emptyset$. Therefore

$$\bigcup_{U_M \in \beta_o} \{U_M : U_M \cap A = \emptyset\} \subset \bigcup_{U_M \in \beta_o} \{M : U_M \cap A = \emptyset\}$$

Which completes the proof of (I) .

(II) To prove that $\bigcup_{U_M \in \beta_o} \{M : U_M \cap A \neq \emptyset\}$ is a closed set we shall prove that

$$X-[\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}]=\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}$$

For $x\in[\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}]^c$ implies that $x\notin M$ for each $M\subset X$ such that $U_M\cap A\neq\varnothing$ which implies that there exists $B\subset X$ such that $U_B\in\beta_o$, $x\in B$ and $U_B\cap A=\varnothing$ such set B exist by Remark 4.3.3, and so $x\in\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}$ which implies that

$$[\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}]^c$$
$$\subset\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}---(1)$$

Conversely, $x\in\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}$ implies that there exist $M\subset X$ such that $U_M\in\beta_o$, $x\in U_M$ and $U_M\cap A=\varnothing$. Now if $B\subset X$ such that $x\in B$, $U_B\in\beta_o$ and $U_B\cap A\neq\varnothing$ then $x\in B$ but $x\in U_M$ implies that $U_B=U_x\subset U_M$ implies that $U_M\cap A\neq\varnothing$ a contradiction which implies that $x\notin\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}$ which implies that

$$\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}$$
$$\subset[\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}]^c---(2)$$

Therefore (1) and (2) implies that

$$[\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}]^c$$
$$=\bigcup_{U_M\in\beta_o}\{U_M:U_M\cap A=\varnothing\}$$

which implies that $\bigcup_{U_M\in\beta_o}\{M:U_M\cap A\neq\varnothing\}$ is a closed set.

(III) The proof is given by an example given after Corollary 4.3.3.

Corollary 4.3.2. *If $G \in \tau$ and $U_A \in \tau$ such that $U_A \subset G$ or equivalently $G \cap A \neq \varnothing$ then*

$$G - \bigcup_{U_B \in \beta_O} \{B : U_B \cap A \neq \varnothing\} \text{ is an open set.}$$

Proof: The Proof is directly since by (II) Proposition 4.3.2,

$$\bigcup_{U_B \in \beta_O} \{B : U_B \cap A \neq \varnothing\} \text{ is a closed set.}$$

Corollary 4.3.3. *If $G \in \tau$ and $U_A, U_B \in \tau$ such that $\mathrm{U}_\mathrm{A} \subset U_B \subset G$ then*

$$U_M \cap B \neq \varnothing \Rightarrow U_A \subset U_B \subset U_M \Rightarrow U_M \cap A \neq \varnothing$$

Equivalently $U_M \cap A = \varnothing \Rightarrow U_M \cap B = \varnothing$ from which

(a) $\bigcup \{M : U_M \cap B \neq \varnothing\} \subset \bigcup \{M : U_M \cap A \neq \varnothing\}$ *Equivalently* $G - \bigcup \{M : U_M \cap A \neq \varnothing\} \subset G - \bigcup \{M : U_M \cap B \neq \varnothing\}$ *and*

(b) $\bigcup \{U_M : U_M \cap B = \varnothing\} \subset \bigcup \{U_M : U_M \cap A = \varnothing\}$.

Example 4.3.2. Let $X = \{1,2,3,4,5,6,7,8,9\}$ and

$$\beta_O = \{\{1\},\{2\},\{2,3\},\{2,3,5,7\},\{1,2,4\},\{1,2,4,6\},\{8\},\{8,9\}\}$$

is the minimal basis for a topology τ on X, $U_2 = \{2\}$, $U_3 = = \{2,3\} \subset \{2,3,5,7\} = U_{\{5,7\}}$ and $G = \{\{1,2,3,4,5,6,7,8\} \in \tau$ then

(1) $U_{\{6\}} - \{6\} = \{1,2,4\} \in \tau$ and $U_{\{5,7\}} - \{5,7\} = \{2,3\} \in \tau$,

(2) $X - \bigcup \{B : U_B \cap \{2\} \neq \varnothing\} = X - \{2,3,4,5,7\} = \{1,8,9\}$

$$= \bigcup \{U_B \in \beta_O : U_B \cap \{2\} = \varnothing\}$$

$\{1,8,9\} \in \tau$ implies that $\bigcup \{B : U_B \cap \{2\} \neq \varnothing\} \in \tau_c$.

(3)

$$\bigcup_{U_B \in \beta_O} \{B : U_B \cap \{2\} \neq \varnothing\} = \{2,3,4,5,6,7\} \neq$$

$$\{1,2,3,4,5,6,7\} = \bigcup_{U_B \in \beta_O} \{U_B \in \beta_O : U_B \cap \{2\} \neq \varnothing\}.$$

(4) $G - \bigcup \{B : U_B \cap \{2\} \neq \varnothing \wedge G \cap \{2\} \neq \varnothing\}$

$$= \{\{1,2,3,4,5,6,7,8\} - \{2,3,4,5,6,7\} = \{1,8\}$$

$$= \bigcup \{U_M \in \beta_O : U_M \cap \{2\} = \varnothing \wedge G \cap M \neq \varnothing\}$$

(5) $\bigcup\{M : U_M \cap \{3\} = \varnothing \wedge G \cap \{3\} \neq \varnothing\}$

$$= \{1,2,4,6\} \subset \{1,2,3,4,6\}$$

$$= \bigcup\{U_M \in \beta_o : U_M \cap \{5,7\} = \varnothing \wedge G \cap M \neq \varnothing\}$$

Proposition 4.3.2. *Suppose that* $U_A, U_B, U_C, \ldots, U_H, \ldots, U_K, U_M, \ldots, U_E \in \tau$ *is completely ordered by the inclusion operator* $"\subset"$ *as a chain* $U_A \subset U_B \subset U_C \subset \ldots, U_S \subset U_E$ *then*

(1) $U_E - (A \cup B \cup C \cup \ldots \cup E) \in \tau$*,*

(2) If U_E *is the greatest element then (i)* $E \in \tau_c$ *.*

(3) if U_H *is an element of the chain such that* $U_H \neq U_E$ *and if* $U_H \neq H$ *then* $H \notin \tau_c$ *and* $H \notin \tau$ *i.e* H *neither open nor closed.*

(4) $U_A - A \subset U_A \subset U_B - B \subset U_B \subset \ldots \subset U_E - E$ *and*

(5) $U_A - A \subset U_B - (A \cup B) \subset U_C - (A \cup B \cup C) \subset \ldots$
$\subset U_E - (A \cup B \cup C \cup \ldots \cup H \cup M \cup \ldots \cup E)$*.*

Proof: (1)

$$x \in U_E - (A \cup B \cup C \cup \ldots \cup E)$$

$$= U_E \cap A^c \cap B^c \cap C^c \cap \ldots \cap E^c = W \Rightarrow$$

$$\Rightarrow x \in U_E,\ x \notin A,\ x \notin B,\ x \notin C,\ \ldots, x \notin E$$

and $U_x \cap A = \varnothing, U_x \cap B = \varnothing, U_x \cap C = \varnothing, \ldots, U_x \cap E = \varnothing$
since $U_x \cap E = \varnothing$ implies that $U_x = U_E$ implies that $x \in E$ a contradiction. If U_M is a member of the chain such that $U_x \cap M \neq \varnothing$ and $M \neq E$ then there is a point $y \in M$ such that $y \in U_x$ which implies that $U_M = U_y \subset U_x$ and since $x \in U_E$ then if $H \subset X$ such that $U_x = U_H$ and $U_A \subset U_M \subset U_H \subset U_E$ from which U_H is an element of the chain $U_A \subset U_B \subset U_C \subset \ldots, \subset U_E$ and $x \in H$ which contradicts that $x \notin S$ for each element U_S of the chain. Then,

$$U_x \cap A = \varnothing, U_x \cap B = \varnothing, U_x \cap C = \varnothing, \ldots, U_x \cap E = \varnothing$$

according to which and since $U_x \subset U_E$ we gets

$$U_x \subset U_E - A, U_x \subset U_E - B,$$

$$U_x \subset U_E - C, \ldots, U_x \subset U_E - E$$

Which implies that $U_x \subset W$ which implies that $W \in \tau$.

As a direct consequence of this result one can show by using the same argument that $U_M -(H \cup K \cup ... \cup M) \in \tau$.

(2) If U_E is the maximal element of the chain then $x \in E^c$ implies that

$$U_x \cap E \neq \varnothing \Rightarrow U_E \subset U_x \Rightarrow U_x = U_E \Rightarrow x \in E$$

which contradicts that $x \in E^c$. Therefore, $U_x \cap E = \varnothing$ from which

$$U_x \subset E^c \Rightarrow E^c \in \tau \Rightarrow E \in \tau_c .$$

(3) $U_H \neq U_E \Rightarrow H \cap E = \varnothing \Rightarrow E \subset H^c$ and if $x \in E$ then $x \in H^c$ and

$$U_x \subset H^c \Rightarrow U_H \subset U_E = U_x \subset H^c \Rightarrow H \subset U_H \subset H^c$$

which is impossible. Hence $U_x \not\subset H^c$ which means that $x \notin H^{co}$ which means that $H^{co} \neq H^c$ which implies that $H^c \notin \tau$. Therefore $H \notin \tau_c$. if $U_H \neq H$ then

$$x \in H \Rightarrow U_x = U_H \not\subset H \Rightarrow x \notin H^o \Rightarrow H \notin \tau$$

(4) The proof is easy and left to the reader.

(5) By using the principal of mathematical induction. Firstly by (6) Remark 4.3.3, $U_A \subset U_B - B$ which implies that

$$U_A - A \subset U_B - (A \cup B) \ ----(I)$$

Secondly $x \in U_H -(A \cup B \cup C \cup ... \cup H)$ and $U_H \subset U_M$ implies that $x \in U_H$, $x \notin (A \cup B \cup C \cup ... \cup H)$, $U_x \subset U_M$ Since $U_H \subset U_M$ and $x \notin M$ because $x \in U_H$ and $U_H \subset U_M$ implies by (6) Remark 4.3.3, that $U_H \cap M = \varnothing$ and $x \in U_H$ which implies that $x \in U_M -(A \cup B \cup C \cup ... \cup H \cup M)$ which implies that

$$U_H -(A \cup B \cup C \cup ... \cup H) \subset U_M -(A \cup B \cup C \cup ... \cup H \cup M)$$

Therefore

$$U_A - A \subset U_B -(A \cup B) \subset U_C -(A \cup B \cup C) \subset ...$$

$$\subset U_E -(A \cup B \cup C \cup ... \cup H \cup M \cup ... \cup E).$$

Example 4.3.3. In the last example we remark that

(a) $U_2 \subset U_3 \subset U_{\{5,7\}} \Rightarrow U_{\{5,7\}} - \{2,3,5,7\} = \varnothing \subset \{1,8\}$,

(b) $U_2 \subset U_4 \subset U_6 \Rightarrow U_6 - \{2,4,6\} = \{1\} \subset \{1,8\}$ and

(c) $U_1 \subset U_4 \subset U_6 \Rightarrow U_6 - \{1,4,6\} = \{2\}$.

Let (X,τ) be a topological space such that each point has a minima open set U_x then we shall prove that

$$\beta_o=\{U_x : x\in X\}=\{U_A : U_A\in\tau\}$$

is the minimal basis for the topology τ. Also $\beta_o^*=\{A : U_A\in\tau\}$ is a partition of X .

Theorem 4.3.2. *Let (X,τ) be a topological space. Then $U\in\tau$ is the minimal open set at a point $x\in U$ iff U can not be written as a union of members of τ each is an improper subset of U .*

Proof: Suppose that $U\in\tau$ is not minimal at any of its points i.e. $U\neq U_x$ for each $x\in U$. Then

$$x\in U\Rightarrow\exists G_x\in\tau : x\in G_x\wedge U\not\subset G_x\Rightarrow x\in U\cap G_x\in\tau\wedge$$
$$U\cap G_x\neq U\Rightarrow U=\bigcup\{U\cap G_x : x\in U\}$$

The contra positive of this result is if $U\in\tau$ can not be written as a union of improper subsets of U members of τ then there exists a point $x\in U$ such that $U=U_x$ that is U is the minimal open set at a point $x\in U$.

Conversely Suppose that $x\in U$, $U=U_x$ and $\{B_i : i\in\Delta\}\subset\tau$ such that $U=\bigcup\{B_i : i\in\Delta\}$ then

$$x\in U_x=U\Rightarrow\exists i\in\Delta : x\in B_i\subset U_x\ ---(I)$$

From the definition of U_x and since $B_i\in\tau$ and $x\in B_i$ then

$$U_x\subset B_i\ ---(II)$$

From (I) and (II), $U=U_x=B_i$ which means that if there exists $x\in U$ such that $U=U_x$ then any family $\{B_i : i\in\Delta\}$ of elements of τ the union of its elements is U implies that there exists $i\in\Delta$ such that $U=B_i$ this means that it is impossible to write U as a union of elements from τ each of which is an improper subset of U .

Theorem 4.3.3. *Let (X,τ) be a topological space such that each point $x\in X$ has the minimal open set U_x i.e. $U_x\in\tau$. Then, $\beta_o=\{U_x : x\in X\}$ is the minimal basis for τ.*

Proof: Firstly $U_x\in\tau$ for each $x\in X$ implies that $\beta_o\subset\tau$ and $G\in\tau : x\in G\Rightarrow U_x\subset G$ which implies that β_o is a basis for the topology τ.

Secondly If β is a basis for τ then by Theorem 4.3.2, U_x can not be written as a union of members of β improper subsets of U_x and so $U_x \in \beta$ which implies that $\beta_o \subset \beta$. In another way $U_x \in \tau \Rightarrow \exists B \in \beta : x \in B \subset U_x$ but from the definition of U_x, $B \in \boldsymbol{\beta} \Rightarrow B \in \boldsymbol{\tau}$ and $x \in B$ implies $U_x \subset B$. Hence $\beta_o \subset \beta$. Therefore, β_o is the minimal basis for τ.

Remark 4.3.4. *If* (X, τ) *is a topological space and* $A \subset X$ *such that* $U_A \in \tau$ *then* $U_A \in \beta_o$.

Example 4.3.4. In Example 4.3.1, where $X = \{1,2,3,4,5,6\}$ and

$$\tau = \{\mathrm{X}, \varnothing, \{1,2\}, \{3\}, \{3,4,5\}, \{1,2,6\}, \{1,2,3\}, \{1,2,3,4,5\}, \{1,2,3,6\}\}$$

$\beta_o = \{\{1,2\}, \{3\}, \{3,4,5\}, \{1,2,6\}\}$ is the minimal basis for τ.

Example 4.3.5. In the discrete topological space (X, D) the minimal basis for D is $\beta_o := \{\{x\} : x \in X\}$ where $U_x = \{x\}$ for each point $x \in X$.

Example 4.3.6. In the excluding topological space (X, E_p) where $p \in X$ is the excluding point, the minimal basis for the topology E_p is $\beta_o = \{\{x\}, X : x \in X - \{p\}\}$ where $U_x = \{x\}$ for each point $x \in X - \{p\}$ and $U_p = \{X\}$. In the topological space (X, τ) the open set $G \in \tau$ is called principal or simple open set if it can not be written as a union of members of τ each is an improper subset of G other wise it is called non principal or non simple. Theorem 4.3.2, says that $G \in \tau$ is a simple open set iff there is a point $x \in G$ such that $G = U_x$. Accordingly the topological spaces can be classifies into two class's principal and non principal topological spaces.

The first kind: The principal topological spaces are which has the minimal basis $\beta_o = \{U_x : x \in X\}$ which consists the minimal open sets for all points of X i.e. (X, τ) is a principal topological space if each point $x \in X$ has the minimal open set U_x.

For example the following topological spaces are principal topological spaces:

(1) (X, D), (X, E_p) and (X, P_p) where $p \in X$.

(2) (X, τ) where τ is a finite family whatever may be the set X.

(3) (X, τ) if X is finite such topological spaces are called finite spaces.

The second kind: The definition of the non principal topological spaces is reading in its name that is a topological space (X, τ) is non principal if at just a point $x \in X$ there is no minimal open set. For example the following topological spaces are non principal spaces:

(1) The co-finite topological space (X, C) where X is infinite

(2) (X, τ) where $\tau = E_p \cup C$ and $p \in X$ where X is infinite

(3) (X, τ) where $\tau = E_p \cup C$ and $p \in X$ where X is infinite

(4) (X, τ) where $\tau = E_{\{p,q\}} \cup C$ and $p, q \in X$ $E_{\{p,q\}} = \{G \subset X : \{p, q\} \cap G = \varnothing\} \cup \{X\}$ where X is infinite and

(5) The usual topological space $(R, \mho)$.

One of these examples will be explained by the following theorem

Theorem 4.3.4. *The topological space* (X, τ) *where* $\tau = E_p \cup C$ *and* $p \in X$ *where* X *is infinite is a non principal space.*

Proof: Suppose that $G \in \tau$ such that $p \in G$ then $G \in C$ and $x \in G - \{p\} \in C$ implies that $G - \{x\} \in \tau$ and $p \in G - \{x\}$. But $G \not\subset G - \{x\}$ which means that G is not minimal at P. Since $G \in \tau$ is any open set hence τ is not principal.

Theorem 4.3.5. *If* (X, τ) *is a topological space then the following Statements are equivalent:*

(1) (X, τ) *is a principal space*

(2) Arbitrary intersection of members of τ *is a member of* τ*,*

(3) The family of the closed sets $\tau_c = \{G^c : G \in \tau\}$ *is a topology on* X *.*

(4) $P^\wedge(X) = \{A^\wedge : A \subset X\} = \tau$.

Proof: We shall prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2): From (1) $\{x\}^\wedge = U_x \in \tau$ for each point $x \in X$ and if $\{G_i : i \in \Delta\} \subset \tau$ and $G = \bigcap_{i \in \Delta} G_i$ then

$$x \in G \Rightarrow x \in G_i \,;\forall i \in \Delta \Rightarrow x \in U_x \subset G_i \,;\forall i \in \Delta$$

$$\Rightarrow U_x \subset G \Rightarrow G = \bigcap_{i \in \Delta} G_i \Rightarrow G \in \tau$$

(2) $\Rightarrow$ (3): From the properties of the closed sets τ_c satisfies (O1) and (O2). Now implies that $\{F_i^c : i \in \Delta\} \subset \tau$, from (2) we find that

$$[\bigcup\{F_i : i \in \Delta\}]^c = \bigcap\{F_i^c : i \in \Delta\} \in \tau$$

Therefore $\bigcup\{F_i : i \in \Delta\} \in \tau_c$ which means that τ_c satisfies (O3).

(3) $\Rightarrow$ (4): By Remark 3.3.2, $\tau \subset P^\wedge(X) -\!-(I)$.

Conversely

$$A \subset X \Rightarrow A^{\wedge} = \bigcap\{G \in \tau : A \subset G\}$$

$$\Rightarrow A^{\wedge c} = \bigcup_{G \in \tau} \{G^{c} : A \subset G\}$$

From (3) τ_c is a topology on X which implies that $A^{\wedge c} \in \tau_c$ and so $A^{\wedge} \in \tau$. Then $P^{\wedge}(X) \subset \tau - -(II)$. From (I) and (II) $P^{\wedge}(X) = \tau$.

(4) $\Rightarrow$ (1): Suppose that $P^{\wedge}(X) = \tau$ then $\{x\}^{\wedge} \in \tau$ which means that $\{x\}^{\wedge} = U_x$ for each point $x \in X$. Hence (X, τ) is a principal space.

Corollary 4.3.4. *As a direct consequence of Theorem 4.3.5. If X is finite then (X, τ) is a principal space whatever may be τ.*

Theorem 4.3.6. *If (X, τ) is a principal space and β_1 and β_2 are two bases for τ then $\beta_1 \cap \beta_2$ is a basis for τ.*

Proof: Since (X, τ) is a principal space then τ has the minimal basis $\beta_o = \{U_x : x \in X\}$ and so $\beta_o \subset \beta_1 \cap \beta_2 \subset \tau$. Hence by Remark 4.1.3, then $\beta_1 \cap \beta_2$ is a basis for τ.

Remark 4.3.5. *Theorem 4.3.6, is not generally valid if (X, τ) is non principal for example the families of all open circles and all open rectangles are bases for the usual topology $\mho^2$ on R^2 while $\beta_1 \cap \beta_2 = \emptyset$.*

Remark 4.3.6. *If (X, τ) is a topological space, $A \subset X$ and U_x is the minimal open set at the point $x \in X$ then $x \in A' \Leftrightarrow U_x - \{x\} \cap A \neq \emptyset$ since $U_x \subset G$ for each $G \in \tau$ such that $x \in G$, $U_x \in \tau$ and $x \in U_x$.*

Remark 4.3.7. *If (X, τ) is a principal topological space then according to Theorem 4.3.5, (X, τ_c) is a topology on X . Remarking that $\tau_{cc} = \tau$, the topology τ_c is the family of the closed sets with respect to τ on X and if $A \subset X$ then*

(1) $A^{\wedge} = \bigcap\{G \in \tau : A \subset G\} = \bigcap\{G \in \tau_{cc} : A \subset G\} = \overline{A}_c$,

(2) From (1) $\overline{\{x\}}_c = \{x\}^{\wedge} = U_x$ for each point $x \in X$,

(3) $ext_c(A) = (\overline{A}_c)^c = A^{\wedge c}$,

(4) $A_c^o = ext_c(A^c) = A^{c \wedge c}$,

(5) $b_c(A) = A^{\wedge} \cap A^{c \wedge} = \overline{A}_c \cap \overline{A^c}_c$,

(6) $U_{xc} = \bigcap\{H \in \tau_c : x \in H\} = \bigcap\{G^c : G \in \tau \wedge x \in G^c\}$
$= \bigcap\{F \in \tau_c : x \in F\} = \overline{\{x\}}$ *and*

(7) Directly from (6) $\beta_{oc} = \{\overline{\{x\}} : x \in X\}$ *is the minimal basis for the topology* τ_c *on* X *where* $\overline{A}_c$, A_c^o, $ext_c(A)$, $b_c(A)$ *and* U_{xc} *respectively are the closure, interior, exterior, boundary of* A *and the minimal open set at* x *with respect to the topology* τ_c *.*

Theorem 4.3.7. *Suppose that* (X, τ) *is a topological space and consider the family* $\beta^\wedge = \{\{x\}^\wedge : x \in X\}$ *of all minimal open sets at the points of* X *. If* $y, z \in X$ *are two distinct points satisfy the conditions*

(1) $y \notin \{z\}^\wedge$

(2) $z \in \{x\}^\wedge$ *and* $x \notin \{z\}^\wedge$ *imply that* $y \in \{x\}^\wedge$ *and*

(3) $x \in \{y\}^\wedge$ *and* $y \notin \{x\}^\wedge$ *imply that* $x \in \{z\}^\wedge$*.*

For each point $x \in X$ *then*

$$\beta^\wedge_{yz} = \{\{x\}^\wedge, \{y\}^\wedge \cup \{z\}^\wedge : \{x\}^\wedge \in \beta^\wedge - \{\{z\}^\wedge\}\}$$
$$= (\beta^\wedge - \{\{z\}^\wedge\}) \cup \{\{y\}^\wedge \cup \{z\}^\wedge\}$$

Is the family of the minimal sets at the points of X *with respect to the topology* $\tau_{yz} = \tau \cap D_{yz}$ *on* X *and if* τ^* *is a topology on* X *such that* $\tau^* \neq \tau_{yz}$ *and* $\tau_{yz} \subset \tau^* \subset \tau$ *then*

(a) $\tau^*_{yz} = \tau^* \cap D_{yz} = \tau_{yz}$ *and (b) The families of the minimal sets at the points of* X *with respect to* τ *and to* τ^* *are coincided.*

Proof: From the condition (1), $y \notin \{z\}^\wedge$ which implies that $\{z\}^\wedge \neq \{y\}^\wedge \cup \{z\}^\wedge$ and from Remark 3.3.3, $\{x\}^\wedge \subset \{x\}^\wedge_{yz}$ for each point $x \in X$ and from Theorem 3.3.4, $\{y\}^\wedge_{yz} = \{y\}^\wedge \cup \{y\}^\wedge = \{y\}^\wedge$ and if $x \in X$ such that $\{x\}^\wedge \notin \{\{y\}^\wedge, \{z\}^\wedge\}$ which means that $\{x\}^\wedge \neq \{z\}^\wedge$ from which we have two possibilities either $z \notin \{x\}^\wedge$ from which by using Theorem 3.3.4 (b) $\{x\}^\wedge_{yz} = \{x\}^\wedge$ or $z \in \{x\}^\wedge$ and $x \notin \{z\}^\wedge$ from which by condition (2) $y \in \{x\}^\wedge$ from which $\{x\}^\wedge_{yz} = \{x\}^\wedge$ this by using Theorem 3.3.4 (b) also by Theorem 3.3.4 part (a) $\{z\}^\wedge_{yz} = \{y\}^\wedge \cup \{z\}^\wedge$. Therefore,

$$\beta^\wedge_{yz} = \{\{x\}^\wedge, \{y\}^\wedge \cup \{z\}^\wedge : \{x\}^\wedge \in \beta^\wedge - \{\{z\}^\wedge\}\}$$

is the family of all minimal sets of the points of X with respect to the topology τ_{yz}. From the condition (1) $y \notin \{z\}^\wedge$ implies that

$\{z\}^{\wedge} \neq \{y\}^{\wedge} \cup \{z\}^{\wedge} = \{z\}^{\wedge}_{yz}$ and from Theorem 3.3.3, this implies that $\tau_{yz} = \tau \cap D_{yz} \neq \tau$. If τ^* is a topology on X such that $\tau^* \neq \tau_{yz}$ then $\tau_{yz} \subset \tau^* \subset \tau$ which implies that $\tau_{yz} \subset \tau^* \cap D_{yz} \subset \tau \cap D_{yz} = \tau_{yz}$ which implies that $\tau^*_{yz} = \tau_{yz}$ where $\tau^*_{yz} = \tau^* \cap D_{yz}$. Hence we gets by Remark 3.3.3, that $\{x\}^{*\wedge}_{yz} = \{x\}^{\wedge}_{yz}$ and so

$$\beta^{*\wedge}_{yz} = \{\{x\}^{*\wedge}, \{y\}^{*\wedge} \cup \{z\}^{*\wedge} : \{x\}^{*\wedge} \in \beta^{*\wedge} - \{\{z\}^{*\wedge}\}\}\}$$
$$= \{\{x\}^{\wedge}, \{y\}^{\wedge} \cup \{z\}^{\wedge} : \{x\}^{\wedge} \in \beta^{\wedge} - \{\{z\}^{\wedge}\}\} = \beta^{\wedge}_{yz}$$

where $\beta^{*\wedge} = \{\{x\}^{*\wedge} : x \in X\}\}$ and by Remark 3.3.3, we finds $\{x\}^{\wedge} \subset \{x\}^{*\wedge} \subset \{x\}^{\wedge}_{yz} = \{x\}^{\wedge}$ from which for each point $x \in X$, $\{x\}^{*\wedge} = \{x\}^{\wedge}$ such that $\{x\}^{\wedge} \neq \{z\}^{\wedge}$ but $\{z\}^{*\wedge}_{yz} = \{z\}^{\wedge}_{yz}$ from which $\{y\}^{*\wedge} \cup \{z\}^{*\wedge} = \{y\}^{\wedge} \cup \{z\}^{\wedge}$ and if $t \in \{z\}^{*\wedge} - \{z\}^{\wedge}$ then $t \in \{z\}^{*\wedge}$ implies that $t \in \{y\}^{\wedge} \cup \{z\}^{\wedge}$ from which $t \in \{y\}^{\wedge}$ because $t \notin \{z\}^{\wedge}$ and we have two cases (1) $y \notin \{t\}^{\wedge}$ which implies by condition (3) that $t \in \{z\}^{\wedge}$ this contradicts that $t \notin \{z\}^{\wedge}$ or the second case (2) $y \in \{t\}^{\wedge}$ which implies by Remark 3.3.4 and since $\tau^* \subset \tau$ and $t \in \{z\}^{*\wedge}$ that $\{y\}^{\wedge} = \{t\}^{\wedge} \subset \{t\}^{*\wedge} \subset \{z\}^{*\wedge}$ then $y \in \{z\}^{*\wedge}$ which implies that $\tau^* \subset D_{yz}$ and this implies that $\tau^* = \tau^* \cap D_{yz} = \tau^*_{yz} = \tau_{yz}$ which contradicts the assumption that $\tau_{yz} \neq \tau^*$ from this contradiction we deduce that such point t where $t \in \{z\}^{*\wedge} - \{z\}^{\wedge}$ does not exists which implies that $\{z\}^{*\wedge} \subset \{z\}^{\wedge}$ but $\{z\}^{\wedge} \subset \{z\}^{*\wedge}$. So $\{z\}^{*\wedge} = \{z\}^{\wedge}$ and the proof is complete.

Remark 4.3.8. *Suppose that* (X, τ) *is a topological space and consider the family of the minimal sets of the points of* X *with respect to the topology* τ *then we may obtain two distinct points* $y, z \in X$ *which does not satisfy the conditions of Theorem 4.3.7, while* $\beta^{\wedge}_{yz} = \{\{x\}^{\wedge}, \{y\}^{\wedge} \cup \{z\}^{\wedge} : \{x\}^{\wedge} \in \beta^{\wedge} - \{\{z\}^{\wedge}\}\}$ *is the family of the minimal open sets about the points of* X *with respect to a topology* τ^* *on* X *different from* $\tau_{yz} = \tau \cap D_{yz}$ *and this does not contradicts Theorem 4.3.7, as illustrated by the following example:*

Example: Let X be a nonempty set, $1, 2, 3, 4 \in X$ and $\tau = \{X, \varnothing, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$ we remark that $3, 1 \in X$ does not satisfies the conditions of Theorem 4.3.7 and

$$\beta^{\wedge}_{31}=\{X,\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$$
$$=\{\{x\}^{\wedge},\{y\}^{\wedge}\cup\{z\}^{\wedge}:\{x\}^{\wedge}\in\beta^{\wedge}-\{\{z\}^{\wedge}\}\}$$

is the family of the minimal sets with respect to the topology $\tau^{*}=\{X,\varnothing,\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$ and clearly $\tau^{*}=\tau_{31}\neq\tau\cap D_{31}=\{X,\varnothing,\{1,2,3\},\{1,2,3,4\}\}$ while the points 1 and 3 fail to satisfies the conditions (2) and (3) of Theorem 4.3.7, also we remark that the points $1,2\in X$ satisfy the three conditions of Theorem 4.3.7 , $\beta^{\wedge}_{31}=\beta^{\wedge}_{21}$ and $\tau^{*}=\tau_{21}=\tau\cap D_{21}$.

Remark 4.3.9. *Suppose that* $X\neq\varnothing$ *and* $y,z\in X$ *are two distinct points and consider the topologies* $\tau=E_{y}\cup(P_{\{y,z\}}\cap C)$ *and* $\tau^{*}=(E_{y}\cap C)\cup(P_{\{y,z\}}\cap C)$ *then*

(1) $\tau_{yz}=\tau\cap D_{yz}=E_{\{y,z\}}\cup(P_{\{y,z\}}\cap C)$ *where*

$$(P_{\{y,z\}}\cap C)\subset D_{yz}\Rightarrow D_{yz}\cap(P_{\{y,z\}}\cap C)=(P_{\{y,z\}}\cap C),$$

and

$$G\in D_{yz}\cap E_{y}\Rightarrow G\in D_{yz}\wedge G\in E_{y}$$
$$\Rightarrow(y\notin G\Rightarrow z\notin G)\wedge y\notin G\Rightarrow y,z\notin G$$
$$\Rightarrow G\in E_{\{y,z\}}\Rightarrow D_{yz}\cap E_{y}\subset E_{\{y,z\}}\quad(i)$$

But $E_{\{y,z\}}\subset D_{yz}\cap E_{y}\quad(ii)$*.From* (i) *and* (ii)*,* $D_{yz}\cap E_{y}=E_{\{y,z\}}$

(2) $\tau^{*}_{yz}=\tau^{*}\cap D_{yz}=(E_{\{y,z\}}\cap C)\cup(P_{\{y,z\}}\cap C)$ *and*

(3) τ_{yz} *is not strictly weaker than* τ *where the topology* $\tau^{+}=E_{\{y,z\}}\cup(E_{y}\cap P_{z}\cap C)\cup(P_{\{y,z\}}\cap C)$ *is the topology on* X *such that* $\tau^{+}\notin\{\tau,\tau_{yz}\}$ *and* $\tau_{yz}\subset\tau^{+}\subset\tau$*.*

(4) τ^{*}_{yz} *is not strictly weaker than* τ^{*}*, for let* $\tau^{**}\subset P(X)$ *be a topology on* X *such that* $\tau^{*}_{yz}\neq\tau^{**}$ *and* $\tau^{*}_{yz}\subset\tau^{**}\subset\tau^{*}$*. Then* $G\in\tau^{**}-\tau^{*}_{yz}$ *implies that either (1)* $z\in G$*,* $y\notin G$ *and* G^{c} *is finite but* $\{y,z\}^{c}\in\tau^{*}_{yz}\subset\tau^{**}$ *which implies that* $G\cup\{y,z\}^{c}=\{y\}^{c}\in\tau^{**}$ *and since* $z\in G$*,* $y\notin G$*,* $(G\cup\{y\})^{c}$ *is finite and* $\{x\}^{c}\in(P_{\{y,z\}}\cap C)\subset\tau^{**}$ *for each* $x\in(G\cup\{y\})^{c}$ *then* $G=\{y\}^{c}\bigcap(\bigcap\{\{x\}^{c}:x\in(G\cup\{y\})^{c}\}$ *is a*

finite intersection of members of τ^{**} *which implies that* $G \in \tau^{**}$ *or (2)* $z \in G$ *and* $y \notin G$ *which leads to the same result that* $G \in \tau^{**}$*. Therefore* $\tau^{**} = \tau^{*}$ *and so* τ^{*}_{yz} *is a strictly weaker than* τ^{*}*.*

Corollary 4.3.5. *Let* (X, τ) *be a principal topological space,* β_o *be the minimal basis for* τ *and* $y, z \in X$ *be two distinct points satisfy the conditions (1), (2) and (3) of Theorem 4.3.7. Then* $\beta_{yz} = \{U_x, U_y \cup U_z : U_x \in \beta - \{U_z\}\}$ *is the minimal basis for the topology* τ_{yz} *which is a principal topology on* X *strictly weaker than the topology* τ*.*

Proof: it is a direct consequence on Theorem 4.3.7.

Lemma 4.3.1. *Let* (X, τ) *be a topological space and* $x, y, z, t \in X$ *such that* $x \notin \{z\}^{\wedge}$ *and* $y \notin \{t\}^{\wedge}$*. Then*

(1) $\tau_{xz} = \tau_{yt} \Rightarrow \{z\}^{\wedge} = \{t\}^{\wedge}$*.*

(2) If $\{z\}^{\wedge} = \{t\}^{\wedge}$ *then* $\{z\}^{\wedge}_{xz} \neq \{t\}^{\wedge}_{yt} \Rightarrow \tau_{xz} \neq \tau_{yt}$

equivalently $\tau_{xz} = \tau_{yt} \Rightarrow \{z\}^{\wedge}_{xz} = \{t\}^{\wedge}_{yt}$*.*

Proof:

(1) By using Theorem 3.3.4, $\{z\}^{\wedge}_{xz} = \{x\}^{\wedge} \cup \{z\}^{\wedge}$ and $\{t\}^{\wedge}_{yt} = \{y\}^{\wedge} \cup \{t\}^{\wedge}$ then $\{z\}^{\wedge} \neq \{t\}^{\wedge}$ implies either (1) $z \notin \{t\}^{\wedge}$ which implies by Theorem 3.3.4, that $\{t\}^{\wedge}_{xz} = \{t\}^{\wedge} \in \beta^{\wedge}_{xz}$ and from Theorem 3.3.3, $\tau_{xz} \neq \tau_{yt}$ or (2) $t \notin \{z\}^{\wedge}$ which leads also to the same result $\tau_{xz} \neq \tau_{yt}$. The contra positive of this result is that $\tau_{xz} = \tau_{yt}$ implies that $\{z\}^{\wedge} = \{t\}^{\wedge}$.

(2) We know that

$$\beta^{\wedge} = \{\{x\}^{\wedge} : x \in X\}, \ \beta^{\wedge}_{xz} = (\beta^{\wedge} - \{\{z\}^{\wedge}\}) \cup \{\{z\}^{\wedge}_{xz}\}$$

and $\beta^{\wedge}_{yt} = (\beta^{\wedge} - \{\{t\}^{\wedge}\}) \cup \{\{t\}^{\wedge}_{yt}\}$. If $\{z\}^{\wedge} = \{t\}^{\wedge}$ then $\beta^{\wedge} - \{\{z\}^{\wedge}\} = \beta^{\wedge} - \{\{t\}^{\wedge}\}$ and then

$$\tau_{xz} = \tau_{yt} \Rightarrow \beta^{\wedge}_{xz} = \beta^{\wedge}_{yt} \Rightarrow \{z\}^{\wedge}_{xz} = \{t\}^{\wedge}_{yt}$$

since $\{z\}^{\wedge}_{xz} \notin (\beta^{\wedge} - \{\{t\}^{\wedge}\})$.

Another proof of (2): If $\{z\}^{\wedge} = \{t\}^{\wedge}$ then $z \in G$ iff $t \in G$ for each $G \in \tau$ and hence

$$\{z\}^{\wedge}_{xz} = \bigcap\{G \in \tau_{xz} : z \in G\} = \bigcap\{G \in \tau : z, t, x \in G\},$$

$$\{t\}^{\wedge}_{yt} = \bigcap\{G \in \tau_{yt} : t \in G\} = \bigcap\{G \in \tau : z, t, y \in G\}$$

And $\{z\}^{\wedge}_{xz} \neq \{t\}^{\wedge}_{yt}$ implies that $\{G \in \tau : z, t, x \in G\} \neq \{G \in \tau : z, t, y \in G\}$ according to which there exists $G \in \tau$ such that $z, t \in G$ and either $x \in G$ and $y \notin G$ from which $G \in \tau_{xz} - \tau_{yt}$ or $y \in G$ and $x \notin G$ which implies that $G \in \tau_{yt} - \tau_{xz}$. In both cases $\tau_{xz} \neq \tau_{yt}$.

Corollary 4.3.6. *Let* (X, τ) *be a topological space and* $x, y, z, t \in X$ *such that* $x \notin \{z\}^{\wedge}$ *and* $y \notin \{t\}^{\wedge}$. *Then* $\tau_{xz} = \tau_{yt}$ *iff* $\{x\}^{\wedge} = \{y\}^{\wedge}$ *and* $\{z\}^{\wedge} = \{t\}^{\wedge}$.

Proof: Suppose that $\{x\}^{\wedge} = \{y\}^{\wedge}$ and $\{z\}^{\wedge} = \{t\}^{\wedge}$ then $y \in \{x\}^{\wedge}$ and $z \in \{t\}^{\wedge}$ imply by Definition 3.2.4(iv) that $\tau \subset D_{yx} \cap D_{zt}$ which implies that $\tau = \tau \cap D_{yx} \cap D_{zt}$ which implies that

$$\tau_{xz} = \tau \cap D_{xz} = \tau \cap D_{yx} \cap D_{xz} \cap D_{zt} \subset \tau \cap D_{yz} \cap D_{zt}$$
$$\subset \tau \cap D_{yt} = \tau_{yt} \quad (I)$$

or $x \in \{y\}^{\wedge}$ and $t \in \{z\}^{\wedge}$ from which in a similar way $\tau_{yt} \subset \tau_{xz} \quad (II)$. From (I) and (II) we gets $\tau_{xz} = \tau_{yt}$. Conversely, by Lemma 4.3.1, we gets

(1) $\tau_{xz} = \tau_{yt} \Rightarrow \{z\}^{\wedge} = \{t\}^{\wedge}$

(2) If $\{z\}^{\wedge} = \{t\}^{\wedge}$ with the given conditions $x \notin \{z\}^{\wedge}$ and $y \notin \{t\}^{\wedge}$ we finds $\{x\}^{\wedge} \neq \{y\}^{\wedge}$ which implies that either $x \notin \{y\}^{\wedge}$ from which with the condition $x \notin \{z\}^{\wedge}$ we finds that $x \notin \{y\}^{\wedge} \cup \{z\}^{\wedge} = \{y\}^{\wedge} \cup \{t\}^{\wedge} = \{t\}^{\wedge}_{yt}$ while $x \in \{x\}^{\wedge} \cup \{z\}^{\wedge} = \{z\}^{\wedge}_{xz}$ accordingly, $\{z\}^{\wedge}_{xz} \neq \{t\}^{\wedge}_{yt}$ or $y \notin \{x\}^{\wedge}$ which in a similar way with the condition $y \notin \{t\}^{\wedge}$ obtain the same result $\{z\}^{\wedge}_{xz} \neq \{t\}^{\wedge}_{yt}$. From Lemma 4.3.1, gets in both cases that $\tau_{xz} \neq \tau_{yt}$ that is if $\{z\}^{\wedge} = \{t\}^{\wedge}$, $x \notin \{z\}^{\wedge}$ and $y \notin \{t\}^{\wedge}$ then

$$\{x\}^{\wedge} \neq \{y\}^{\wedge} \Rightarrow \{z\}^{\wedge}_{xz} \neq \{t\}^{\wedge}_{yt} \Rightarrow \tau_{xz} \neq \tau_{yt}$$

Equivalently the contra positive of this result is

$$\tau_{xz} = \tau_{yt} \Rightarrow \{z\}^{\wedge}_{xz} = \{t\}^{\wedge}_{yt} \Rightarrow \{x\}^{\wedge} = \{y\}^{\wedge}.$$

Therefore if $x \notin \{z\}^{\wedge}$ and $y \notin \{t\}^{\wedge}$ then $\tau_{xz} = \tau_{yt}$ implies that $\{z\}^{\wedge} = \{t\}^{\wedge}$ and $\{x\}^{\wedge} = \{y\}^{\wedge}$ and the proof is complete.

Remark 4.3.10. *The converse of Lemma 4.3.1, generally incorrect as illustrated by the following example: Let* X *be an infinite set,* $x,y,z,t\in X$ *be distinct points and let* $\tau=E_{\{t,z\}}\cup(P_{\{x,z\}}\cap E_t\cap C)$*. Then* $\{x\}^\wedge=\{x\}$*,* $\{z\}^\wedge=\{x,z\}$*,* $\{y\}^\wedge=\{y\}$ *and* $\{t\}^\wedge=X$ *from which* $x\in\{z\}^\wedge$ *and* $y\in\{t\}^\wedge$ *which implies that* $\tau_{xz}=\tau\cap D_{xz}=\tau=\tau\cap D_{yt}=\tau_{yt}$ *according to which the conditions* $x\notin\{z\}^\wedge$ *and* $y\notin\{t\}^\wedge$ *must be given in both lemmas 4.3.1 and Corollary 4.3.6.*

Theorem 4.3.8. *Let* (X,τ) *be a topological space and* $y,z\in X$ *be such that* $\tau_{yz}=\tau\cap D_{yz}$ *is a strictly weaker topology than the topology* τ*. Then* y *and* z *satisfy the conditions (1), (2) and (3) of Theorem 4.3.7.*

Proof: (1) If $y\in\{z\}^\wedge$ then $\tau=\tau\cap D_{yz}$ and so $y\notin\{z\}^\wedge$. (2) If $z\in\{x\}^\wedge$ then $\tau=\tau\cap D_{zx}$ and so $\tau\cap D_{yz}=\tau\cap D_{yz}\cap D_{zx}\subset\tau\cap D_{yx}\subset\tau$ and if $x\notin\{z\}^\wedge$ then by Theorem 3.3.4, $\{z\}^\wedge_{yx}=\{z\}^\wedge$ implies that $\{z\}^\wedge_{yz}\neq\{z\}^\wedge_{yx}$ because $\{z\}^\wedge_{yz}\neq\{z\}^\wedge$ and $y\notin\{z\}^\wedge$ which implies that by Theorem 3.3.3, $\tau\cap D_{yz}\neq\tau\cap D_{yx}$ which implies that $\tau\cap D_{yx}=\tau$ since τ_{yz} is a strictly weaker topology than τ from which $y\in\{x\}^\wedge$.

(3) From Theorem 3.3.4, we finds that $\{z\}^\wedge_{yz}=\{y\}^\wedge\cup\{z\}^\wedge$ and if $x\in\{y\}^\wedge$ then $x\in\{z\}^\wedge_{yz}$ which implies that $\tau\cap D_{yz}\subset D_{xz}$ and so $\tau\cap D_{yz}\subset\tau\cap D_{xz}\subset\tau$. Now if $\tau_{yz}=\tau_{xz}$ then $\{z\}^\wedge_{yz}=\{z\}^\wedge_{xz}$ which implies that $\{y\}^\wedge\cup\{z\}^\wedge=\{x\}^\wedge\cup\{z\}^\wedge$ and so $y\notin\{x\}^\wedge$ implies that $y\in\{z\}^\wedge$ implies that $\tau\cap D_{yz}=\tau$ which contradicts that $\tau\cap D_{yz}$ is strictly weaker than τ and hence $\{z\}^\wedge_{yz}\neq\{z\}^\wedge_{xz}$ from which by Theorem 3.3.3, $\tau\cap D_{yz}\neq\tau\cap D_{xz}$ which implies that $\tau\cap D_{xz}=\tau$ which implies that $x\in\{z\}^\wedge$ and the proof is complete.

Theorem 4.3.9. *Let* (X,τ) *and* (X,τ^*) *be two topological spaces such that* τ^* *is strictly weaker than* τ *and let* τ *and* τ^* *have distinct families* $\beta^\wedge$ *and* $\beta^{*\wedge}$ *of minimal sets respectively. Then there are two distinct points* $y,z\in X$ *satisfying the conditions of Theorem 4.3.7, such that* $\tau^*=\tau\cap D_{yz}=\tau_{yz}$.

Proof: Let $\{x\}^{\wedge}$ and $\{x\}^{*\wedge}$ be the minimal sets with respect to τ and τ^* respectively for each point $x \in X$ since $\beta^{\wedge} \neq \beta^{*\wedge}$ then there is a point $z \in X$ such that $\{z\}^{\wedge} \neq \{z\}^{*\wedge}$ which implies that there is a point $y \in \{z\}^{*\wedge} - \{z\}^{\wedge}$ because $\{z\}^{\wedge} \subset \{z\}^{*\wedge}$ because $\tau^* \subset \tau$ this by Theorem 3.3.3 and since $y \notin \{z\}^{\wedge}$ and so $\tau \neq \tau \cap D_{yz}$ and $y \in \{z\}^{*\wedge}$ implies that $\tau^* \subset D_{yz}$. Hence we finds $\tau^* \subset \tau$ implies that $\tau^* \subset \tau \cap D_{yz} \subset \tau$. If τ^* is strictly weaker than τ then $\tau^* = \tau \cap D_{yz} = \tau_{yz}$. If $t \in X - \{z\}$ and $\{t\}^{\wedge} \neq \{t\}^{*\wedge}$ then by the same way there is a point $x \in \{t\}^{*\wedge} - \{t\}^{\wedge}$ such that $\tau^* = \tau \cap D_{xt} = \tau_{xt}$ and by Lemma 4.3.1, $\{x\}^{\wedge} = \{y\}^{\wedge}$ and $\{z\}^{\wedge} = \{t\}^{\wedge}$. Clearly by Theorem 4.3.8. y and z satisfy the conditions (1), (2) and (3) of Theorem 4.3.7.

Corollary 4.3.7. *Let (X, τ) and (X, τ^*) be two principal topological spaces. Then τ^* is a strictly weaker topology than τ iff there are two distinct points $y, z \in X$ satisfying the conditions (1), (2) and (3) of Theorem 4.3.7.*

Proof: It is a direct consequence of Theorems 4.3.7.

4.4. Topological subspaces.

Theorem 4.4.1. *Let $(X, \tau) \neq$ be a topological space and Y be a nonempty subset of X then the family $\tau_Y = \{G \cap Y : G \in \tau\}$ is a topology on Y.*

Proof: (1) Clearly $X \cap Y = Y$ and $\varnothing \cap Y = \varnothing$ which implies that $Y, \varnothing \in \tau_Y$. So T_1 of the definition (2 – 1 – 1) of the topology is satisfied.

(2) If $U_1, U_2 \in \tau_Y$ then there are $G_1, G_2 \in \tau$ such that $U_1 = G_1 \cap Y$ and $U_2 = G_2 \cap Y$ and so

$$U_1 \cap U_2 = (G_1 \cap Y) \cap (G_2 \cap Y) = (G_1 \cap G_2) \cap Y$$

And since $(G_1 \cap G_2) \in \tau$ then $U_1 \cap U_2 \in \tau_Y$. So T_2 is satisfied.

(3) Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of nonempty members of τ_Y. Then for each $\alpha \in \Delta$ there exists $G_\alpha \in \tau$ such that $U_\alpha = G_\alpha \cap Y$ and then

$$\bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta}(G_\alpha \cap Y) = (\bigcup_{\alpha\in\Delta} G_\alpha)\cap Y$$

and $(\bigcup_{\alpha\in\Delta} G_\alpha)\in\tau$ implies that $(\bigcup_{\alpha\in\Delta} U_\alpha)\in\tau_Y$. Hence τ_Y satisfies T_3.

From (1), (2) and (3), τ_Y is a topology on Y.

Theorem 4.4.2. *Let (X,τ) be a topological, Y be a nonempty subset of X and $i:Y\to X$ be the inclusion function i.e. $i(x)=x$ for each $x\in Y$. Then $\tau_Y=\{i^{-1}(G):G\in\tau\}$.*

Proof: We need to prove $i^{-1}(G)=G\cap Y$ for each $G\in\tau$. For, if $G\in\tau$ then $i^{-1}(G)\subset Y$ and

$$x\in i^{-1}(G) \Rightarrow x=i(x)\in G \Rightarrow x\in G\cap Y$$

$$\Rightarrow i^{-1}(G)\subset G\cap Y \qquad (I)$$

Conversely

$$x\in G\cap Y \Rightarrow x\in G \wedge x\in Y \Rightarrow x=i(x)\in i(G)$$

$$\Rightarrow x\in i^{-1}(G) \Rightarrow G\cap Y\subset i^{-1}(G) \qquad (II)$$

Hence (I) and (II) implies that $i^{-1}(G)=G\cap Y$.

Remark 4.4.1. *Let (X,τ) be a topological space and Y be a nonempty subset of X. Then*

(1) If $G\in\tau$ such that $G\subset Y$ then $G=G\cap Y$ which implies that $G\in\tau_Y$ This implies that $\{G\subset Y: G\in\tau\}\subset\tau_Y$.

(2) From the definition of τ_Y if $U\in\tau_Y$ then there is $G\in\tau$ such that $U=G\cap Y$. If $Y\in\tau$ then $U\in\tau$ which means that?

$$Y\in\tau \Rightarrow \tau_Y\subset\tau.$$

Definition 4.4.1. Let (X,τ) be a topological space and Y be a nonempty subset of X. Then the topology $\tau_Y=\{G\cap Y : G\in\tau\}$ on Y is called a relative or a partial topology of the topology τ on X and (Y,τ_Y) is called a relative topological space or a topological subspace of the topological space (X,τ). Briefly we write Y is a topological subspace of the topological space X.

Example 4.4.1. Let $X=\{\{a,b,c,d,e,g\}$, $Y=\{b,c,d,e\}$ and

$$\tau=\{\{X,\varnothing,\{a,b\},\{c\},\{c,d,e\},\{a,b,g\},\{a,b,c\},\{a,b,c,d,e\},\{a,b,c,g\}\}$$

Then $\tau_Y=\{Y,\varnothing,\{b\},\{c\},\{c,d,e\},\{b,c\}\}$.

Example 4.4.2. Consider the co-finite topological space (X,C) and let Y be a nonempty subset of X. Then

(a) $U\in C_Y-\{\varnothing\}$ implies that there is $G\in C$ such that $U=G\cap Y$, since $G^c=X-G$ is finite because $G\in C$ then

$$Y-U=Y-G\cap Y=Y\cap G^c\subset G^c$$

which implies that $Y-U$ is finite this for each element $U\in C_Y-\{\varnothing\}$ which means that C_Y is a subfamily of the co-finite topology on Y.

(b) If U is a subset of Y such that $Y-U$ is finite then $Y-U$ is a finite subset of X which implies that $(Y-U)^c\in C$ this means that $(Y-U)^c\cap Y\in C_Y$. But

$$(Y-U)^c\cap Y=(Y^c\cup U)\cap Y=(Y^c\cap Y)\cup(U\cap Y)=U$$

This means that $U\in C_Y$ for each subset U of Y such that $Y-U$ is finite which implies that the co-finite topology on Y is a subfamily of C_Y. From (a) and (b), C_Y is the co-finite topology on Y.

Theorem 4.4.3. *Let* (X,τ) *be a topological space,* Y *be a nonempty subset of* X *and* β *be a basis for* τ. *Then the family* $\beta_Y = \{B \cap Y : B \in \beta\}$ *is a basis for* τ_Y.

Proof: (1) $U \in \beta_Y \Rightarrow \exists B \in \beta : U = B \cap Y$, since β is a basis for τ then $\beta \subset \tau$ which implies that $B \in \tau$ which implies that $U \in \tau_Y$ and so $\beta_Y \subset \tau_Y$.

(2) If $U \in \tau_Y$ and $y \in U$ then there is $G \in \tau$ such that $U = G \cap Y$ which implies that $y \in G$ and since β is a basis for τ then there exists $B \in \beta$ such that $y \in B \subset G$ this implies that $y \in B \cap Y \subset G \cap Y = U$. So for each $U \in \tau_Y$ and each point $y \in U$ there exists $B^* \in \beta_Y$ ($B^* = B \cap Y$) such that $y \in B^* \subset U$. Therefore β_Y is a basis for τ_Y.

Example 4.4.3. Consider the discrete topological space (X,D) and let $Y \subset X$ be nonempty. Then D_Y is the discrete topology on Y since $y \in Y \Rightarrow y \in X \Rightarrow \{y\} \cap Y = \{y\} \in D_Y$ which implies that

$$\beta_{oY} = \{Y \cap \{x\} : x \in X\} \supset \{\{y\} : y \in Y\} \supset D_Y$$

is a basis for D_Y. Since $\{\{x\} : x \in X\}$ is a basis for D. Then D_Y is the discrete topology on Y.

Another solution:

$$A \subset Y \Rightarrow A \subset X \Rightarrow A \in D \Rightarrow A = A \cap Y \in D_Y$$
$$\Rightarrow P(Y) \subset D_Y \Rightarrow P(Y) = D_Y$$

Therefore D_Y is the discrete topology on Y.

Theorem 4.4.4. *Let* (Y,τ_Y) *be a subspace of a topological space* (X,τ). *If* $A \subset Y$ *then*

(1) A *is a closed set in* Y *iff there is a closed set* $F \in \tau_C$ *such that* $A = F \cap Y$ *i.e.* $A \in \tau_{Y^c} \Leftrightarrow \exists\, F \in \tau_C : A = F \cap Y$ *where*

$$\tau_{Y^c} = \{Y - U : U \in \tau_Y\},$$

(2) $\tau Y_C = \{F \cap Y : F \in \tau_C\}$*,*

(3) $A'_Y = A' \cap Y$*,*

(4) $\overline{A}_Y = \overline{A} \cap Y$*,*

(5) $A^{\wedge}_Y = A^{\wedge} \cap Y$*,*

(6) $A^o \subset A^o_Y$ *and* $A^o \neq A^o_Y$ *,*

(7) $b_y(A) \subset b(A)$ *and* $b_Y(A) \neq b(A)$ *and*

(8) $A^o = A^o_Y \cap Y^o$.

Where A'_Y, $\overline{A}_Y$, $A^{\wedge}_Y$, A^o_Y *and* $b_Y(A)$ *are the derived set, the closure, the minimal set, the interior and the boundary of* A *with respect to the topology* τ_Y *on* Y.

Proof: (1) If $F \in \tau_C$ such that $A = F \cap Y$ then

$F \in \tau_C \Rightarrow F^C \in \tau \Rightarrow Y \cap F^C \in \tau_Y$ and so

$$Y \cap F^C = Y - F = Y - (F \cap Y) = Y - A \Rightarrow Y - A \in \tau_Y$$
$$\Rightarrow A \in \tau_{Y^c}$$

Conversely if $A \in \left(\tau_Y\right)_C$ i.e. A is a closed set with respect to τ_Y then $Y - A \in \tau_Y$ and there exists $G \in \tau$ such that $Y - A = G \cap Y$. Hence

$$A = Y - G \cap Y = Y - G = Y \cap G^C = Y \cap F$$

where $F = G^c \in \tau_c$.

(2) It is a direct consequence of (1).

(3) If $y \in A'_Y$ and $G \in \tau$ such that $y \in G$ then $G \cap Y \in \tau_Y$ and hence

$$\begin{aligned}\emptyset \neq (G \cap Y) - \{y\} \cap A &= (G \cap Y) \cap \{y\}^c \cap A \\ &= G \cap \{y\}^c \cap (Y \cap A) = (G - \{y\}) \cap A \Rightarrow y \in A' \\ &\Rightarrow A'_Y \subset A' \Rightarrow A'_Y \subset A' \cap Y \quad (I)\end{aligned}$$

Conversely If $U \in \tau_Y$ such that $y \in U$ then there exists $G \in \tau$ such that $U = G \cap Y$ which implies that $y \in G$ and if $y \in A' \cap Y$ then $y \in A'$ and so

$$\begin{aligned}U - \{y\} \cap A &= G \cap Y - \{y\} \cap A = G - \{y\} \cap A \neq \emptyset \\ &\Rightarrow y \in A'_Y \Rightarrow A' \cap Y \subset A'_Y \quad (II)\end{aligned}$$

From (I) and (II) $A'_Y = A' \cap Y$.

(4) Directly from (3) .

$$\begin{aligned}A'_Y = A' \cap Y &\Rightarrow A \cup A'_Y = A \cup (A' \cap Y) \\ &\Rightarrow \overline{A}_Y = (A \cup A') \cap (A \cup Y) = (A \cup A') \cap Y \\ &\Rightarrow \overline{A}_Y = \overline{A} \cap Y\end{aligned}$$

Another proof: Regarding that $A \subset Y \Leftrightarrow A \cap Y = A$ we finds

$$\begin{aligned}x \in \overline{A}_Y &\Leftrightarrow U \cap Y \neq \emptyset; \forall U \in \tau_Y : x \in U \Leftrightarrow \\ &\Leftrightarrow G \cap Y \cap A \neq \emptyset; \forall G \in \tau : x \in G \Leftrightarrow \\ &\Leftrightarrow G \cap A \neq \emptyset; \forall G \in \tau : x \in G \Leftrightarrow x \in \overline{A} \\ &\Leftrightarrow x \in \overline{A} \cap Y \Rightarrow \overline{A}_Y = \overline{A} \cap Y\end{aligned}$$

Third proof:

$$\overline{A} = \bigcap\{F \in \tau_C : A \subset F\} \Rightarrow \overline{A} \cap Y$$
$$= [\bigcap\{F \in \tau_C : A \subset F\}] \cap Y = \bigcap\{F \cap Y : F \in \tau_C, A \subset F \cap Y\}$$
$$= \bigcap\{M \in \tau_{Y\ c} : A \subset M\} = \overline{A}_Y \Rightarrow \overline{A}_Y = \overline{A} \cap Y$$

(5) Since $A \subset Y$ implies that for each $G \in \tau$, $A \subset G$ iff $A \subset G \cap Y$ then

$$A_Y^{\wedge} = \bigcap\{U \in \tau_Y : A \subset U\} = \bigcap_{G \in \tau} \{G \cap Y : A \subset G \cap Y\}$$
$$= [\bigcap\{G \in \tau : A \subset G\}] \bigcap Y = A^{\wedge} \cap Y$$

(6) Since $A \subset Y$ then

$$x \in A^o \Rightarrow \exists G \in \tau : x \in G \subset A \Rightarrow x \in G \cap Y \subset A \cap Y = A$$
$$\Rightarrow x \in A_Y^o \Rightarrow A^o \subset A_Y^o$$

On the other hand If $X = \{\{a,b,c,d\}$, $Y = \{b,c,d\}$, $A = \{c,d\}$ and

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{c\}, \{a,d\}, \{a,c\}, \{a,b,c\},$$
$$\{a,b,d\}, \{a,c,d\}\}$$

Then $\tau_Y = \{Y, \varnothing, \{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}\} = D_Y$ which is the discrete topology on Y. So $A^o = \{c\}$ and $A_Y^o = \{c,d\}$ which implies that $A^o \neq A_Y^o$.

(7) Since $Y - A \subset A^c$ so

$$b_Y(A) = \overline{A}_Y \cap \overline{(Y-A)}_Y = \overline{A} \cap \overline{(Y-A)} \cap Y$$
$$\subset \overline{A} \cap \overline{A^c} \cap Y = b(A) \cap Y \Rightarrow b_Y(A) \subset b(A)$$

On the other hand consider the example given in (6) we finds

(a) $\overline{A} = \{c,d\}$ and $\overline{A^c} = \{a,b,d\}$ which implies that $b(A) = \overline{A} \cap \overline{A^c} = \{d\}$.

(b) $\overline{A}_Y=\{c,d\}$ and $\overline{(Y-A)}_Y=\{b\}$ which implies that $b_Y(A)=\overline{A}_Y\cap\overline{(Y-A)}_Y=\varnothing$.

From (a) and (b), $b(A)\not\subset b_Y(A)$ and so $b_Y(A)\neq b(A)$.

(8) From (6), $A^o\subset A^o_Y$ and $A\subset Y\Rightarrow A^o\subset Y^o$ from which

$$A^o\subset A^o_Y\cap Y^o \quad (I)$$

Conversely $y\in A^o_Y\cap Y^o$ implies that $y\in A^o_Y$ and $y\in Y^o$ and we finds

$$\begin{aligned} y\in A^o_Y &\Rightarrow \exists U\in\tau_Y: y\in U\subset A\Rightarrow \exists G\in\tau: U=G\cap Y\\ &\Rightarrow y\in G\cap Y\subset A \quad (1)\end{aligned}$$

Also since $A\cap Y=A$ then

$$y\in Y^o\Rightarrow \exists H\in\tau: y\in H\subset Y\Rightarrow y\in G\cap H\cap Y\subset A$$

But $G\cap H\subset H\subset Y\Rightarrow G\cap H=G\cap H\cap Y \quad (2)$.

From (1) and (2) and since $G\cap H\in\tau$ we gets

$$\begin{aligned} y\in G\cap H=G\cap H\cap Y\subset A\Rightarrow y\in A^o\\ \Rightarrow A^o_Y\cap Y^o\subset A^o \quad (II)\end{aligned}$$

From (I) and (II), $A^o=A^o_Y\cap Y^o$.

Theorem 4.4.5. *Let* (X,τ) *be a topological space and* $Z\subset Y\subset X$ *be such that* Z *is nonempty. Then* $\tau_Z=(\tau_Z)_Y$.

Proof: Suppose that $H\in\tau_Z$ then there exist $G\in\tau$ such that $H=G\cap Z$ but $Z\subset Y\Leftrightarrow Z=Y\cap Z$ which implies that

$H = G \cap (Y \cap Z) = (G \cap Y) \cap Z$ but $U = G \cap Y \in \tau_Y$ then $H \in (\tau_Y)_Z$ which implies that $\tau_Z \subset (\tau_Y)_Z \quad (I)$

On the other hand if $H \in (\tau_Y)_Z$ then there exists $U \in \tau_Y$ such that $H = U \cap Z$ and there exists $G \in \tau$ such that $U = G \cap Y$. Hence

$$H = (G \cap Y) \cap Z = G \cap (Y \cap Z) = G \cap Z \Rightarrow H \in \tau_Z .$$

Therefore $(\tau_Y)_Z \subset \tau_Y \quad (II)$.

From (I) and (II), we have that $\tau_Z = (\tau_Z)_Y$.

Exercises

1 – Consider the set of the real numbers R and show that: $\beta = \{(a,b) : a,b \in Q\}$ is a basis for the usual topology $\mho$ on R .

2 – Consider the set R^2 where R is the set of the real numbers and show that (a) The family of all open squares which has parallel sides to the coordinates axes is a basis for $\mho^2$ on R^2. (b) The family of all triangles which have equals sides form a basis for $\mho^2$ on R^2.

3 – If $X \neq \varnothing$ and $p,q \in X$ show that $\beta_1 = C - \{X - \{p\}\}$, $\beta_2 = C - \{G\}$ where $G \in C$ and $\beta_3 = C - \{X - \{p,q\}\}$ are bases for the co-finite topology C on X . If X is infinite and if $|X| = 4$ then β_3 fail to be a basis for C .

4 – If $X \neq \varnothing$ and $p \in X$ write the minimal basis for each of D, E_p and P_p . If $X = \{a,b,c,d\}$ write the minimal basis for each of I, D, E_p and P_p .

5 – Consider the set R of the real numbers and Prove that if $\beta = \{(a,b] : a,b \in R\}$ is a basis for the upper limit point topology τ on R then $\mho \subset \tau$.

6 – If $X = \{a,b,c,d,e\}$ and $\sigma = \{a,b,c\},\{a,c,d\},\{b,c\}\}$ find the topology on X generated by σ. Also write a subbasis for the discrete topology D consist no single and no two points subsets of X .

7 – Write the topology generated by the family

(a) $\sigma = \{[n, n+1] : n \in N\}$ on R ,

(b) $\sigma=\{[a,a+1]:a\in R\}$ on R ,

(c) $\sigma=\{\{1\},\{n,n+1\}:n\in N\}$ on N and also on R and

(e) $\sigma=\{[a,b]:a\in Q\}$ on R . In this case show that $\beta=\sigma\cup\{\{x\}:x\in Q\}$ is a basis for the same topology on R which generated by σ (hint: show that $\beta\subset\beta(\sigma)\subset\tau(\beta(\sigma))$).

8 – Show that the topology τ on a nonempty set X is finite if X is finite but not conversely.

9 – If (X,τ) is a topological space, $p\in X$ and N_p is the neighborhood system of the point p show that for each $U\in N_p$ there exists $B\in\beta_p$ such that $B\subset U$ where β_p is a local basis for the point p .

10 – If σ is a local basis for a topology τ on a set X and $p\in X$ show that $\sigma_p=\{S\in\sigma:p\in S\}$ may be not forms a local basis for τ and show that the family of finite intersection of the members of σ_p forms a local basis for τ .

11 – Show that if σ is a subbasis for a topology τ on a set X then $\sigma-\{X,\varnothing\}$ is also.

12 – If X is a nonempty set show that $\sigma=\{\{x\}^c:x\in X\}$ is a subbasis for the co-finite topology C on X where $\{x\}^c=X-\{x\}$.

13 - Consider the topological spaces (X,τ_1) and (X,τ_2) and let β_1, β_2 be two bases for τ_1 and τ_2 respectively. Then show that $\beta_1\cup\beta_2$ is not necessary a basis for any topology on X and find the topology τ on X which generated by $\beta_1\cup\beta_2$ and show the relation between τ and each of τ_1 and τ_2 .

15 – Consider the topological spaces (X,C) and (X,τ) where $\tau=E_p\cup C$ and $p\in X$ find $\{x\}^\wedge=\bigcap\{G\in C:x\in G\}$ and $\{x\}^\wedge=\bigcap\{G\in\tau:x\in G\}$ for each point $x\in X$ and show is $\{x\}^\wedge$ the minimal open set for x or no? Why?. Prove that $\beta=\{\{x\}^\wedge:x\in X\}$ is a basis for a topology $\tau^\wedge$ on X in both cases and discuss the relation between $\tau^\wedge$ and each of C and τ .

19 – Consider the set $X=\{1,2,3,4,5,6\}$ and let $\tau=\{X,\varnothing,\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\}\}$ then show that:

(a) $\beta_{32}^\wedge=\beta_{42}^\wedge=\beta_{52}^\wedge$ which is the minimal basis given by Theorem 4.3.7.

(b) the topology $\tau_{\overset{\wedge}{52}}$ on X which has the minimal basis $\beta_{\overset{\wedge}{52}}$ does not equals to $\tau_{52} = \tau \cap D_{52}$.

(c) the topology $\tau_{\overset{\wedge}{42}}$ on X which has the minimal basis $\beta_{\overset{\wedge}{42}}$ does not equals to $\tau_{42} = \tau \cap D_{42}$.

(d) The topologies $\tau_{\overset{\wedge}{32}}, \tau_{\overset{\wedge}{42}}$ and $\tau_{\overset{\wedge}{52}}$ generated by the bases $\beta_{\overset{\wedge}{32}}, \beta_{\overset{\wedge}{42}}$ and $\beta_{\overset{\wedge}{52}}$ respectively are equals and equal to the topology $\tau_{32} = \tau \cap D_{32}$

(e) any of the two points 5 and 2 and the points 4 and 2 does not satisfy the conditions of Theorem (21) but the points 3 and 2 satisfy these conditions.

20 – Consider the usual topological space $(R, \mho)$ and the basis $\beta\{(a,b) : a,b \in R\}$ of the topology $\mho$. If $Y = [c,d] \subset R$, write the basis $\beta_Y = \{B \cap Y : B \in \beta\}$.

21 – Consider the space topological space (X, τ), $p \in X$ and the set $\varnothing \neq Y \subset X$ write τ_Y in each of the following cases:

(a) (X, E_p), $Y = X - \{p\}$.

(b) (X, P_p), $Y = X - \{p\}$.

(c) (X, τ) where $\tau = E_p \cup C, Y = X - \{p\}$.

(d) (R, τ) where τ is (1) the right rays, (2) the left rays topological space and $Y = N$, $Y = Z$, $Y = [0, \infty)$ and $(-\infty, 0)$.

(e) (X, τ) where $X = \{1,2,3,4,5,6,7,8\}$ and τ is the topology on X which has the basis $\beta = \{\{1\},\{1,2,5\},\{3,4\},\{1,6\},\{7,8\}\}$ if $Y = \{1,2,6,4,8\}$ and if $Y = \{1,3,4,5,7\}$.

22 – If (X, τ) is a topological space, Y and Z are nonempty subsets of X prove that

(a) $Y \neq Z$ does not implies that $\tau_Y - \{Y\} \neq \tau_Z - \{Z\}$.

(b) There is no comparison between $\tau_{(Y\cap Z)}$ and $\tau_{(Y\cup Z)}$.

(c) $(\tau_Y)_Z = \tau_{(Y\cap Z)} = (\tau_Z)_Y$.

(d) $\tau_Y \cap \tau_Z \subset \tau_{(Y\cup Z)}$.

(e) If $W \in \tau_{(Y\cup Z)}$ then there are $U \in \tau_Y$ and $V \in \tau_Z$ such that $W = U \cup V$.

(g) If $W \in \tau_{(Y\cap Z)}$ then there are $U \in \tau_Y$ and $V \in \tau_Z$ such that $W = U \cap V$.

23 – If (X,τ) is a topological space and Y is a nonempty subset of X prove that $Y \in \tau_c \Rightarrow \left(\tau_Y\right)_c \subset \tau_c$.

23– If (X,τ) is a topological space prove that

(a) $\tau_Y = I_Y$ is the indiscrete topology on Y such that $|Y| = 2$ where $|Y|$ is the cardinal number of Y iff $\tau = I$ that is $\tau_Y = I_Y \; \forall Y \subset X : |Y| = 2 \Leftrightarrow \tau = I$.

(b) $\tau_Y = D_Y$ is the discrete topology on Y for each subset Y of X such that $|Y| \geq 2$ where $|Y|$ is the cardinal number of Y iff $\{x\}^{\wedge} = \{x\}$ for each point $x \in X$ that is

$$\tau_Y = D_Y \;\forall\; Y \subset X : |Y| = 2 \Leftrightarrow \{x\}^{\wedge} = \{x\};\; \forall\; x \in X.$$

If τ is a principal topology on X then

$$\tau_Y = D_Y \;\forall\; Y \subset X : |Y| = 2 \Leftrightarrow \tau = D$$

24 – If (X,τ) is a topological space and Y is a nonempty set such that $X \cap Y = \varnothing$ and $X^* = X \cup Y$, consider the family $\tau^* \subset P(X^*)$ which

consists the empty set and the sets $G^* \subset X^*$ such that $G^* = G \cup Z$ where $G \in \tau$ and Z be such that $Y - Z$ is finite prove that τ^* is a topology on X^* and Y is dense in X^*. If $A \subset X^*$ prove that $\overline{A}_{X^*} = (\overline{X \cap A}) \cap (Y \cap A)$ if $Y \cap A$ is finite and $\overline{A}_{X^*} = X^*$ if $Y \cap A$ is infinite.

25 – Prove that if (Y, τ_Y) is a topological subspace of (X, τ) and

(a) $A \subset Y$ Then $ext(A) = (Y - A)^o = ext_Y(A) \cap Y^o$.

(b) $A \subset Y \Rightarrow isd.(A) = A \Rightarrow isd_Y(A) = A$.

(c) $A \subset X$ Then $isd.(A \cap Y) = isd_Y(A \cap Y)$.

Chapter V

Continuous Functions in Topological Spaces

Topics:

- Continuity.
- Homeomorphism and Topological Properties.

5.1. Continuity.

Definition 5.1.1. Consider the topological spaces (X,τ) and (Y,τ^*) then the function $f:X\to Y$ is said to be continuous at the point $x\in X$ if for each open subset V of Y such that $f(x)\in V$ there is an open subset U of X such that $x\in U$ and $f(U)\subset V$ i.e.

$$V\in\tau^*: f(x)\in V \Rightarrow \exists U\in\tau : x\in U, f(U)\subset V$$

If the function f is continuous at each point of X then f is said to be continuous function on X .

The following theorem gives us equivalent definitions of the continuity of the function f from a topological space to another one.

Theorem 5.1.1. *Let (X,τ) and (Y,τ^*) be two topological spaces and $f:X\to Y$ be any function. Then the following statements are equivalent:*

(1) f is continuous,

(2) The inverse of any open set by the function f is open that is $f^{-1}(V)\in\tau$ for each $V\in\tau^*$,

(3) The inverse of a closed set by the function f is a closed set that is $f^{-1}(F)\in\tau_c$ for each $F\in\tau_c^*$,

(4) The closure of the inverse by f of a subset of Y is a subset of the inverse of its closure that is $\overline{f^{-1}(B)}\subset f^{-1}(\overline{B})$ for each $B\subset Y$ and

(5) The image of the closure of a subset of X by f is a subset of the closure of its image that is $f(\overline{A})\subset\overline{f(A)}$ for each $A\subset X$.

Proof: We shall prove that

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and there are another ways for the proof which the reader can be used.

(1) $\Rightarrow$ (2): Let $x \in X$ and $V \in \tau^*$ then $x \in f^{-1}(V) \Rightarrow f(x) \in V$ and from (1) where f is continuous there is $U \in \tau$ such that $x \in U$ and $f(U) \subset V$ from which

$$x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V) \Rightarrow x \in (f^{-1}(V))^o \Rightarrow f^{-1}(V)$$
$$\subset (f^{-1}(V))^o \Rightarrow (f^{-1}(V))^o = f^{-1}(V) \Rightarrow f^{-1}(V) \in \tau \; .$$

(2) $\Rightarrow$ (3): From (2) we get

$$F \in \tau^*_c \Rightarrow F^c = Y - F \in \tau^* \Rightarrow f^{-1}(Y - F) \in \tau$$
$$\Rightarrow f^{-1}(Y - F) = X - f^{-1}(F) \in \tau \Rightarrow f^{-1}(F) \in \tau_c$$

(3) $\Rightarrow$ (4): Suppose that $B \subset Y$ then $\overline{B} \in \tau^*_c$ and from (3), $f^{-1}(\overline{B}) \in \tau_c$ from which $\overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$ then

$$B \subset \overline{B} \Rightarrow f^{-1}(B) \subset f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subset \overline{f^{-1}(\overline{B})} =$$
$$= f^{-1}(\overline{B}) \Rightarrow \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$$

(4) $\Rightarrow$ (5): Suppose that $A \subset X$ then $f(A) \subset Y$, then by substituting $B = f(A)$ in (4) we gets

$$A \subset f^{-1}(f(A) \Rightarrow \overline{A} \subset \overline{f^{-1}(f(A))} \subset f^{-1}(\overline{f(A)})$$
$$\Rightarrow f(\overline{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)} \Rightarrow f(\overline{A}) \subset \overline{f(A)}$$

(5) $\Rightarrow$ (1): Let $x \in X$ be an arbitrary point and $V \in \tau^*$ be such that $f(x) \in V$ then $(i)\; V^c = Y - V \in \tau^*_c$ from which $\overline{V^c} = V^c$ and $x \in f^{-1}(V)$ then by substituting $A = f^{-1}(V^c)$ in (5),

$$f(\overline{f^{-1}(V^c)}) \subset \overline{f(f^{-1}(V^c))} \subset \overline{V^c} = V^c$$
$$\Rightarrow \overline{f^{-1}(V^c)} \subset f^{-1}(f(\overline{f^{-1}(V^c)})) \subset f^{-1}(V^c)$$

$$\Rightarrow \overline{f^{-1}(V^c)} = f^{-1}(V^c) \Rightarrow f^{-1}(V^c) \in \tau_c$$

$$\Rightarrow f^{-1}(Y - V) = X - f^{-1}(V) \in \tau_c \Rightarrow f^{-1}(V) \in \tau$$

Set $U = f^{-1}(V)$ then $U \in \tau$, $x \in U$ and $U = f^{-1}(V) \Rightarrow f(U) \subset V$ which implies that f is a continuous function at the arbitrary point $x \in X$ and so is continuous.

Theorem 5.1.2. *Let (X, τ) and (Y, τ^*) be two topological spaces and $f : X \to Y$. Then the following statements are equivalent:*

(1) f is continuous,

(2) $f^{-1}(B^o) \subset (f^{-1}(B))^o$ for each $B \subset Y$,

(3) If β^* is a basis for τ^* then $f^{-1}(B) \in \tau$ for each $B \in \beta^*$ and

(4) If σ^* is a subbais for τ^* then $f^{-1}(S) \in \tau$ for each $S \in \sigma^*$.

Proof: The statement (1) of this theorem is equivalent to the statement (2) of Theorem 5.1.1, and we can prove that each of the statements (2), (3) and (4) of this theorem equivalent to (2) of Theorem 5.1.1. Now we shall prove that (2) of this theorem is equivalent to (1) i.e. equivalent to (2) of Theorem 5.1.1, and left the others to the reader.

Firstly (1) $\Rightarrow$ (2): Suppose that $B \subset Y$ and f is continuous, by (2) of Theorem 5.1.1, we gets

$B^o \in \tau^* \Rightarrow f^{-1}(B^o) \in \tau \Rightarrow f^{-1}(B^o) = (f^{-1}(B^o))^o$ and so

$$B^o \subset B \Rightarrow f^{-1}(B^o) \subset f^{-1}(B)$$

$$\Rightarrow f^{-1}(B^o) = (f^{-1}(B^o))^o \subset (f^{-1}(B))^o .$$

$$\Rightarrow f^{-1}(B^o) \subset (f^{-1}(B))^o$$

(2) $\Rightarrow$ (1): Suppose that $f^{-1}(B^o) \subset (f^{-1}(B))^o$ for each $B \subset Y$ then

$$V \in \tau^* \Rightarrow V^o = V \Rightarrow f^{-1}(V) = f^{-1}(V^o) \subset (f^{-1}(V))^o$$

$$\Rightarrow f^{-1}(V) \subset (f^{-1}(V))^o \Rightarrow (f^{-1}(V))^o = f^{-1}(V).$$

$$\Rightarrow f^{-1}(V) \in \tau$$

Theorem 5.1.3. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces,* $f: X \to Y$ *be a continuous function at the point* $x \in X$ *and* $< x_n >$ *be a sequence of points of* X *convergent to* x *i.e.* $x_n \to x$. *Then the sequence* $< f(x_n) >$ *convergent to the point* $f(x)$ *that is* $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Proof: Suppose that $V \in \tau^*$ such that $f(x) \in V$ then $x \in f^{-1}(V)$ and since f is continuous then $f^{-1}(V) \in \tau$. If $x_n \to x$ then there is a positive integer $n_o \in N$ such that $n \geq n_o \Rightarrow x_n \in f^{-1}(V)$ and so $n \geq n_o \Rightarrow f(x_n) \in f(f^{-1}(V)) \subset V$ which implies that $f(x_n) \to f(x)$.

Theorem 5.1.4. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces and*

(1) $f: X \to Y$ *is a continuous function.*

(2) $f(A^\wedge) \subset (f(A))^\wedge$ *for each* $A \subset X$ *and*

(3) $(f^{-1}(B))^\wedge \subset f^{-1}(B^\wedge)$ *For each* $B \subset Y$.

Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3) and if* (X,τ) *is a principal topological space then the three statements are equivalent.*

Proof: (1) $\Rightarrow$ (2): If $V \in \tau^*$ then by Theorem 5.1.1,2

$$A \subset f^{-1}(V) \Rightarrow f(A) \subset f(f^{-1}(V)) \subset V \Rightarrow f(A) \subset V$$
$$\Rightarrow A \subset f^{-1}(f(A)) \subset f^{-1}(V)$$

that is $A \subset f^{-1}(V) \Leftrightarrow f(A) \subset V$.

Now $A^\wedge = \bigcap\{G \in \tau : A \subset G\}$ and if f is continuous then

$$\{f^{-1}(V) : V \in \tau^* \wedge A \subset f^{-1}(V)\} \subset \{G \in \tau : A \subset G\}$$

And so $A^\wedge \subset \bigcap\{f^{-1}(V) : V \in \tau^*, A \subset f^{-1}(V)\}$ which implies that

$$f(A^\wedge) \subset \bigcap\{f(f^{-1}(V)) : V \in \tau^*, A \subset f^{-1}(V)\}$$

But $f(f^{-1}(V)) \subset V$ for each $V \in \tau^*$ from which

$$f(A^\wedge) \subset \bigcap\{V \in \tau^* : A \subset f^{-1}(V)\} = \bigcap\{V \in \tau^* : f(A) \subset V\}$$
$$= (f(A))^\wedge$$

Another proof of (1) $\Rightarrow$ (2): Suppose that $A \subset X$, since $f^{-1}(V) \in \tau$ for each $V \in \tau^*$ and since f is continuous then

$$V \in \tau^* \wedge f(A) \subset V \Rightarrow A \subset f^{-1}(f(A)) \subset f^{-1}(V)$$

$$\Rightarrow A^{\wedge} \subset f^{-1}(V) \Rightarrow f(A^{\wedge}) \subset f(f^{-1}(V)) \subset V$$

$$\Rightarrow f(A^{\wedge}) \subset \bigcap \{V \in \tau^* : f(A) \subset V\} = (f(A))^{\wedge}$$

$$\Rightarrow f(A^{\wedge}) \subset (f(A))^{\wedge}$$

(2) $\Rightarrow$ (3): Suppose that $B \subset Y$ then

$$A = f^{-1}(B) \Rightarrow f(A) = f(f^{-1}(B)) \subset B \Rightarrow f(A) \subset B$$

$$\Rightarrow (f(A))^{\wedge} \subset B^{\wedge}$$

From (2)

$$f(A^{\wedge}) \subset (f(A))^{\wedge} \subset B^{\wedge} \Rightarrow f(A^{\wedge}) \subset B^{\wedge}$$

$$\Rightarrow A^{\wedge} \subset f^{-1}(f(A^{\wedge})) \subset f^{-1}(B^{\wedge}) \Rightarrow A^{\wedge} \subset f^{-1}(B^{\wedge})$$

$$\Rightarrow (f^{-1}(B))^{\wedge} \subset f^{-1}(B^{\wedge})$$

One can prove (2) $\Rightarrow$ (3) directly since if $B \subset Y$ then by substituting $A = f^{-1}(B) \subset X$ in (2) we gets

$$f((f^{-1}(B))^{\wedge}) \subset [f((f^{-1}(B))]^{\wedge} \subset B^{\wedge}$$

$$\Rightarrow (f^{-1}(B))^{\wedge} \subset f^{-1}[f((f^{-1}(B))^{\wedge})] \subset f^{-1}(B^{\wedge})$$

$$\Rightarrow (f^{-1}(B))^{\wedge} \subset f^{-1}(B^{\wedge})$$

If (X, τ) is a principal topological space then to prove that the three statements are equivalent we needs only to prove that

(3) $\Rightarrow$ (1) for let $(f^{-1}(B))^{\wedge} \subset f^{-1}(B^{\wedge})$ for each $B \subset Y$ then

$$V \in \tau^* \Rightarrow V^{\wedge} = V \Rightarrow (f^{-1}(V))^{\wedge} \subset f^{-1}(V^{\wedge}) = f^{-1}(V)$$

$$\Rightarrow (f^{-1}(V))^{\wedge} = f^{-1}(V)$$

Then by Theorem 5.1.2, (X,τ) is principal implies that $(f^{-1}(V))^{\wedge}\in\tau$ and so $f^{-1}(V)\in\tau$. Hence f is continuous.

(1) $\Rightarrow$ (3) if $B\subset Y$ then

$$B^{\wedge}=\bigcap\{V\in\tau^{*}:B\subset V\}$$

$$\Rightarrow f^{-1}(B^{\wedge})=\bigcap_{V\in\tau^{*}}\{f^{-1}(V):B\subset V\}$$

But $B\subset V\Rightarrow f^{-1}(B)\subset f^{-1}(V)$ which implies that

$\{V\in\tau^{*}:B\subset V\}\subset\{V\in\tau^{*}:f^{-1}(B)\subset f^{-1}(V)\}$ which implies that

$$\{f^{-1}(V):V\in\tau^{*},B\subset V\}\subset\{f^{-1}(V):V\in\tau^{*},f^{-1}(B)\subset f^{-1}(V)\}$$

And then

$$\bigcap_{V\in\tau^{*}}\{f^{-1}(V):f^{-1}(B)\subset f^{-1}(V)\}$$

$$\subset\bigcap_{V\in\tau^{*}}\{f^{-1}(V):B\subset V\}=f^{-1}(B^{\wedge})$$

If f is continuous then $f^{-1}(V)\in\tau$ for each $V\in\tau^{*}$ and then

$$(f^{-1}(B))^{\wedge}=\bigcap\{G\in\tau:f^{-1}(B)\subset G\}$$

$$\subset\bigcap_{V\in\tau^{*}}\{f^{-1}(V):f^{-1}(B)\subset f^{-1}(V)\}$$

$$\subset\bigcap_{V\in\tau^{*}}\{f^{-1}(V):B\subset V\}=f^{-1}(B^{\wedge})$$

$$\Rightarrow(f^{-1}(B))^{\wedge}\subset f^{-1}(B^{\wedge})$$

Example 5.1.1. If (X,τ) is a topological space then the inclusion functioning $i:X\to X$ where $i(x)=x$ for each $x\in X$ is continuous since $i^{-1}(V)=V\in\tau$ for each $V\in\tau$.

Example 5.1.2. If (X,τ) and (Y,τ^{*}) are two topological spaces then the constant function $f:X\to Y$ where $f(x)=b$ for each $x\in X$ and $b\in Y$ is any point is a continuous function since

$$V \in \tau^* \Rightarrow f^{-1}(V) = \begin{cases} \varphi; & b \notin V \\ X\,; & b \in V \end{cases} \quad \Rightarrow f^{-1}(V) \in \tau$$

Example 5.1.3. Consider the topological spaces (X, D) and (Y, τ) then the function $f : X \to Y$ is continuous whatever may be the topology τ on Y since $V \in \tau \Rightarrow f^{-1}(V) \in P(X) = D$.

Example 5.1.4. The function $f : (-1,1) \to R$ where $f(x) = \dfrac{x}{1-x^2}$ for each point $x \in (-1,1)$ is continuous since it is division of two continuous functions defined on $(-1,1)$ and its denominator is not equals to zero for any point $x \in (-1,1)$ by considering the usual topology $\mho$ on R and the relative topology on $(-1,1)$ of the topology $\mho$, see the figure of the curve $f(x) = \dfrac{x}{1-x^2}$

Example 5.1.5. Consider the usual topological space $(R, \mho)$ and the function $f : R \to R$ which is given by

$$f(x) = \begin{cases} -1; & x \in (-\infty, 0] \\ \ \ 1; & x \in (0, \infty) \end{cases}$$

For each point $x \in R$, f is not continuous since $(-\infty, 0) \in \mho$ while $f^{-1}((-\infty, 0)) = f^{-1}(\{-1\}) = (-\infty, 0] \notin \mho$.

Example 5.1.6. Let $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}, \{b, c\}\}$, $Y = \{1,2,3\}$ and $\tau^* = \{Y, \varnothing, \{1\}, \{1,2\}\}$. Then the function $f : X \to Y$ where $f = \{(a,1), (b,1), (c,2)\}$ is not continuous since $\{1\} \in \tau^*$ while $f^{-1}(\{1\}) = \{a, b\} \notin \tau$.

Remark 5.1.1. *The converse of Theorem 5.1.3, is incorrect, for consider the co-countable topological space* (R, τ) *if* $< x_n >$ *is a sequence of points of* R *convergent to the point* $p \in R$ *then* $< x_n >$ *is of the form* $< x_n > = \{x_1, x_2, \ldots\ldots, x_n, p, p, p, \ldots\ldots\}$ *see Example 5.1.1,2,3 . By considering the function* $f : (R, \tau) \to (R, \mho)$ *given by* $f(x) = x$ *for each* *point* $x \in R$ *then* $f(x_n) \to f(p)$ *and* $(0,1) \in \mho$ *while* $f^{-1}((0,1)) = (0,1) \notin \tau$ *since* $(0,1)^c = R - (0,1)$ *is uncountable which implies that* f *is not continuous.*

Theorem 5.1.5. *If* (X,τ_1), (Y,τ_2) *and* (Z,τ_3) *are three topological spaces and* $f : X \to Y$ *and* $g : Y \to Z$ *are two continuous functions then* $g \circ f : X \to Z$ *is continuous.*

Proof: Suppose that $W \in \tau_3$ then $g^{-1}(W) \in \tau_2$ since g is continuous and $f^{-1}(g^{-1}(W)) \in \tau_1$ since f is continuous. But

$$(g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W) = f^{-1}(g^{-1}(W))$$

Which means that $g \circ f$ is continuous.

Theorem 5.1.6. *If* $X \neq \emptyset$, τ *and* τ^* *are two topologies on* X *then* τ *is stronger than* τ^* *iff the inclusion function* $i : X \to X$ *is continuous where* $i(x) = x$ *for each* $x \in X$ *.*

Proof: Suppose that i is continuous then

$$U \in \tau^* \Rightarrow i^{-1}(U) = U \in \tau \Rightarrow \tau^* \subset \tau$$

Conversely if $\tau^* \subset \tau$ then $U \in \tau^* \Rightarrow U \in \tau$ but $i^{-1}(U) = U$ for each $U \in \tau^*$. Hence i is continuous.

Theorem 5.1.7. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces and* $A \subset X$ *, if the function* $f : X \to Y$ *is surjective and continuous. Then*

(1) $f(x)$ *is an interior point of* $f(A)$ *implies that* x *is an interior point of* A *that is*

$$f(x) \in (f(A))^o \Rightarrow x \in A^o$$

(2) The point x *is a limit point of* A *implies that* $f(x)$ *is a limit point of* $f(A)$ *that is*

$$x \in A' \Rightarrow f(x) \in (f(A))'.$$

(3) The point x *is a boundary point of* A *implies that* $f(x)$ *is a boundary point of* $f(A)$ *that is*

$$x \in b(A) \Rightarrow f(x) \in b(f(A)).$$

Proof: we shall prove (3) and left (1) and (2) to the reader, for if $f(x) \notin b(f(A))$ then there exists an open set $V \in \tau^*$ such that $f(x) \in V$ and $V \cap f(A) = \varnothing$ or $V \cap (Y - f(A)) = \varnothing$ if f is surjective then by Theorem , $f^{-1}(f(A)) = A$ and

$f^{-1}(Y - f(A)) = f^{-1}(Y) - f^{-1}(f(A)) = X - A = A^c$ and since f is continuous then $f^{-1}(V) \in \tau$ and therefore

$$\begin{aligned} V \cap f(A) = \varnothing \Rightarrow f^{-1}(V \cap f(A)) &= f^{-1}(V) \cap f^{-1}(f(A)) \\ &= f^{-1}(V) \cap A = \varnothing \end{aligned}$$

or

$$\begin{aligned} V \cap (Y - f(A)) = \varnothing \Rightarrow & f^{-1}(V) \cap f^{-1}((Y - f(A))) \\ &= f^{-1}(V) \cap A^c = \varnothing \end{aligned}$$

both cases implies that $x \notin b(A)$, the contra positive of this result is the required aim which is $x \in b(A) \Rightarrow f(x) \in b(f(A))$.

Remark 5.1.2. *if (X, τ) is a topological space, Y is a non empty set and $f : X \to Y$ is any function then the family $f^*(\tau) = \{V \subset Y : f^{-1}(V) \in \tau\}$ forms a topology on Y, the function $f : (X, \tau) \to (Y, f^*(\tau))$ is continuous and $f^*(\tau)$ is the strongest topology on Y such that the function f is continuous that is if τ^* is a topology on Y such that if $f : (X, \tau) \to (Y, \tau^*)$ is continuous then $\tau^* \subset f^*(\tau)$ that is if f is continuous then*

$$V \in \tau^* \Rightarrow f^{-1}(V) \in \tau \Rightarrow V \in f^*(\tau) \Rightarrow \tau^* \subset f^*(\tau)$$

and we have the following three Propositions:

Proposition 5.1.1. *If (X, τ) is a topological space, Y is a nonempty set and $f : (X, \tau) \to (Y, f^*(\tau))$. Then $f^{-1}(F)$ is a closed set iff F is a closed set for each $F \subset Y$ that is*

$$F \in (f^*(\tau))_c \Leftrightarrow f^{-1}(F) \in \tau_c$$

Proof: Since by Remark 5.1.2, if the function f is continuous then

$$F \in (f^{*}(\tau))_c \Leftrightarrow Y - F \in f^{*}(\tau) \Leftrightarrow X - f^{-1}(F) =$$

$$f^{-1}(Y - F) \in \tau \Leftrightarrow f^{-1}(F) \in \tau_c$$

Proposition 5.1.2. *Consider the topological spaces* (X, τ_1), $(Y, f^{*}(\tau_1))$ *and* (Z, τ_2) *and* $f : (X, \tau_1) \to (Y, f^{*}(\tau_1))$. *If* $g : (Y, f^{*}(\tau_1)) \to (Z, \tau_2)$ *is any function then* g *is continuous iff* $g \circ f : (X, \tau_1) \to (Z, \tau_2)$ *is continuous.*

Proof: Clearly by Remark 5.1.2, f is continuous and so by Theorem 5.1.7, $g \circ f$ is continuous.

Conversely, suppose that $g \circ f$ is continuous then

$$V \in \tau_2 \Rightarrow f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in \tau_1$$

$$\Rightarrow g^{-1}(V) \in f^{*}(\tau_1)$$

Proposition 5.1.3. *Let* (X, τ_1) *be a topological space,* Y *and* Z *be two nonempty sets and consider the functions* $f : (X, \tau_1) \to (Y, f^{*}(\tau_1))$ *and* $g : (Y, f^{*}(\tau_1)) \to (Z, g^{*}(f^{*}(\tau_1)))$. *Then*

$$(g \circ f)^{*}(\tau_1) = g^{*}(f^{*}(\tau_1)).$$

Proof: Clearly f and g are continuous and the function also by Theorem 5.1.6, $g \circ f : (X, \tau_1) \to (Y, g^{*}(f^{*}(\tau_1)))$ is continuous which implies that

$$g^{*}(f^{*}(\tau_1)) \subset (g \circ f)^{*}(\tau_1) \quad (I)$$

Conversely,

$$V \in (g \circ f)^{*}(\tau_1) \Rightarrow (g \circ f)^{-1}(V) \in \tau_1 \Rightarrow f^{-1}(g^{-1}(V)) \in \tau_1$$

$$\Rightarrow g^{-1}(V) \in f^{*}(\tau_1) \Rightarrow V \in g^{*}(f^{*}(\tau_1)$$

$$\Rightarrow (g \circ f)^{*}(\tau_1) \subset g^{*}(f^{*}(\tau_1) \quad (II)$$

From (I) and (II), $(g \circ f)^{*}(\tau_1) = g^{*}(f^{*}(\tau_1))$.

(b) if (Y, τ^{*}) is a topological space, X is a nonempty set and $f : X \to Y$ is any function then the family $f^{-1}(\tau^{*}) = \{f^{-1}(V) : V \in \tau^{*}\}$ forms a topology on X and clearly the function $f : (X, f^{-1}(\tau^{*})) \to (Y, \tau^{*})$ is continuous and $f^{-1}(\tau^{*})$ is the weakest topology on X such that f is continuous i.e. if τ is a topology on X such that $f : (X, \tau) \to (Y, \tau^{*})$ is continuous then $f^{-1}(\tau^*) \subset \tau$ that is if $G \in f^{-1}(\tau^*)$ then there is $V \in \tau^*$ such that $G = f^{-1}(V)$ and if f is continuous then $G = f^{-1}(V) \in \tau$ which implies that $f^{-1}(\tau^*) \subset \tau$ and we have the following proposition

Proposition 5.1.4. *Let* (X, τ_1), (Z, τ_2) *be two topological spaces,* Y *be a nonempty set and* $g : (Y, g^{-1}(\tau_2)) \to (Z, \tau_2)$. *If* $f : (X, \tau_1) \to (Y, g^{-1}(\tau_2))$ *then* f *is continuous iff the function* $g \circ f : (X, \tau_1) \to (Z, \tau_2)$ *is continuous.*

Proof: Clearly g is continuous and so if f is continuous then by Theorem 5.1.5, $g \circ f$ is also.

Conversely $V \in g^{-1}(\tau_2)$ implies that there exists $W \in \tau_2$ such that $V = g^{-1}(W)$. So $f^{-1}(V) = f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ then $g \circ f$ is continuous implies that $f^{-1}(V) \in \tau_1$ which implies that f is continuous.

Definition 5.1.2. If (X, τ) is a topological space, Y is a nonempty set and $f : X \to Y$ is a surjective function the topology $f^*(\tau)$ on Y given by Remark 5.1.2, i.e. $f^*(\tau) = \{V \subset Y : f^{-1}(V) \in \tau\}$ is called the identification topology. If (X, τ) and (Y, τ^*) are two topological spaces and $f : X \to Y$ is a surjective and continuous function such that $\tau^* = f^*(\tau)$ then f is called the identification function.

Example 5.1.7. If $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$ and $\tau = \{X, \varnothing, \{1\}, \{2\}, \{1,3\}, \{1,2\}, \{1,2,3\}\}$ and

(1) $f_1 : X \to Y$ such that $f_1 = \{(1,a),(2,a),(3,b),(4,b)\}$ then $f_1(\tau^*) = \{Y, \varnothing, \{a\}, \{a,b\}, \{c\}, \{a,c\}\}$ is not the identification topology on Y since f_1 is not surjective.

(2) $f_2 : X \to Y$ such that $f_2 = \{(1,a),(2,a),(3,b),(4,c)\}$ then $f_2(\tau^*) = \{Y, \varnothing, \{a\}, \{a,b\}\}$ is an identification topology on Y and so f_2 is the identification function.

5.2. Homeomorphism and Topological Properties:

Definition 5.2.1. If (X, τ) and (Y, τ^*) are two topological spaces then $f : X \to Y$ is called

(a) an open function if $f(U) \in \tau^*$ for each $U \in \tau$ i.e. f is an open function if the images of the open sets by the function f are open.

(b) a closed function if $f(F) \in \tau^*_c$ for each $F \in \tau_c$ i.e. f is a closed function if the images of the closed sets by f are closed sets.

Remark 5.2.1. *The continuous functions need not be open or closed and the open and the closed functions may be not continuous as illustrated by the following examples:*

Example 5.2.1. If $|X| \geq 2$ then the inclusion function $i:(X,D) \to (X,I)$ is continuous and neither open nor closed.

Example 5.2.2. The function $i^{-1}:(X,I) \to (X,D)$ which is the inverse of the function i given in Example 4.2.1, is open and closed but not continuous.

Example 5.2.3. The function given in Example 5.1.5, which is not continuous is also not open since $(-\infty,0) \in \mho$ while $f((-\infty,0)) = \{-1\} \notin \mho$ but it is a closed function since if F is a nonempty closed set i.e. $F \in \mho_c - \{\varnothing\}$ then either (1) $F \subset (-\infty,0]$, (2) $F \subset (0,\infty)$ or (3) $F \cap (-\infty,0] \neq \varnothing$ and $F \cap (o,-\infty) \neq \varnothing$ which implies respectively that $f(F) = \{-1\}$ or $f(F) = \{1\}$ or $f(F) = \{-1,1\}$ which in all cases $f(F) \in \mho_c$ and also $f(\varnothing) = \varnothing$.

Example 5.2.4. Let $X = \{a,b,c\}$, $Y = \{x,y,z\}$ and let $\tau = \{X,\varnothing,\{a\},\{b,c\}\}$ and $\tau^* = \{Y,\varnothing,\{x,z\},\{y,z\},\{z\}\}$. If $f:X \to Y$ is such that $f = \{(a,x),(b,y),(c,y)\}$ then f is continuous and closed but not open. It is not open because $\{a\} \in \tau$ while $f(\{a\}) = \{x\} \notin \tau^*$. Show that it is continuous and closed.

Theorem 5.2.1. *Let (X,τ) and (Y,τ^*) be two topological spaces and $f:X \to Y$ be an injective function. Then f^{-1} is continuous iff f is open iff f is closed.*

Proof: Since f is injective then, $f(A-B) = f(A) - f(B)$ and $(f^{-1})^{-1}(A) = f(A)$ for subsets $A, B \subset X$. Then f^{-1} is continuous iff $(f^{-1})^{-1}(G) = f(G) \in \tau^*$ for each $G \in \tau$ iff f is open iff $f(X-F) = Y - f(F) \in \tau^*$ iff $f(F) \in \tau^*_c$ for each $F \in \tau_c$ iff f is a closed function.

Theorem 5.2.2. *If (X,τ) and (Y,τ^*) are two topological spaces, and $f:X \to Y$ is an injective and an open function and A is an isolated subset of X. Then f is a closed function.*

Theorem 5.2.3. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces and* $f:X\rightarrow Y$ *. Then*

(a) The following statements are equivalent:

(1) f *is open.*

(2) $f(A^o)\subset(f(A))^o$ *for each subset* A *of* X *.*

(3) $(f^{-1}(B))^o\subset f^{-1}(B^o)$ *for each subset* B *of* Y *.*

(b) f *is closed iff* $\overline{f(A)}\subset f(\overline{A})$ *for each subset* A *of* X *.*

Proof: We shall prove (a) by proving that that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2): Suppose that A is a subset of X then by (1) f is open implies that $f(A^o)\in\tau^*$ which implies that $f(A^o)=(f(A^o))^o$ and hence

$$A^o\subset A\Rightarrow f(A^o)\subset f(A)\Rightarrow f(A^o)=(f(A^o))^o\subset(f(A))^o$$

$$\Rightarrow f(A^o)\subset(f(A))^o$$

(2) $\Longrightarrow$ (3): If $B\subset Y$ then $A=f^{-1}(B)\subset X$, by substituting in (2) we gets

$$f((f^{-1}(B))^o)\subset(f(f^{-1}(B)))^o\subset B^o$$

$$\Rightarrow(f^{-1}(B))^o\subset f^{-1}[f((f^{-1}(B))^o)]\subset f^{-1}(B^o)$$

$$\Rightarrow(f^{-1}(B))^o\subset f^{-1}(B^o)$$

(3) $\Longrightarrow$ (1): Suppose that $G\in\tau$ then $G=G^o$ and $f(G)\subset Y$ then by setting in (3) $B=f(G)$ we gets

$$G=G^o\subset(f^{-1}(f(G)))^o\subset f^{-1}((f(G))^o)$$

$$\Rightarrow f(G)\subset f[f^{-1}((f(G))^o)]\subset(f(G))^o$$

$$\Rightarrow f(G)=(f(G))^o\Rightarrow f(G)\in\tau^*$$

Therefore f is open.

(3) $\Longrightarrow$ (2): If $A\subset X$ then $B=f(A)\subset Y$, by substituting in (3) we gets

$$A^o\subset[f^{-1}(f(A))]^0\subset f^{-1}((f(A))^o)$$

$$\Rightarrow f(A^o)\subset f[f^{-1}((f(A))^o)]\subset(f(A))^o$$

$$\Rightarrow f(A^o)\subset(f(A))^o$$

(b) If f is a closed function then $A\subset X\Rightarrow f(\overline{A})\in\tau^*_c$ and so

$$A \subset \overline{A} \Rightarrow f(A) \subset f(\overline{A}) \Rightarrow \overline{f(A)} \subset \overline{f(\overline{A})} = f(\overline{A})$$
$$\Rightarrow \overline{f(A)} \subset f(\overline{A})$$

Conversely, If $\overline{f(A)} \subset f(\overline{A})$ for each $A \subset X$ then if $F \subset X$ so

$$F \in \tau_c \Rightarrow \overline{F} = F \Rightarrow f(\overline{F}) = f(F) \Rightarrow \overline{f(F)} \subset f(\overline{F}) = f(F)$$
$$\Rightarrow \overline{f(F)} = f(F) \Rightarrow f(F) \in \tau^*_c$$

Therefore f is a closed set.

Proposition 5.2.1. *Let (X,τ), (Y,τ^*) be two topological spaces and β be a basis for the topology τ on X then f is open if $f(B) \in \tau$ for each $B \in \beta$.*

Proof: If $G \in \tau$ then there exists a family $\{B_i : i \in \Delta\}$ of members of β such that $G = \bigcup_{i \in \Delta} B_i$ and $f(B) \in \tau$ for each $B \in \beta$ implies that

$$f(G) = f(\bigcup_{i \in \Delta} B_i) = \bigcup_{i \in \Delta} f(B_i) \in \tau$$

Which implies that f is an open function.

Theorem 5.2.4. *Let (X,τ), (Y,τ^*) be two topological spaces, $f : X \to Y$ is an open and bijective function and A be an isolated subset of X. Then $f(A)$ is an isolated subset of Y.*

Proof: Let $A \subset X$ be isolated $f : X \to Y$ is an open and bijective function. Then if $y \in f(A)$, there exists $x \in A$ such that $y = f(x)$ because f is surjective and $x \in A$ implies that $x \in isd(A)$ because A is isolated which implies that there exists $G \in \tau$ such that $G \cap A = \{x\}$. Since f is injective then

$$f(G \cap A) = f(G) \cap f(A) = \{f(x)\} = \{y\}$$

Since f is open then $f(\mathrm{G}) \in \tau^*$ which implies that $\mathrm{y} \in \mathrm{isd}(f(A)$. Then $\mathrm{isd}(f(A)) = A$. Therefore $f(A)$ is an isolated subset of Y.

Theorem 5.2.5. *Let (X,τ), (Y,τ^*) be two topological spaces and $f : X \to Y$ and let*

(1) f is open

(2) $(f(A))^{\wedge} \subset f(A^{\wedge}); \forall A \subset X$. Then

(a) If (X,τ) is a principal space then (1) $\Longrightarrow$ (2).

(b) If (Y,τ^*) is a principal space then (2) $\Longrightarrow$ (1).

If (X,τ) and (Y,τ^*) are principal spaces then (1) and (2) are equivalent.

Proof: (a) If (X,τ) is a principal space then $A^{\wedge} \in \tau$ for each subset A of X and so

$$A \subset A^{\wedge} \Rightarrow f(A) \subset f(A^{\wedge}) \Rightarrow (f(A))^{\wedge} \subset (f(A^{\wedge}))^{\wedge}$$

Then if f is open then $A^{\wedge} \in \tau$ implies that $f(A^{\wedge}) \in \tau^*$ which implies that $(f(A^{\wedge}))^{\wedge} = f(A^{\wedge})$ and then

$$(f(A))^{\wedge} \subset f(A^{\wedge})$$

(b) We know that $G \in \tau \Rightarrow G^{\wedge} = G \Rightarrow f(G^{\wedge}) = f(G)$ and so from (2) we gets

$$(f(G))^{\wedge} \subset f(G^{\wedge}) = f(G) \Rightarrow (f(G))^{\wedge} = f(G)$$

If (Y, τ^*) is a principal topological space then $(f(G))^{\wedge} \in \tau^*$ which implies that $f(G) \in \tau^*$. Hence f is an open function.
If (X, τ) and (Y, τ^*) are principal spaces (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1).

Remark 5.2.2. *Suppose that* $\{(X_i, \tau_i) : i \in \Lambda\}$ *is a family of topological spaces such that* $\{X_i : i \in \Lambda\}$ *is a family of pairwise disjoint nonempty sets i.e.* $X_i \cap X_j = \varnothing$ *for each* $i, j \in \Lambda$ *such that* $i \neq j$ *and* $X = \bigcup_{i \in \Lambda} X_i$ *then the family* $\tau = \{U \subset X : X_i \cap U \in \tau_i ; \forall i \in \Lambda\}$ *forms a topology on* X. *In this case the inclusion functions* $i_i : X_i \subset X$ *are well defined and satisfies the properties given by the following theorems:*

Theorem 5.2.6. $i_\lambda : X_\lambda \subset X$ *are continuous, open and closed.*

Proof: Firstly according to the definition of the topology τ clearly if $U \in \tau$ then $i_\lambda^{-1}(U) = U \cap X_\lambda \in \tau_\lambda$ and so i_λ is continuous. Secondly Suppose that $U \in \tau_\lambda$ then since the family $\{X_\lambda : \lambda \in \Lambda\}$ is a family of pairwise disjoint members then $U \cap X_\lambda = U$ and $U \cap X_\mu = \varnothing$ for each $\mu \in \Lambda$ such that $\lambda \neq \mu$ which implies that $U \in \tau$ for each $U \in \tau_\lambda$ and each $\lambda \in \Lambda$ i.e. $i_\lambda(U) = U \in \tau$ this means tat i_λ is an open function for each $\lambda \in \Lambda$.

Thirdly, Suppose that $F \in \tau_{\lambda c}$ then $V = X_\lambda - F \in \tau_\lambda$ where $\lambda \in \Lambda$ and as before in secondly, $i_\lambda(X_\lambda - F) = X_\lambda - F \in \tau$ and since $X_\lambda, X_\lambda - F \in \tau$ for each $\lambda \in \Lambda$ then

$$X - F = \bigcup\{X_\mu : \mu \in \Delta - \{\lambda\}\} \bigcup (X_\lambda - F)$$

which implies that $X - F \in \tau$ which implies that $F \in \tau_c$. Hence i_λ is a closed set.

Theorem 5.2.7. *Let* (X,τ), (Y,τ^*) *be two topological spaces,* $f : X \to Y$ *and* $A \subset X$ *then*

(1) If f *is open then* $f(x)$ *is an interior point of* $f(A)$ *if* $x \in A$ *is an interior point of* A *i.e.*

$$x \in A^o \Rightarrow f(x) \in (f(A))^o .$$

(2) If f *is an open and injective function then* $x \in A$ *is a limit point of* A *if* $f(x)$ *is a limit point of* $f(A)$ *i.e.*

$$f(x) \in (f(A))' \Rightarrow x \in A' .$$

(3) If f *is an open and bijective function then* $x \in X$ *is a boundary point of* A *if* $f(x)$ *is a boundary point of* $f(A)$ *i.e.*

$$f(x) \in b((f(A)) \Rightarrow x \in b(A) .$$

Proof: We shall prove (2) and left (1) and (3) to the exercises, for let $x \in X$ and $x \notin A'$ then there is an open set $G \in \tau$ such that $x \in G$ and $\left(G - \{x\}\right) \cap A = \varnothing$. Then $f(x) \in f(G)$ and if f is injective then

$$G - \{x\} \cap A = \varnothing \Rightarrow f[(G - \{x\}) \cap A] = [f(G - \{x\})] \cap f(A)$$
$$= [f(G) - \{f(x)\}] \cap f(A) = \varnothing$$

and if f is open then $f(G) \in \tau^*$ and so $f(x) \notin (f(A))'$ then the contra positive of this result is the required aim that is

$$f(x) \in (f(A))' \Rightarrow x \in A' .$$

Theorem 5.2.8. *Let* (X,τ), (Y,τ^*) *be two topological spaces and* $f : X \to Y$ *be open (closed) bjective and continuous function then* $\tau^* = f^*(\tau)$.

Proof: We shall prove the case when f is a closed function, for since f is continuous then by Remark 5.1.2, $\tau^* \subset f^*(\tau)$ $\quad (I)$.

Conversely

$$V \in f^*(\tau) \Rightarrow f^{-1}(V) \in \tau \Rightarrow X - f^{-1}(V) \in \tau_c$$
$$\Rightarrow f(X - f^{-1}(V)) = Y - f(f^{-1}(V)) = Y - V \in \tau^*_c$$
$$\Rightarrow V \in \tau^* \Rightarrow f^*(\tau) \subset \tau^* \quad (II)$$

From (I) and (II) we gets $\tau^* = f^*(\tau)$.

Theorem 5.2.9. *Let* (X,τ) *be a topological space and* Y *be nonempty set. Then* $f:(X,\tau)\to(Y,f^*(\tau))$ *is an open (closed) function iff,* $f^{-1}(f(U))$ *is an open (closed) set for each* $U\in\tau$ *(*$U\in\tau_C$*).*

Proof: We shall prove the theorem in the case when f is a closed set and left the other case to the exercises. For, if f is a closed function then

$$\begin{aligned} U\in\tau_C &\Rightarrow f(U)\in f^*(\tau)_C \Rightarrow Y-f(U)\in f^*(\tau) \\ &\Rightarrow X-f^{-1}(f(U))=f^{-1}(Y-f(U))\in\tau \\ &\Rightarrow f^{-1}(f(U))\in\tau_C \end{aligned}$$

Conversely if $f^{-1}(f(U))\in\tau_C$ for each $U\in\tau_C$ then

$$\begin{aligned} U\in\tau_C &\Rightarrow f^{-1}(f(U))\in\tau_C \Rightarrow X-f^{-1}(f(U)) \\ &=f^{-1}(Y-f(U))\in\tau \Rightarrow Y-f(U)\in f^*(\tau) \\ &\Rightarrow f(U)\in f^*(\tau)_C \end{aligned}$$

Definition 5.2.2. Let (X,τ), (Y,τ^*) be two topological spaces. Then $f:X\to Y$ is called a homeomorphism if it satisfies the following conditions:

(a) f is a bijective function.

(b) f and f^{-1} are continuous functions.

Example 5.2.5. Consider the relative topologies on the intervals $[a,b]$ and $[c,d]$ of the usual topology $\mho$ on R then $f:[a,b]\to[c,d]$ given by $f(x)=c+\frac{d-c}{b-a}(x-a)$ for each point $x\in[a,b]$ is a homeomorphism between the intervals $[a,b]$ and $[c,d]$ since

(1) f is injective and surjective function.

(2) f is a linear function i.e. it is a polynomial of the first degree and accordingly a continuous. Also the inverse function of f, $f^{-1}:[c,d]\to[a,b]$ given by $f^{-1}(x)=a+\frac{b-a}{d-c}(x-c)$ for each point $x\in[c,d]$ is linear and continuous.

Remark 5.2.3. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces, if* $f:X\to Y$ *is a homeomorphism between them then we say that the topological spaces* X *and* Y *are homeomophic by the homeomorphism* f*. Considering the family of all topological spaces and define the relation "* X *is homeomorphic to* Y *" one can easily prove that this relation is an equivalence relation, the family of all topological spaces* *divides into a pairwise disjoint subfamilies that is the equivalence classes* $\left[X\right]$ *which obtained by the equivalence relation where* $\left[X\right]$ *is the family of all topological spaces homeomorphic to the topological space* X *.*

Theorem 5.2.10. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces and* $f:X\to Y$ *be a bijective function. Then the following statements are equivalent.*

(1) f *is a homeomorphism.*

(2) f *is continuous and open.*

(3) f *is continuous and closed.*

(4) f *is continuous and* $f(G)\in\tau^*\Rightarrow G\in\tau$ *for each* $G\subset X$ *.*

Proof: (1) $\Rightarrow$(2): From (1) f is homeomorphism implies that f^{-1} is continues and so $G\in\tau\Rightarrow f(G)=(f^{-1})^{-1}(G)\in\tau^*$ which implies that f is open.

(2) $\Rightarrow$ (3): From (2) f is open implies that $F\in\tau_c\Rightarrow X-F\in\tau\Rightarrow f(X-F)\in\tau^*$ and since f is bijective then $f(X-F)=Y-f(F)\in\tau^*$ from which $f(F)\in\tau^*_c$ and so f is closed.

(3) $\Rightarrow$ (4): From (3) f is a closed function and then

$$F\in\tau_c\Rightarrow(f^{-1})^{-1}(F)=f(F)\in\tau^*_c\ .$$

Then f^{-1} is continues. Also if $G\subset X$ such that $f(G)\in\tau^*$ then because f is continuous from (3), $f^{-1}(f(G))\in\tau$ but $f^{-1}(f(G))=G$ since f is bijective and so $G\in\tau$. Hence we proved that for each $G\subset X$, $f(G)\in\tau^*\Rightarrow G\in\tau$

(4) $\Rightarrow$ (1): Since f is bijective then if $V\in\tau^*$, there exists $G\subset X$ such that $V=f(G)$ and from (4) $f(G)\in\tau^*\Rightarrow G\in\tau$. Again since f is injective then $f^{-1}(f(G))=G$. Hence

$$V\in\tau^*\Rightarrow f^{-1}(V)=f^{-1}(f(G))=G\in\tau$$

this means that f is continuous.

Definition 5.2.3. The property P is called an invariant topological property if for each topological space X satisfying the property P , P is a common property of all members of the equivalence class $\left[X\right]$ i.e. the topological space X satisfies P implies that each member $Y\in[X]$ also satisfies P .

Remark 5.2.4. *Not all properties of the topological spaces are invariant topological properties, for example consider the two real sets* $I_1=(1,3)$

and $I_2 = (3,9)$ *and consider the relative topologies of the usual topology* $\mho$ *on* R *on both then* $f : I_1 \to I_2$ *given by* $f(x) = 3x$ *is homeomorphism and we remark that the distance between two points in* I_1 *less than* 2 *and in* I_2 *less than* 6 *i.e. the distance between the points in topological spaces is not invariant topological property.*

Example 5.2.6. The property that the topological space is separable is an invariant topological property.

Proof: Let (X,τ) and (Y,τ^*) be two topological spaces and $f : X \to Y$ be homeomorphism and let X be a separable space then there exists a countable subset A of X such that $\overline{A} = X$

and since f is continuous then $V \in \tau^* \Rightarrow f^{-1}(V) \in \tau$. So

$y \in V \Rightarrow \exists x \in X : y = f(x)$ and $f(f^{-1}(V) \cap A) = V \cap f(A)$ and we finds

$$\overline{A} = X \Rightarrow f^{-1}(V) \cap A \neq \varnothing$$

$$\Rightarrow f(f^{-1}(V) \cap A) = V \cap f(A) \neq \varnothing \Rightarrow \overline{f(A)} = Y$$

Since f is bijective then $g : A \to f(A)$ given by $g(x) = f(x)$ for each point $x \in A$ is also bijective which implies that $f(A)$ is countable and so Y is separable and the property that the topological space is separable is an invariant topological property.

Remark 5.2.5. *Regarding Theorems 5.1.7 and Theorem 5.2.4, we remarks that if* (X,τ), (Y,τ^*) *are two homeomorphic topological spaces,* $f : X \to Y$ *and* $A \subset X$ *then*

(1) $x \in A^o \Leftrightarrow f(x) \in (f(A))^o$.

(2) $x \in A' \Leftrightarrow f(x) \in (f(A))'$.

(3) $x \in b(A) \Leftrightarrow f(x) \in b(f(A))$.

Remark 5.2.6. *If* (X,τ) *and* (Y,τ^*) *are two topological spaces and* $\Psi \subset P(X)$ *is such that* $X = \bigcup\{A : A \in \Psi\}$. *As we know if* $f : X \to Y$ *is continuous then* $f / A : A \to Y$ *is continuous for each* $A \in \Psi$. *Consider the topological space* (A, τ_A) *for each* $A \in \Psi$ *and by defining the functions* $f_A : A \to Y$ *for each* $A \in \Psi$ *such that for each* $A, B \in \Psi$, $f_A(A \cap B) = f_B(A \cap B)$. *Now we can define the function* $f : X \to Y$ *such that for each* $x \in X$, $f(x) = f_A(x)$ *if* $x \in A$. *It should be remarked that by this definition of* f, $f_A = f / A$ *for each* $A \in \Psi$ *and logically we* *have the question what are the conditions for f to be continuous?. The answer is given by the following theorems.*

Theorem 5.2.11. *If Ψ is a family of open sets i.e. $\Psi \subset \tau$ then $f : X \to Y$ given by Remark 5.2.6, is continuous iff f_A is continuous for each $A \in \Psi$.*

Proof: $f_A = f / A : (A, \tau_A) \to (Y, \tau^*)$ is continuous where $f_A^{-1}(V) = A \cap f^{-1}(V) \in \tau_A$.

Conversely, Suppose that $V \in \tau^*$ since $f_A : A \to Y$ is continuous then $f_A^{-1}(V) \in \tau_A$ for each $A \in \Psi$. Since $\Psi \subset \tau$ then $A \in \tau$ implies that $\tau_A \subset \tau$ implies that $f_A^{-1}(V) \in \tau$ for each $A \in \Psi$ from which $f^{-1}(V) = \bigcup_{A \in \Psi} A \cap f^{-1}(V) = \bigcup \{ f_A^{-1}(V) : A \in \Psi \} \in \tau$. Hence f is continuous.

Theorem 5.2.12. *If Ψ is a finite family of closed sets i.e. $\Psi \subset \tau_C$ and is finite then $f : X \to Y$ given by Remark 5.2.6, is continuous iff f_A is continuous for each $A \in \Psi$.*

Proof: it is similar to the proof of Theorem 5.2.11, regarding that Ψ is a finite family of closed sets.

Theorem 5.2.13. *If (X, τ) and (Y, τ^*) are two topological spaces, $A \subset X$ such that $x \in A$ and $f : X \to Y$ is a continuous function at the point x. Then $f_A : (A, \tau_A) \to (Y, \tau^*)$ is continuous at x.*

Proof: Let $V \in \tau^*$ and $f(x) \in V$. Since f is continuous at x then there exists $U \in \tau$ such that $x \in U$ and $f(U) \subset V$. Hence

$$x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V) \Rightarrow x \in U \subset f^{-1}(V)$$

$$\Rightarrow x \in A \cap U \subset A \cap f^{-1}(V) = f_A^{-1}(V)$$

From which $x \in A \cap U$ and $f_A(A \cap U) \subset f_A(f_A^{-1}(\mathrm{V})) \subset \mathrm{V}$ and clearly $A \cap U \in \tau_A$. Hence f_A is continuous at the point x.

Theorem 5.2.14. *If* (X,τ) *and* (Y,τ^*) *are two topological spaces,* $A \subset X$, $x \in A^o$ *and* $f_A : (A,\tau_A) \to (Y,\tau^*)$ *is continuous. Then the function* $f : X \to Y$ *is continuous at the point* x.

Proof: Suppose that $V \in \tau^*$ such that $f(x) \in V$ if the function $f_A : (A,\tau_A) \to (Y,\tau^*)$ is continuous at the point x then there is $W \in \tau_A$ such that $x \in W$ and $f(W) \subset V$. But $A \cap A^o = A^o \in \tau_A$, $x \in U$ where $W \cap A^o = U \in \tau_A$ and $U \cap A^o = U \in (\tau_A)_{A^o} = \tau_{A^o}$. Since $A^o \in \tau \Rightarrow \tau_{A^o} \subset \tau$ then $U \in \tau$, $x \in U$ and $U \subset W \Rightarrow f(U) \subset f(W) \subset V$ which implies that f is continuous at x.

(Hint: $W \in \tau_A$ and $A^o \in \tau$ implies that $\exists G \in \tau : W = G \cap A \Rightarrow U = W \cap A^o = (G \cap A^o) \cap A \Rightarrow U \in \tau_A$)

Remark 5.2.6. *Theorem 5.2.11, can considered to be a consequence of Theorem 5.2.14, since if* $x \in X$ *then there exists* $A \in \Psi$ *such that* $x \in A$ *and since* $\Psi \subset \tau$ *then* $A \in \tau$ *and* $A^o = A$. *From Theorem 5.2.14,* f *is continuous at* x *and hence* f *is continues.*

Theorem 5.2.15. *Let* $f : (X,\tau) \to (Y,\tau^*)$ *be an injective and open (closed) function. Then* $f_A : (A,\tau_A) \to ((f(A),\tau^*_{f(A)})$ *is open (closed) for each* $A \in \Psi$.

Proof: We prove that $f : (X,\tau) \to (Y,\tau^*)$ be an injective and open function implies that $f_A : (A,\tau_A) \to ((f(A),\tau^*_{f(A)})$ is open and left the other case to the reader. For let $U \subset A$ be such that $U \in \tau_A$. Then there is $G \in \tau$ such that $U = A \cap G$. Since f is injective then,

$f_A(U) = f(U) = f(A \cap G) = f(A) \cap f(G)$ and f is open implies that $f(G) \in \tau^*$ implies that $f_A(U) \in \tau^*_{f(A)}$. Hence f_A is open.

Definition 5.1.4. The property P of a topological space (X,τ) is said to be hereditary if (X,τ) is P implies that each topological subspace (Y,τ_Y) of (X,τ) is P also.

Example 5.1.7. Consider the real set R and let $p \in R$ and $Y = R - \{p\}$. Then (R, P_p) is a separable space while $(Y, (P_p)_Y)$ is not separable since $(P_p)_Y$ is the discrete topology on Y and the unique dense subset of Y is Y itself and Y is uncountable.

Exercises

1 – Show that $f : (R,\mho) \to (R,\mho)$ given by $f(x) = \begin{cases} x; & x \le 1 \\ x+2; & x > 1 \end{cases}$ is not continuous and write its inverse.

(Hint: $f^{-1}(x) = \begin{cases} x; & x \le 1 \\ 1; & 1 < x < 3 \\ x-2; & x \ge 3 \end{cases}$ and $f^{-1}((-3,2)) \notin \mho$)

2 – Show that $f : (R,\mho) \to (R,\mho)$ given by

$f(x) = \begin{cases} 0; & x \in Q \\ 1; & x \in Q^c \end{cases}$ is not continuous while the function $f/Q : (Q, \mho_Q) \to (R,\mho)$ is continuous.

3 – Prove that the constant function $f : (R,\mho) \to (R,\mho)$ is continuous, closed and not open.

4 – Consider the function $f : (X,\tau) \to (Y,\tau^*)$ and prove that it is continuous iff the function $g : (X,\tau) \to (f(X), \tau^*_{f(X)})$ is continuous where $g(x) = f(x)$ for each $x \in X$.

5 – Consider the function $f : (X,\tau) \to (Y,\tau^*)$ and let $A \subset X$. If $B \subset Y$ show that f is continuous if $f/A : (A,\tau_A) \to (Y,\tau^*)$. Then prove that the function $f_A : (A,\tau_A) \to (Y,\tau^*_B)$ if $f(A) = B$ is injective and open and if f is homeomorphism then f_A is also.

6 – Consider the function $f : (X,\tau) \to (Y,\tau^*)$ if $\{p\} \in \tau$ then f is continuous at p .

7 – Prove that if (X,τ) and (Y,τ^*) are two topological spaces then the following statements are equivalent:

(a) $f:(X,\tau)\rightarrow(Y,\tau^*)$ is continuous.

(b) $f:(X,\tau_c)\rightarrow(Y,\tau_c^*)$ is continuous,

(c) $f(A^{\wedge})\subset(f(A))^{\wedge}$ for each $A\subset X$.

Where f is the same in all cases (a), (b) and (c).

8 – Consider the function $f:(X,\tau)\rightarrow(Y,\tau^*)$ and if $A\subset X$, $p\in A$ and f is continuous at p then the function $f/A:(A,\tau_A)\rightarrow(Y,\tau^*)$ is continuous at the point p.

9 – Consider the usual topology $(R,\mho)$ and $f:R\rightarrow R$ given by $f(x)=|x|$ for each $x\in R$ describe the elements of the topology $f^*(\mho)$ on R.

10 – Let $X=\{1,2,3,4\}$, $Y=\{a,b,c,d\}$, (Y,τ^*) and let $g,f:X\rightarrow(Y,\tau^*)$ such that $f=\{(1,a),(2,a),(3,d),(4,b)\}$ and $g=\{(1,b),(2,c),(3,c),(4,d)\}$. Find the topology generated on X by the two functions f and g i.e. the topology generated by the family $\sigma=\{f^{-1}(V),g^{-1}(V):V\in\tau^*\}$

Chapter VI

Separation axioms

Topics:

- Hausdorff or T_2 – spaces.
- T_1 – spaces:
- T_o – spaces:
- Regular and T_3 – spaces.
- Normal and T_4 – spaces.
- Axioms of countability:

Some important properties of the topological spaces are the separation axioms. The main purpose of these axioms is to make the points and sets of a topological space topologically distinguishable, It is possible that in a topological space there exist distinct points, x and y and distinct open sets U and V, so that $x \in U$ and $y \in V$, such points are topologically distinguishable. The first three separation axioms T_o, T_1 and T_2 from which we are studied in this chapter give successively greater distinguish ability between points, while the axioms T_3 and T_4 do the same for certain closed subsets of the topological spaces.

6.1. Hausdorff or T_2 – spaces.

Definition 6.1.1. A topological space (X,τ) is called a Hausdorff space or T_2 – space if for each two distinct points x and y of X there are two members U and V of τ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Example 6.1.1. The discrete topological space (X,D) is a T_2 – space, since

$$x,y \in X : x \neq y \Rightarrow \{x\},\{y\} \in D, x \in \{x\}, y \in \{y\}, \{x\} \cap \{y\} = \varnothing$$

Example 6.1.2. The usual topological space $(R,\mho)$ is a T_2 – space, since $x,y \in R : x \neq y \Rightarrow (i)\, x < y \vee (ii)\, y < x$

Then by using Archimedes property,

$$x < y \Rightarrow \exists q \in Q : x < q < y \Rightarrow x \in (x-1,q), y \in (q, y+1),$$

$$(x-1,q) \cap (q, y+1) = \varnothing, (x-1,q), (q, y+1) \in \mho$$

Similarly there are two disjoint open intervals one contains the point x and the other contains y.

Example 6.1.3. The topological space (X,τ) is a T_2-space where $\tau = E_p \cup C$ and $p \in X$ since,

$$x, y \in X - \{p\} : x \neq y \Rightarrow \{x\}, \{y\} \in E_p \Rightarrow \{x\}, \{y\} \in \tau, \{x\} \cap \{y\} = \varnothing,$$

as well as

$$x \in X - \{p\} \Rightarrow \{x\} \in E_p, \{x\}^c \in C \Rightarrow \{x\}, \{x\}^c \in \tau,$$

$$x \in \{x\}, p \in \{x\}^c, \{x\} \cap \{x\}^c = \varnothing$$

Another way, if $x, y \in X$ such that $x \neq y$ then there are two cases

(a) $x \neq p$ which implies that $\{x\} \in E_p, \{x\}^c \in C$ which implies that $\{x\}, \{x\}^c \in \tau$, $x \in \{x\}$, $y \in \{x\}^c$ and $\{x\} \cap \{x\}^c = \varnothing$ or

(b) $y \neq p$ in this case we gets the same result as in case (a) by replacing y instead of x.

Example 6.1.4. The co-finite topological space (X,C) if X is infinite is not T_2 since the intersection of any two members of C is nonempty for if $G_1, G_2 \in C$ then G_1^c and G_2^c are finite sets and so $G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subset {G_2}^c$, which is impossible because G_1 is infinite and G_2^c is finite or another way $G_1 \cap G_2 = \varnothing \Rightarrow G_1^c \cup G_2^c = X$ which also is impossible because X is infinite and $G_1^c \cup G_2^c$ is finite. Then $G_1 \cap G_2 \neq \varnothing$ for each $G_1, G_2 \in C$.

Example 6.1.5. Let $X = \{a,b,c\}$ and $\tau = \{X, \varnothing, \{a\}, \{a,c\}\}$. Then (X,τ) is not T_2 since each open set containing the point c contains a which means that we can not separates a and c by two disjoint members of τ

Theorem 6.1.1. *Let (X,τ) be a topological space, Then the following two statements are equivalent*

(1) (X,τ) is a T_2-space.

(2) $x, y \in X : x \neq y \Rightarrow \exists U \in \tau : x \in U, y \in \overline{U}^c$.

Proof: (1) $\Rightarrow$ (2): If (X,τ) is a T_2-space then

$$x, y \in X : x \neq y \Rightarrow \exists U, V \in \tau : x \in U, y \in V, U \cap V = \varnothing$$

But, $V \in \tau, U \cap V = \varnothing \Rightarrow \overline{U} \cap V = \varnothing \Rightarrow y \notin \overline{U} \Rightarrow y \in \overline{U}^c$.

Secondly (2) $\Rightarrow$ (1): $x, y \in X : x \neq y \Rightarrow \exists U \in \tau : x \in U, y \in \overline{U}^c$ and clearly $\overline{U}^c \in \tau$ and $U \cap \overline{U}^c = \tau$. Hence (X,τ) is a T_2-space.

Theorem 6.1.2. *The property that (X,τ) is a T_2-space is hereditary.*

Proof: Let (X,τ) be a T_2-space and $Y \subset X$ be such that $Y \neq \varnothing$. Then by using Theorem 6.1.1, we finds

$x, y \in Y : x \neq y \Rightarrow x, y \in X \Rightarrow \exists U \in \tau : x \in U, y \in \overline{U}^c$ which implies that $x \in Y \cap U$, $y \in Y \cap \overline{U}^c$, $Y \cap U, Y \cap \overline{U}^c \in \tau_Y$ and

$$(Y \cap U) \cap (Y \cap \overline{U}^c) = \varnothing .$$

Therefore by the Definition 6.1.1, (Y, τ_Y) is a T_2-space.

Another way $x \in Y \cap U$, $Y \cap U \subset Y \cap \overline{U}$ and $Y \cap \overline{U} \in \tau_{Yc}$ implies that $(\overline{Y \cap U})_Y \subset (Y \cap \overline{U})$ and so

$$y \in Y \cap \overline{U}^c = Y - \overline{U} = Y - (Y \cap \overline{U}) \subset Y - (\overline{Y \cap U})_Y$$

$$\Rightarrow y \in Y - (\overline{Y \cap U})_Y$$

Hence by Theorem 6.1.1, (Y, τ_Y) is a T_2-space.

Theorem 6.1.3. *The property that the topological space (X,τ) is T_2 is an invariant topological property.*

Proof: Let (X,τ) and (Y,τ^*) be two topological spaces and $f : X \to Y$ be homeomorphism. If $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ then $f^{-1}(y_1), f^{-1}(y_2) \in X$ are distinct, then (X,τ) is T_2 implies that there is an open set $U \in \tau$ such that $f^{-1}(y_1) \in U$ and $f^{-1}(y_2) \in \overline{U}^c$. Since f is homeomorphism then $f(U) \in \tau^*$ and $f(\overline{U}) = \overline{f(U)}$ Theorems 5.1.1 and 5.2.1. But

$$f(\overline{U}^c) = f(X - \overline{U}) = Y - f(\overline{U}) = Y - \overline{f(U)}$$

Therefore, $y_1 \in f(U)$ and $y_2 \in Y - \overline{f(U)}$ which implies by Theorem 6.1.1, that (Y,τ^*) is T_2.

Remark 6.1.1. *A sequences in topological spaces may be convergent to more than one point as it given by Example 6.1.1(2,3). The convergence of sequences in* T_2-*topological spaces is ruling by the following Theorem.*

Theorem 6.1.4. *The sequences in* T_2- *spaces does not converges to more than one point.*

Proof: Let (X,τ) be a T_2-space and $<x_n>$ be a sequence of points of X convergent to two distinct points $x,y\in X$. Then there are two open sets $U,V\in\tau$ such that $x\in U$, $y\in V$ and $U\cap V=\varnothing$. Hence

(i) $\lim\limits_{n\to\infty} x_n = x \Rightarrow \exists n_1\in N: x_n\in U; \forall n\geq n_1$ and

(ii) $\lim\limits_{n\to\infty} x_n = y \Rightarrow \exists n_2\in N: x_n\in V; \forall n\geq n_2$.

Then $n_o=\max.\{n_1,n_2\}$ implies that $x_{n_o}\in U\cap V$ which contradicts that $U\cap V=\varnothing$. Hence $<x_n>$ does not convergent to more than one point.

Remark 6.1.2. *The converse of Theorem 6.1.4, under some conditions is valid as it explains by the following Theorem.*

Theorem 6.1.5. *The topological space* (X,τ)*which satisfies the first axiom of countability is* T_2 *if each sequence of points of* X *converges to a unique point.*

Proof: Let (X,τ) be not T_2. Then there are two distinct point $x,y\in X$ such that each two open sets $U,V\in\tau$, $x\in U$ and $y\in V$ implies that $U\cap V\neq\varnothing$. If (X,τ) satisfies the first axiom of countability then there are two nested countable local bases $\beta_x=\{B_1,B_2,B_3,...,B_n,...\}$ and $\beta_y=\{U_1,U_2,U_3,...,U_n,...\}$ for x and y respectively such that $B_1\supset B_2\supset B_3 ...\supset B_n\supset...$ and $U_1\supset U_2\supset U_3 ...\supset U_n\supset...$, then $B_n\cap U_n\neq\varnothing$ for each $n\in N$ which implies that there exists a point $x_n\in B_n\cap U_n$ for each $n\in N$. Accordingly we obtained a sequence $<x_n>$ of points of X. If $x,y\in X$ and $G,H\in\tau$ such that $x\in G$ and $y\in H$ then there are $n_1,n_2\in N$ such that $x_n\in B_n\subset G$, whenever $n\geq n_1$ and $x_n\in U_n\subset H$, whenever $n\geq n_2$. Therefore, $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} x_n = y$. The contrapositive of this result is "If each sequence in X converges to a unique point then (X,τ) is T_2".

Theorem 6.1.6. *Let X be an infinite set and (X,τ) be T_2. Then there are a sequence of distinct points of X and a sequence of pairwise disjoint members of τ such that each point in the first sequence belongs to a corresponding member in the second sequence.*

Proof: Since X is infinite and (X,τ) is T_2 then there are two points $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and there are two members $V_{12}, V_{21} \in \tau$ such that $x_1 \in V_{12}$, $x_2 \in V_{21}$ and $V_{12} \cap V_{21} = \varnothing$. By putting $U_1 = V_{12}$ and $U_2 = V_{21}$ we gets $\{U_1, U_2\} \subset \tau$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$. Again since X is an infinite set and (X,τ) is T_2 , there is $x_3 \in X - \{x_1, x_2\}$ and a family $\{V_{13}, V_{31}, V_{23}, V_{32}\} \subset \tau$ such that $x_1 \in V_{13}$, $x_2 \in V_{23}$, $x_3 \in V_{31} \cap V_{32}$, $V_{13} \cap V_{31} = \varnothing$ and $V_{23} \cap V_{32} = \varnothing$. By putting $U_1^* = U_1 \cap V_{13}$, $U_2^* = U_2 \cap V_{23}$ and $U_3^* = V_{31} \cap V_{32}$ we gets $U_1^* \cap U_2^* \subset U_1 \cap U_2 = \varnothing$, $U_1^* \cap U_3^* \subset V_{13} \cap V_{31} = \varnothing$ and $U_2^* \cap U_3^* \subset V_{23} \cap V_{32} = \varnothing$ which implies that the family $\{U_1^*, U_2^*, U_3^*\}$ is a family of pairwise disjoint members such that $x_i \in U_i^*$ for each $i \in \{1,2,3\}$. By using mathematical principal let $n \in N$ and $\{U_1, U_2, U_3, ..., U_n\} \subset \tau$ be a family of pairwise disjoint members and $\{x_1, x_2, x_3, ..., x_n\} \subset X$ be a set of distinct points such that $x_i \in U_i$ for each $i \in \{1,2,3,...n\}$. Since X is infinite and (X,τ) is T_2 then there exists a point $x_{n+1} \in X - \{x_1, x_2, x_3, ..., x_n\}$ and a family

$$\{V_{1(n+1)}, V_{2(n+1)}, V_{3(n+1)}, ..., V_{n(n+1)}, V_{(n+1)1},$$
$$V_{(n+1)2}, V_{(n+1)3}, ...; V_{(n+1)n}\} \subset \tau$$

of members of τ such that for each $i \in \{1,2,3,...n\}$, $x_i \in V_{i(n+1)}$, $x_{n+1} \in V_{(n+1)i}$ and $V_{i\,(n+1)} \cap V_{(n+1)i} = \varnothing$. Then by putting $U_i^* = U_i \cap V_{i(n+1)}$ for each $i \in \{1,2,3,...n\}$ and $U_{n+1}^* = \cap\{V_{(n+1)i} : i \in \{1,2,3,...,n\}\}$ we obtains $U_i^* \cap U_k^* \subset U_i \cap U_k = \varnothing$ for each $i,k \in \{1,2,3,...,n\}$ such that $i \neq k$ also $U_i^* \cap U_{n+1}^* \subset V_{i\,(n+1)} \cap V_{(n+1)i} = \varnothing$ for each $i \in \{1,2,3,...n\}$. Accordingly we obtains a sequence of distinct points

$\{x_1,x_2,x_3,...,x_n,...,...\}\subset X$ and a sequence of disjoint members $\{U_1^*,U_2^*,U_3^*,...,U_n^*,......\}\subset\tau$ such that $x_n\in U_n^*$ for each $n\in N$.

Corollary 6.1.1. *Let* X *be an infinite set and* (X,τ) *be* T_2. *Then there exists an infinite isolated subset of* X.

Proof: From Theorem 6.1.6, we obtained the subset of X $A=\{x_1,x_2,x_3,...,x_n,...,...\}$ and the family of members of τ, $\{U_1^*,U_2^*,U_3^*,...,U_n^*,......\}\subset\tau$ such that $U_n^*\cap A=\{x_n\}$ for each $n\in N$ which implies that A is an isolated set i.e. $isd(A)=A$.

Theorem 6.1.7. *Let* (X,τ) *be a* T_2-*space and* $f:X\to X$ *be a continuous function. Then* $A=\{x\in X:f(x)\neq x\}$ *is open.*

Proof: If (X,τ) is T_2, then

$$x\in A\Rightarrow f(x)\neq x$$

$$\Rightarrow\exists U,V\in\tau:x\in U,f(x)\in V,\ U\cap V=\varnothing$$

$$\Rightarrow x\in(U\cap f^{-1}(V))$$

If f is continuous then $f^{-1}(V)\in\tau$ which implies that $(U\cap f^{-1}(V))\in\tau$ and since $U\cap V=\varnothing$ then

$$t\in(U\cap f^{-1}(V))\Rightarrow t\in U,\ f(t)\in V\Rightarrow f(t)\neq t$$

$$\Rightarrow t\in A\Rightarrow x\in(U\cap f^{-1}(V))\subset A$$

$$\Rightarrow A\in\tau$$

Another proof:

$$x\in A\Rightarrow f(x)\neq x$$

$$\Rightarrow\exists U_x,V_x\in\tau:x\in U_x,f(x)\in V_x\wedge U_x\cap V_x=\varnothing$$

$$\Rightarrow x\in(U_x\cap f^{-1}(V_x))$$

$$\Rightarrow A\subset\bigcup_{x\in A}(U_x\cap f^{-1}(V_x))\qquad(I)$$

Also since $U_x\cap V_x=\varnothing$ for each point $x\in A$ then

$$t\in \bigcup_{x\in A} (U_x \cap f^{-1}(V_x)) \Rightarrow \exists\, x \in A : t\in(U_x \cap f^{-1}(V_x))$$

$$\Rightarrow t\in U_x,\ f(t)\in V_x \Rightarrow t\neq f(t) \Rightarrow t\in A$$

$$\Rightarrow (U_x \cap f^{-1}(V_x))\subset A$$

$$\Rightarrow \bigcup_{x\in A} (U_x \cap f^{-1}(V_x))\subset A \qquad (II)$$

Therefore from (I) and (II)

$$A=\bigcup_{x\in A} (U_x \cap f^{-1}(V_x))$$

which implies that A is open since f is continuous implies that $U_x \cap f^{-1}(V_x) \in \tau$ for each point $x \in A$.

6.2. T_1-spaces:

Definition 6.2.1. A topological space (X,τ) is called a T_1-space if for each two distinct points $x,y\in X$ such that $x\neq y$ there are two open sets $U,V \in \tau$ such that U contains x and not y and V contains y and not x that is

$$x,y\in X : x\neq y \Rightarrow \exists U,V\in\tau : x\in U, y\notin U \wedge y\in V, x\notin V\,.$$

Example 6.2.1. The co-finite space (X,C) is T_1 since if $x,y\in X$ such that $x\neq y$ then $\{x\}^c,\{y\}^c \in C$, $x\in\{y\}^c, y\notin\{y\}^c \wedge y\in\{x\}^c, x\notin\{x\}^c$, here $U=\{y\}^c$ and $V=\{x\}^c$.

Remark 6.2.1. *Clearly from the Definition 6.2.1, of* T_2*,* T_2 *gives* T_1 *which is expressed by the implication*

$$T_2\rightarrow T_1.$$

Accordingly the discrete topological space (X,D) *and the usual topological space* $(R,\mho)$ *which are given by Examples 6.1.1, and 6.1.2, as* T_2-*spaces are also* T_1-*spaces. Also the topological space* (X,τ) *is a* T_1-*space where* $\tau=E_p\cup C$ *and* $p\in X$ *which is given by Example 6.1.3, to be* T_2*. The converse of the implication* $T_2\rightarrow T_1$ *need not be valid for example if* X *is infinite then* (X,C) *is* T_1 *by Example 6.2.1, and is not* T_2 *by Example 6.1.4.*

The topological space which is given by Example 6.1.5, is neither T_2 *nor* T_1.

Example 6.2.2. The left rays topological space (R,τ) is not T_1 where $\tau=\{R,\varnothing,(a,\infty):a\in R\}$ since $x,y\in R$ such that $x\neq y$ implies that either (1) $x<y$ from which

$$a\in R: x\in(a,\infty)\Rightarrow a<x<y\Rightarrow y\in(a,\infty)$$

Which means that each open set containing x contains y or

(2) $y<x$ in which we can show that each open set containing y contains x.

Theorem 6.2.1. *Let* (X,τ) *be a topological space,* $x\in X$ *and* $\{x\}^{\wedge}=\bigcap\{G\in\tau:x\in G\}$. *Then the following statements are equivalent:*

(1) (X,τ) *is a* T_1-*space,*

(2) $\overline{\{x\}}=\{x\};\forall x\in X$*,*

(3) $\{x\}^{\wedge}=\{x\};\forall x\in X$*,*

(4) $\{x\}^{\wedge}\cap\{y\}^{\wedge}=\varnothing\ \ \forall x,y\in X:x\neq y$ *,*

(5) $\overline{\{x\}}\cap\overline{\{y\}}=\varnothing\ \ \forall x,y\in X:x\neq y$ *,*

(6) (Y,τ_Y) *is a* T_1-*space for each subset* $Y\subset X$ *such that* $Y\neq X$ *and* $|X|\geq 3$ *and*

(7) τ_Y *is the discrete topology on* Y *for each finite subset* Y *of* X

Proof: One can prove that:

(1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1). At the following we shall prove that (1) ⇒ (2) and (4) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1) and left (2) ⇒ (3) and (3) ⇒ (4) to the reader.

(1) ⇒ (2): Suppose that (X,τ) is a T_1-space and $x\in X$ is an arbitrary point then by using Proposition 3.2.1(ii),

$$y\in X-\{x\}\Rightarrow x\neq y\Rightarrow\exists V\in\tau:y\in V,x\notin V\Rightarrow V\cap\{x\}=\varnothing$$

$$\Rightarrow V\cap\overline{\{x\}}=\varnothing\Rightarrow y\notin\overline{\{x\}}\Rightarrow\overline{\{x\}}=\{x\}$$

(4) ⇒ (5): Suppose that $x,y\in X$ such that $x\neq y$ and $t\in\overline{\{x\}}\cap\overline{\{y\}}$ then $t\neq x$ or $t\neq y$ but $t\neq x$ implies that

$$t\in\overline{\{x\}}\Rightarrow x\in\{t\}^{\wedge}\Rightarrow\{x\}^{\wedge}\cap\{t\}^{\wedge}\neq\varnothing.$$

and $t\neq y$ Implies that

$$t\in\overline{\{y\}}\Rightarrow y\in\{t\}^{\wedge}\Rightarrow\{y\}^{\wedge}\cap\{t\}^{\wedge}\neq\varnothing.$$

the results in both cases contradicts (3). Hence such point t does not exist and so $\overline{\{x\}}\cap\overline{\{y\}}=\varnothing$ for each $x,y\in X$ such that $x\neq y$.

(5) $\Rightarrow$ (6): Suppose that $Y\subset X$ and $x,y\in Y$ such that $x\neq y$ then $x,y\in X$ and by (5) $\overline{\{x\}}\cap\overline{\{y\}}=\varnothing\Rightarrow(\overline{\{x\}}\cap Y)\cap(\overline{\{y\}}\cap Y)=\varnothing$, then by using Theorem 6.1.2, $\overline{\{x\}}_Y\cap\overline{\{y\}}_Y=\varnothing$ and clearly

(i) $x\in Y-\overline{\{y\}}_Y$, $y\notin Y-\overline{\{y\}}_Y$,

(ii) $y\in Y-\overline{\{x\}}_Y$, $x\notin Y-\overline{\{x\}}_Y$ and

(iii) $Y-\overline{\{y\}}_Y, Y-\overline{\{x\}}_Y\in\tau_Y$. Therefore (Y,τ_Y) is a T_1 $-$space.

(6) $\Rightarrow$ (7): Let Y be a finite subset of X and $x\in Y$ be an arbitrary point. Then by using (6), (Y,τ_Y) is a T_1 $-$space and since $x\in Y$ is an arbitrary point we gets

$$y\in Y-\{x\}\Rightarrow\exists\, G_y\in\tau_Y : x\in G_y\wedge y\notin G_y\Rightarrow$$

$$\bigcap_{y\in Y-\{x\}} G_y=\{x\}\Rightarrow\{x\}\in\tau_Y\Rightarrow\{\{x\}:x\in Y\}\subset\tau_Y$$

Therefore τ_Y is the discrete topology on Y.

(7) $\Rightarrow$ (1): Let $x,y\in X$ be such that $x\neq y$ and $Y\subset X$ be finite such that $x,y\in Y$. Then by (7) $\{x\},\{y\}\in\tau_Y$ and so there are two open sets $G,H\in\tau$ such that (i) $G\cap Y=\{x\}$ and (ii) $H\cap Y=\{y\}$. Hence $(i)\Rightarrow x\in G,\ y\notin G$ and $(ii)\Rightarrow y\in H,\ x\notin H$ which implies that (X,τ) is a T_1 $-$space.

Remark 6.2.2. *In Theorem 6.2.1, one can prove that each two statements are equivalent.*

Remark 6.2.3. *If (X,τ) is T_1 by using (3) of Theorem 6.2.1, if $A\subset X$ then*

$$A^{\wedge}=\bigcup\{\{x\}^{\wedge}:x\in A\}=\bigcup\{\{x\}:x\in A\}=A$$

Which implies that $\{x\}^{\wedge}=x$ for each point $x\in X$. Accordingly (X,τ) is T_1 iff $A^{\wedge}=A$ for each $A\subset X$.

Corollary 6.2.1. *The property T_1 of the topological spaces is hereditary.*

Proof: In Theorem 6.2.1, where the statements (6) and (1) are equivalent means that (X,τ) is T_1 implies that (Y,τ_Y) is T_1 for each $Y\subset X$.

Corollary 6.2.2. *If (X,τ) is T_1 then each finite subset of X is a closed set.*

Proof: From (2) of Theorem 6.2.1, $\{x\}\in\tau_c$ for each point $x\in X$. If F is a finite subset of X then it can be written as a union of a finite number of closed sets in the form $F=\bigcup\{\{x\}:x\in F\}$. Hence F is a closed set.

As a consequence on Corollary 6.2.2, we obtain the following two corollaries:

Corollary 6.2.3. *Let (X,τ) be a topological space. Then the following statements are equivalent:*

(1) (X,τ) is T_1.

(2) $C\subset\tau$ where C is the co-finite topology on X and

(3) $P^{\wedge}(X)=\{A^{\wedge}:A\subset X\}=D$.

Proof: (1) $\Rightarrow$ (2): If (X,τ) is a T_1-space and $G\in C$ then G^c is finite which implies by Corollary 6.2.2, that $G^c\in\tau_c$ which implies that $G\in\tau$. Hence $C\subset\tau$.

(2) $\Rightarrow$ (3): By (2) $\{x\}^c\in\tau$ for each $x\in X$, if $A\subset X$ then

$$\{\{x\}^c : A\subset\{x\}^c\}\subset\{G\in\tau : A\subset G\}$$

According to which

$$A\subset A^{\wedge}=\bigcap\{G\in\tau : A\subset G\}\subset\bigcap\{\{x\}^c : A\subset\{x\}^c\}=A.$$

$$\Rightarrow A^{\wedge}=A\Rightarrow P^{\wedge}(X)=D$$

(3) $\Rightarrow$ (1): Suppose that $P^{\wedge}(X)=D$ then

$$x\in X\Rightarrow\{x\}\in P^{\wedge}(X)\Rightarrow\exists A\in P(X):A^{\wedge}=\{x\}$$

Then $A\subset A^{\wedge}\subset\{x\}$ and then either $A=\varnothing$ or $A=\{x\}$, but $A=\varnothing$ is negligible since in this case $A^{\wedge}=\varnothing^{\wedge}=\varnothing\neq\{x\}$ a contradiction and so $A=\{x\}$. Then $\{x\}^{\wedge}=\{x\}$ for each $x\in X$. Therefore by (3) of Theorem 6.2.1, (X,τ) is T_1.

Corollary 6.2.4. *Let (X,τ) be a principal topological space. Then the following two statements are equivalent:*

(1) (X,τ) is T_1.

(2) $\tau=D$ where D is the discrete topology on X.

Proof: As we know (X,D) is T_1.

Conversely if (X,τ) is a T_1-space, then by Corollary 6.2.3, $C\subset\tau$ and if τ is a principal topology then

$$A \subset X \Rightarrow A = \bigcap\{\{x\}^c : x \in A^c\} \in \tau \Rightarrow D \subset \tau \Rightarrow \tau = D$$

It should be noted that if X is finite and $A \subset X$ then A^c is finite and by Corollary 6.2.3, $A^c \in \tau_c$ which implies that $A \in \tau$ which implies that $\tau = D$

Theorem 6.2.2. *The property* T_1 *is an invariant topological property*.

Proof: One can use Theorem 6.2.1, to prove this theorem which we left to the reader.

Theorem 6.2.3. *Let* (X, τ) *be a* T_1*−space and* A *be any subset of* X*. Then* A' *is a closed set.*

Proof: If (X, τ) is a topological space and $A \subset X$ then

$$x \in A'^c \Rightarrow x \notin A' \Rightarrow \exists G \in \tau : x \in G, (G - \{x\}) \cap A = \varnothing .$$

If (X, τ) is a T_1−space then $\{x\} \in \tau_c$ and so $G - \{x\} = G \cap \{x\}^c \in \tau$. Hence by Remark 3.2.1, and since $x \notin A'$ we gets

$$G - \{x\} \cap A = \varnothing \Rightarrow (G - \{x\}) \cap A' = \varnothing \Rightarrow G \cap A' = \varnothing$$

$$\Rightarrow x \in G \subset A'^c \Rightarrow A'^c \in \tau \Rightarrow A' \in \tau_c$$

Theorem 6.2.4. *Let* (X, τ) *be a* T_1*−space,* $A \subset X$ *and* $G \in \tau$ *such that* $x \in G$*. Then* $x \in A'$ *implies that* $G \cap A$ *is an infinite set.*

Proof: If $x \in X$, $A \subset X$, $G \in \tau$ and $x \in G$, let $G \cap A$ be finite. Then by Corollary 6.2.1, $(G \cap A) - \{x\}$ is a closed set so $U = G - [(G \cap A) - \{x\}] \in \tau$ and $x \in U$ and to complete the proof it is typical to the proof of Theorem 1.4.7.

6.3. T_o −spaces:

Definition 6.3.1. A topological space (X, τ) is called T_o if for each two distinct points there exists an open set contains one of them and not the other that is

$$x, y \in X : x \neq y \Rightarrow \exists U \in \tau : x \in U, y \notin U \vee y \in U, x \notin U .$$

Remark 6.3.1. *Directly from the Definition 6.3.1, we remarks that each* T_1*−space is* T_o *and then we gets the implication*

$$T_2 \to T_1 \to T_o$$

Example 6.3.1. According to Remark 6.3.1, all examples of T_1 or T_2− spaces are examples of T_o.

Example 6.3.2. The following topological spaces are all T_o:

(1) The excluding point topological space (X, E_p) where $p \in X$.

(2) The particular point topological space (X,P_p) where $p\in X$.

(3) The left rays topological space (X,τ) where.

$$\tau=\{(-\infty,a):a\in R\}\cup\{R,\varnothing\}$$

(4) The right rays topological space (X,τ) where.

$$\tau=\{(a,\infty):a\in R\}\cup\{R,\varnothing\}$$

We shall prove (1) and left (2), (3) and (4) to the reader. For if $x\in X-\{p\}$ then $\{x\}\in E_p$ and $y\notin\{x\}$ for each point $y\in X-\{x\}$. Hence (X,E_p) is T_o. Also $\overline{\{x\}}=\{x,p\}\neq\{y,p\}=\overline{\{y\}}$ for any two distinct pints $x,y\in X-\{p\}$ and for any point $x\in X-\{p\}$, $\overline{\{p\}}=\{p\}\neq\{x,p\}=\overline{\{x\}}$ which implies that (X,E_p) is T_o.

Example 6.3.3. All topological spaces which are given in Example 6.3.2, are not T_1. In Example 6.3.2, we showed that the right rays topological space is not T_1 and (X,P_p) does not satisfied T_1 since $x\in X-\{p\}$ and $G\in\tau$ such that $x\in G$ implies that $p\in G$. This means that there is no any open set contains x and not p this means that (X,P_p) is not T_1. Also $\overline{\{p\}}=X\neq\{p\}$ or $\{x\}^{\wedge}=\{x,p\}\neq\{x\}$ or $\overline{\{x\}}\cap\overline{\{y\}}=\{x,p\}\cap\{y,p\}=\{p\}\neq\varnothing$ for any two distinct points $x,y\in X-\{p\}$ each one of them implies that (X,P_p) is not T_1.

Example 6.3.4. Let $X=\{a,b,c,d\}$ and

$$\tau=\{X,\varnothing,\{a,b\},\{c\},\{c,d\},\{a,b,c\}\}.$$

Clearly each member of τ containing a contains b and each member of τ containing b contains a and so there is no any open set contains a (b) and not b (a). Hence (X,τ) is not T_o.

Remark 6.3.2. *From Examples 6.3.2 and 6.3.3 clearly the inverse of the implication* $T_1\rightarrow T_o$ *need not be true.*

Theorem 6.3.1. *The following statements are equivalents:*

(1) (X,τ) *is a* T_o-*space,*

(2) $\overline{\{x\}}\neq\overline{\{y\}}$ *for each* $x,y\in X$ *such that* $x\neq y$*,*

(3) $\{x\}^{\wedge}\neq\{y\}^{\wedge}$ *for each* $x,y\in X$ *such that* $x\neq y$*,*

(4) $\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}$ *for each* $x\in X$ *and*

(5) (Y,τ_Y) *is a* T_o-*space for each subset* Y *of* X *such that* $Y\neq X$*.*

Proof: One can prove that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (1). At the following we shall prove (1) ⇒ (2), (3) ⇒ (4) and (4) ⇒ (5) and left the others to the reader.

(1) ⇒ (2): Suppose that (X,τ) is a T_o −space then

$$x,y\in X : x\neq y \Rightarrow \exists U\in\tau : x\in U, y\notin U \vee y\in U, x\notin U$$

and there are two possibilities

(i) $x\in U, y\notin U \Rightarrow U\cap\{y\}=\varnothing \Rightarrow U\cap\overline{\{y\}}=\varnothing \Rightarrow x\notin\overline{\{y\}}$ or

(ii) $y\in U, x\notin U \Rightarrow U\cap\{x\}=\varnothing \Rightarrow U\cap\overline{\{x\}}=\varnothing \Rightarrow y\notin\overline{\{x\}}$ both cases lead to $\overline{\{x\}}\neq\overline{\{y\}}$.

(3) ⇒ (4): let $x\in X$ be an arbitrary point then $y\in\overline{\{x\}}-\{x\}\Rightarrow x\in\{y\}^{\wedge}$ and from (3) since $\{x\}^{\wedge}\neq\{y\}^{\wedge}$ then $x\in\{y\}^{\wedge}$ implies that $y\notin\{x\}^{\wedge}$. Hence $\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}$.

Another way: let $x\in X$ be an arbitrary point then

$$y\in X-\{x\}\Rightarrow\{x\}^{\wedge}\neq\{y\}^{\wedge}\Rightarrow x\notin\{y\}^{\wedge}\vee y\notin\{x\}^{\wedge}$$

$$\Rightarrow y\notin\overline{\{x\}}\vee y\notin\{x\}^{\wedge}\Rightarrow y\notin\{x\}^{\wedge}\cap\overline{\{x\}}$$

$$\Rightarrow\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}$$

Equivalently, If $x\in X$ is an arbitrary point then $y\in X-\{x\}$ implies that by (3) that $\{x\}^{\wedge}\neq\{y\}^{\wedge}$ and $y\notin\{x\}^{\wedge}$ which implies that $y\notin\{x\}^{\wedge}\cap\overline{\{x\}}$ or

$$y\in\{x\}^{\wedge}\Rightarrow x\notin\{y\}^{\wedge}\Rightarrow y\notin\overline{\{x\}}\Rightarrow y\notin\{x\}^{\wedge}\cap\overline{\{x\}}.$$

and hence $\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}$.

Third way: $x\in X$ be an arbitrary point then

$$y\in(\{x\}^{\wedge}\cap\overline{\{x\}})-\{x\}\Rightarrow y\in\{x\}^{\wedge}\wedge y\in\overline{\{x\}}$$

$$\Rightarrow y\in\{x\}^{\wedge}\wedge x\in\{y\}^{\wedge}\Rightarrow\{x\}^{\wedge}=\{y\}^{\wedge}$$

which contradicts (3) which means that such point y does not exists and so $\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}$.

(4) ⇒ (5): Suppose that $Y\subset X$ and $x,y\in Y$ such that $x\neq y$ then $x,y\in X$ and by (4)

$$\{x\}^{\wedge}\cap\overline{\{x\}}=\{x\}\Rightarrow y\notin\{x\}^{\wedge}\cap\overline{\{x\}}\Rightarrow y\notin\{x\}^{\wedge}\vee y\notin\overline{\{x\}}$$

Then

$$y \notin \{x\}^{\wedge} \Rightarrow \exists\, G \in \tau : x \in G, y \notin G$$
$$\Rightarrow G \cap Y \in \tau_Y, x \in G \cap Y, y \notin G \cap Y$$

and

$$y \notin \overline{\{x\}} \Rightarrow \exists G \in \tau : y \in G, x \notin G$$
$$\Rightarrow G \cap Y \in \tau_Y, y \in G \cap Y, x \notin G \cap Y$$

The results in both cases means that (Y, τ_Y) is T_o.

Corollary 6.3.1. *In the topological spaces the axiom T_o is hereditary.*

Proof: In Theorem 6.3.1, the statements (5) and (1) are equivalents.

Corollary 6.3.2. *Let (X,τ) be a T_o-topological space and $\beta^{\wedge} = \{\{x\}^{\wedge} : x \in X\}$ be the family of the minimal sets with respect to τ and $\overline{\beta} = \{\overline{\{x\}} : x \in X\}$. Then the cardinal numbers of X, $\beta^{\wedge}$ and $\overline{\beta}$ are equal i.e. $\left|\beta^{\wedge}\right| = \left|\overline{\beta}\right| = |X|$. If X is finite then $\beta^{\wedge} = \beta_o = \{U_x : x \in X\}$ and the converse of the theorem is also true i.e. $\left|\beta^{\wedge}\right| = |\beta_o| = |X| = n$, $n \in N$ implies that (X,τ) is T_o.*

Proof: Consider the function $f : X \to \beta^{\wedge}$ defined by $f(x) = \{x\}^{\wedge}$ for each point $x \in X$. Clearly f is surjective since $\{x\}^{\wedge}$ is well defined for each point $x \in X$ and if (X,τ) is T_o then by Theorem 6.3.1,

$$x, y \in X : x \neq y \Rightarrow \{x\}^{\wedge} \neq \{y\}^{\wedge} \Rightarrow f(x) \neq f(y)$$

which implies that f is injective. Hence X and $\beta^{\wedge}$ has the same cardinal number i.e. $\left|\beta^{\wedge}\right| = |X|$. In a similar way one can show that $\left|\overline{\beta}\right| = |X|$.

Conversely, let X be a finite set, $|X| = n$ where $n \in N$ and (X,τ) is not T_o. Then by Theorem 6.3.1, there are two points $x, y \in X$ such that $x \neq y$ and $U_x = U_y$ which implies that $|\beta_o| \leq n-1 < n$ i.e. $n = |X| \neq |\beta_o|$. The contra positive of this result is $|\beta_o| = |X| = n$ implies that (X,τ) is T_o.

Also in a similar way on can show that $\left|\overline{\beta}\right|=\left|X\right|=n$ implies that (X,τ) is T_o.

Theorem 6.3.2. *The axiom T_o is an invariant topological property.*

Proof: Let (X,τ) and (Y,τ^*) be two homeomorphic topological spaces by the homeomorphism $f:X\to Y$. Then

$$y_1,y_2\in Y:y_1\neq y_2$$

$$\Rightarrow\exists x_1,x_2\in X:\ y_1=f(x_1)\wedge y_2=f(x_2)$$

By the definition of the function $f(x)$ is singleton which implies that $x_1\neq x_2$. If (X,τ) is T_o then there is an open set $U\in\tau$ such that $x_1\in U,\, x_2\notin U$ or $x_2\in U,\, x_1\notin U$. Since f is homeomorphism the $f(U)\in\tau^*$ and $f^{-1}(f(U))=U$. So $y_1\in f(U)$, $y_2\notin f(U)$ or $y_2\in f(U)$, $y_1\notin f(U)$. Therefore (Y,τ^*) is T_o.

Theorem 6.3.3. *Let (X,τ) be a principal T_o-space. Then A' is a closed set for each subset A of X.*

Proof: If $A\subset X$ then $x\in A'^{c}$ implies that $x\notin A'$ implies that $\left(U_x-\{x\}\right)\cap A=\varnothing$. Accordingly if (X,τ) is T_o then by Theorem 6.3.1, $U_x\neq U_y$ for each point $y\in X-\{x\}$. So

$$y\in U_x\Rightarrow x\notin U_y\Rightarrow U_y\subset\left(U_x-\{x\}\right)\Rightarrow U_y\cap A=\varnothing$$

$$\Rightarrow y\notin A'\Rightarrow U_x\cap A'=\varnothing\Rightarrow U_x\subset A'^{c}\Rightarrow A'^{c}\in\tau$$

$$\Rightarrow A'\in\tau_c$$

Theorem 6.3.4. *Let (X,τ) be a T_o-space and $y,z\in X$ be two distinct points satisfying the conditions (1) and (2) of Theorem 4.3.7, then (X,τ_{yz}) is a T_o-space iff $z\notin\{y\}^{\wedge}$ where $\tau_{yz}=\tau\cap D_{yz}$.*

Proof: If (X,τ_{yz}) is a T_o-space then by Theorem 6.3.1(3) $\{y\}^{\wedge}_{yz}\neq\{z\}^{\wedge}_{yz}$ and by Corollary 3.3.1, $\{y\}^{\wedge}_{yz}=\{y\}^{\wedge}$ and $\{z\}^{\wedge}_{yz}=\{y\}^{\wedge}\cup\{z\}^{\wedge}$ which implies that $\{y\}^{\wedge}\neq\{y\}^{\wedge}\cup\{z\}^{\wedge}$ according to which $z\notin\{y\}^{\wedge}$.

Conversely let $z \notin \{y\}^{\wedge}$. Then $\{y\}_{yz}^{\wedge} = \{y\}^{\wedge} \neq \{y\}^{\wedge} \cup \{z\}^{\wedge}$ from which $\{y\}_{yz}^{\wedge} \neq \{z\}_{yz}^{\wedge}$. If (X,τ) is T_o then $\{x\}^{\wedge} \neq \{z\}^{\wedge}$ and by Theorem 4.3.7,

$$\beta_{yz}^{\wedge} = \{\{x\}^{\wedge}, \{y\}^{\wedge} \cup \{z\}^{\wedge} : \{x\}^{\wedge} \in \beta^{\wedge} - \{\{z\}^{\wedge}\}\}$$

From which $\{x\}_{yz}^{\wedge} = \{x\}^{\wedge}$ for each point $x \in X - \{z\}$ and for each two points $t, x \in X - \{z\}$, $\{x\}^{\wedge} = \{x\}_{yz}^{\wedge} \neq \{t\}_{yz}^{\wedge} = \{t\}^{\wedge}$. If there exists a point $x \in X - \{y,z\}$ such that $\{x\}_{yz}^{\wedge} = \{z\}_{yz}^{\wedge}$ then $\{x\}^{\wedge} = \{y\}^{\wedge} \cup \{z\}^{\wedge}$ which implies that $y, z \in \{x\}^{\wedge}$ and either $x \in \{y\}^{\wedge}$ which implies that $\{x\}^{\wedge} = \{y\}^{\wedge}$ or $x \in \{z\}^{\wedge}$ which implies that $\{x\}^{\wedge} = \{z\}^{\wedge}$. Both cases contradict that (X,τ) is T_o hence $\{x\}_{yz}^{\wedge} \neq \{z\}_{yz}^{\wedge}$ for each point $x \in X - \{y,z\}$. Therefore (X, τ_{yz}) is a T_o − space.

6.4. Regular and T_3 − spaces:

Definition 6.4.1. The topological space (X,τ) is called regular if for each closed set $F \in \tau_c$ and each point $x \in F^c$ there are two disjoint open sets $U, V \in \tau$ such that one contains F and the other contains the point x that is

$$F \in \tau_c, x \in F^c \Rightarrow \exists U, V \in \tau : x \in U, F \subset V, U \cap V = \varnothing.$$

Example 6.4.1. The discrete topological space (X,D) is regular.

Example 6.4.2. Let $X = \{1,2,3,4\}$ and $\tau = \{X, \varnothing, \{1\}, \{2,3\}, \{4\}, \{1,2,3\}, \{1,4\}, \{2,3,4\}\}$ which implies that $\tau_c = \{\varnothing, X, \{2,3,4\}, \{1,4\}, \{1,2,3\}, \{4\}, \{2,3\}, \{1\}\} = \tau$

Clearly $F \in \tau_c \Rightarrow F \in \tau \Rightarrow F^c \in \tau_c \Rightarrow F^c \in \tau$ and so

$F \in \tau_c, x \in F^c \Rightarrow F, F^c \in \tau : x \in F^c, F \subset F, F \cap F^c = \varnothing$. Hence (X,τ) is regular.

Example 6.4.3. The topological space (X,τ) is regular where $\tau = E_p \cup C$ since if $x \in X - \{p\}$ and $F \in \tau_c$ such that $x \in F^c$ then either $p \in F$ or $p \notin F$ and we find that

$$p \in F \Rightarrow \{x\} \in E_p \subset \tau, \{x\}^c \in C \subset \tau \wedge x \in \{x\}, F \subset \{x\}^c \ (1) \text{ or}$$

$$p \notin F \Rightarrow F \in E_p \subset \tau, F^c \in \tau \wedge x \in F^c, F \subset F \ (2)$$

Another way: if $x \in X$ and $F \in \tau_c$ such that $x \in F^c$ then either $x = p$ or $x \neq p$ and we find that

$$x \neq p \Rightarrow \{x\} \in E_p \subset \tau, \{x\}^c \in C \subset \tau \wedge x \in \{x\}, \qquad (i)$$

$$F \subset \{x\}^c, \{x\} \cap \{x\}^c = \varnothing$$

or

$$x = p \Rightarrow p \notin F \Rightarrow F \in E_p \subset \tau, F^c \in \tau, x \in F^c \qquad (ii)$$

$$F \subset F, F \cap F^c = \varnothing$$

Example 6.4.4. Let $X = \{1,2,3,4\}$. Then

$$\tau = \{X, \varnothing, \{1\}, \{1,2,3\}, \{4\}, \{1,4\}\} \Rightarrow$$
$$\tau_c = \{\varnothing, X, \{2,3,4\}, \{4\}, \{1,2,3\}, \{2,3\}\}$$

and we remark that $\{2,3\} \in \tau_c$**,** $1 \in \{2,3\}^c$ and the unique member of τ different from X which contains $\{2,3\}$ is $\{1,2,3\}$ which contains also the point 1. Hence (X,τ) is not regular.

Example 6.4.5. The co-finite topological space (X,C) if X is infinite is not regular because the intersection of any two members of C is nonempty.

Theorem 6.4.1. *The topological space (X,τ) is regular iff for each open set $U \in \tau$ and each point $x \in U$ there exists an open set $V \in \tau$ such that $x \in V$ and $\overline{V} \subset U$ i.e. $U \in \tau, x \in U \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset U$.*

Proof: Let (X,τ) be a regular space. Then

$$U \in \tau, x \in U \Rightarrow U^c \in \tau_c, x \notin U^c \Rightarrow \exists V, W \in \tau :$$

$$x \in V, U^c \subset W, V \cap W = \varnothing$$

And we find that

(i) $x \in V$ (ii) $V \cap W = \varnothing \Rightarrow \overline{V} \cap W = \varnothing \Rightarrow \overline{V} \subset W^c$

(iii) $U^c \subset W \Rightarrow W^c \subset U$

From (i), (ii) and (iii) we gets $x \in V \subset \overline{V} \subset U$.

Conversely Suppose that the condition of the theorem is satisfied then

$$U \in \tau, x \in U \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset U$$

and hence

$$F \in \tau_c, x \in F^c \Rightarrow F^c \in \tau, x \in F^c \Rightarrow \exists V \in \tau :$$

$$x \in V \subset \overline{V} \subset F^c \Rightarrow x \in V, F \subset \overline{V}^c, V \cap \overline{V}^c = \varnothing$$

Therefore (X, τ) is regular.

Corollary 6.4.1. *If (X, τ) is a regular space then each finite open set $G \in \tau$ is a closed set.*

Proof: Let $G \in \tau$ be finite. Then by Theorem 6.4.1, we gets

$$x \in G \Rightarrow \exists V_x \in \tau : x \in \overline{V}_x \subset G \Rightarrow G = \bigcup\{\overline{V}_x : x \in G\}$$

$$\Rightarrow G \in \tau_c$$

because G could be written as a finite union of closed sets.

Corollary 6.4.2. *A principal topological (X, τ) is regular iff $\tau = \tau_c$.*

Proof: Similar to that in Corollary 6.4.1, suppose that (X, τ) is a regular principal space then

$$G \in \tau \Rightarrow G = \bigcup\{\overline{V}_x : x \in G\} \Rightarrow G \in \tau_c \Rightarrow \tau \subset \tau_c$$

and so

$$F \in \tau_c \Rightarrow F^c \in \tau \Rightarrow F^c \in \tau_c \Rightarrow F \in \tau \Rightarrow \tau_c \subset \tau .$$

Then $\tau = \tau_c$.

Conversely let $\tau = \tau_c$ then if $F \in \tau_c$ and $x \in F^c$ then

$F, F^c \in \tau$, $x \in F^c$, $F \subset F$ and $F \cap F^c = \varnothing$. Hence (X, τ) is regular.

Theorem 6.4.2. *The property that (X, τ) is regular is a hereditary property.*

Proof: Let (X, τ) be a regular space $Y \subset X$ and if $U \in \tau_Y$, then there exists $G \in \tau$ such that $U = G \cap Y$, so

$$y \in U \Rightarrow y \in G \Rightarrow \exists V \in \tau : y \in V \subset \overline{V} \subset G$$

$$\Rightarrow y \in V \cap Y \subset \overline{V} \cap Y \subset G \cap Y = U$$

But $V \cap Y \in \tau_Y$ and so

$$\overline{V} \cap Y \in \tau_{Yc}, V \cap Y \subset \overline{V} \cap Y \Rightarrow V \cap Y \subset \overline{(V \cap Y)}_Y$$

$$\subset \overline{V} \cap Y \subset U \Rightarrow y \in V \cap Y \subset \overline{(V \cap Y)}_Y \subset U$$

Where $\tau_{Yc} = \{Y - U : U \in \tau_Y\}$ is the family of the closed sets with respect to the topology τ_Y on Y and by Theorem 6.4.1, (Y, τ_Y) is regular.

Another way: Suppose that $M \in \tau_{Yc}$ and $y \in Y - M$ then $M \in \tau_{Yc} \Rightarrow \exists F \in \tau_c : M = F \cap Y$ which implies that

$$y \in Y - M = Y - F \cap Y = Y - F \Rightarrow y \notin F \qquad (*)$$

If (X,τ) is regular then there exist $U, V \in \tau$ such that $y \in U$, $F \subset U$ and $U \cap V = \varnothing$ and then (1) $y \in U \Rightarrow y \in U \cap Y$
(2) $F \subset V \Rightarrow M = F \cap Y \subset V \cap Y$,
(3) $(U \cap Y) \cap (V \cap Y) = \varnothing$ and (4) $(U \cap Y), (V \cap Y) \in \tau_Y$. Hence (Y, τ_Y) is regular.

Third way: starting from the result (*) we gets

$$y \notin F \Rightarrow \exists V \in \tau : y \in V \subset \overline{V} \subset F^c \Rightarrow y \in V \wedge F \subset \overline{V}^c$$

$$\Rightarrow y \in V \cap Y, M = F \cap Y \subset \overline{V}^c \cap Y$$

But $(V \cap Y) \cap (\overline{V}^c \cap Y) = (V \cap \overline{V}^c) \cap Y = \varnothing$ and $V \cap Y, \overline{V}^c \cap Y \in \tau_Y$ which implies that (Y, τ_Y) is regular.

Theorem 6.4.3. *The property that (X,τ) is regular is an invariant topological property.*

Proof: Let (X,τ) and (Y,τ^*) be two homeomorphic topological spaces by the homeomorphism $f : X \to Y$ and (X,τ) be regular. To prove that (Y,τ^*) is regular suppose that $U \in \tau^*$ and $y \in U$ then there exists a point $x \in X$ such that $y = f(x)$, f is continuous because it is homeomorphism and (X,τ) is regular which implies that

$$f^{-1}(U) \in \tau, x \in f^{-1}(U) \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset f^{-1}(U)$$

$$\Rightarrow y = f(x) \in f(V) \subset f(\overline{V}) \subset f(f^{-1}(U))$$

From which and since f is homeomorphism we find that

(1) f is correspondence which implies that $f(f^{-1}(U)) = U$.
(2) f is continuous and closed which implies that $f(\overline{V}) = \overline{f(V)}$.
(3) f is open which implies that $f(V) \in \tau^*$.

Therefore $y \in f(V) \subset \overline{f(V)} \subset U$ and by Theorem 6.4.1, (Y,τ^*) is regular.

Theorem 6.4.4. *If (X,τ) is a regular space and F is a closed set i.e. $F \in \tau_c$ then $F = \bigcap\{G \in \tau : F \subset G\} = F^{\wedge}$ but not conversely.*

Proof: Let (X,τ) be a regular space and $F \in \tau_c$. Then

$$x \in F^c \Rightarrow \exists W_x, V_x \in \tau : x \in W_x, F \subset V_x, W_x \cap V_x = \varnothing$$

So,

(1) $F^c \subset \bigcup\{W_x : x \in F^c\}$ and

(2) $F \subset \bigcap\{V_x : x \in F^c\}$.

Since $W_x \cap V_x = \varnothing$ for each point $x \in F^c$ then

$$y \in F^c \Rightarrow y \in W_y \Rightarrow y \notin V_y \Rightarrow y \notin \bigcap\{V_x : x \in F^c\} \Rightarrow$$

$$[\bigcap\{V_x : x \in F^c\}] \cap F^c = \varnothing \Rightarrow \bigcap\{V_x : x \in F^c\} \subset F$$

From which and from (2), $F = \bigcap\{V_x : x \in F^c\}$. Also $\{V_x : x \in F^c\} \subset \{G \in \tau : F \subset G\}$ since $F \subset V_x$ for each point $x \in F^c$ and hence

$$F \subset \bigcap\{G \in \tau : F \subset G\} \subset \bigcap\{V_x : x \in F^c\} = F \Rightarrow$$

$$F = \bigcap\{G \in \tau : F \subset G\} = F^\wedge$$

Conversely consider the co-finite space (X, C) where X is infinite or any non-regular T_1 – space we find that

$\{\{x\}^c : x \in F^c\} \subset \{G \in C : F \subset G\}$ for any $F \in C_c$ and so

$$F \subset \bigcap\{G \in C : F \subset G\} \subset \bigcap\{\{x\}^c : x \in F^c\} = F \Rightarrow$$

$$F = \bigcap\{G \in C : F \subset G\} = F^\wedge$$

While (X, C) is not regular. In fact if (X, τ) is T_1 then

$$F^\wedge = \bigcap\{G \in C : F \subset G\} = \bigcap\{\{x\}^\wedge : x \in F\}$$

$$= \bigcap\{\{x\} : x \in F\} = F$$

Theorem 6.4.5. *If (X, τ) is a regular space then $\{x\}^\wedge = \overline{\{x\}}$ for each point $x \in X$.*

Proof: If (X, τ) is regular and $G \in \tau$ then

$$x \in G \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset G \Rightarrow \overline{\{x\}} \subset G \Rightarrow x \in G$$

This means that $\overline{\{x\}} \subset G$ iff $x \in G$ for each $G \in \tau$ and from Theorem 6.4.4 ,

$$\overline{\{x\}} = \bigcap\{G \in \tau : \overline{\{x\}} \subset G\} = \bigcap\{G \in \tau : x \in G\} = \{x\}^\wedge$$

Another way: If $G \in \tau$ then clearly

$$x \in G \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset G \Rightarrow \overline{\{x\}} \subset G$$

$$\Rightarrow \overline{\{x\}} \subset \{x\}^\wedge \quad (I)$$

Secondly, $y \notin \overline{\{x\}} \Rightarrow \exists G \in \tau : y \in G, x \notin G$ and so $y \in G \Rightarrow \exists V \in \tau : y \in V \subset \overline{V} \subset G$ which implies that

(1) $x \notin G \Rightarrow x \notin \overline{V} \Rightarrow x \in \overline{V}^{c}$,

(2) $y \in \overline{V} \Rightarrow y \notin \overline{V}^{c}$ and

(3) $\overline{V}^{c} \in \tau$

from which $y \notin \{x\}^{\wedge}$ then $\{x\}^{\wedge} \subset \overline{\{x\}}$ (II).

From (I) and (II), $\{x\}^{\wedge} = \overline{\{x\}}$.

Remark 6.4.1. *Corollary 6.4.2, can be considered as a corollary of Theorem 6.4.5, since if* (X,τ) *is a principal regular space then* $\{x\}^{\wedge} = U_x$ *for each point* $x \in X$ *and so*

$$G \in \tau \Rightarrow G = \bigcup\{U_x : x \in G\} = \bigcup\{\overline{\{x\}} : x \in G\} \Rightarrow G \in \tau_c$$

$$\Rightarrow \tau \subset \tau_c \Rightarrow \tau = \tau_c$$

Conversely $\tau = \tau_c$ *implies that* (X,τ) *is regular.*

Corollary 6.4.3. *The principal topological space* (X,τ) *is regular iff* $U_x = \overline{\{x\}}$ *for each point* $x \in X$.

Proof: It is a direct consequence of Theorem 6.4.5 and remark 6.4.1

Remark 6.4.2. *Theorem 6.4.4, can be considered as a corollary of Theorem 6.4.5 . For if* (X,τ) *is a regular space and* $F \in \tau_c$ *then*

$$F = \bigcup \overline{\{x\}} : x \in F\} = \bigcup\{\{x\}^{\wedge} : x \in F\} = F^{\wedge}$$

$$= \bigcap\{G \in \tau : F \subset G\}$$

Theorem 6.4.6. *If* (X,τ) *is a regular space then*

(1) $x, y \in X : \overline{\{x\}} \neq \overline{\{y\}} \Rightarrow \overline{\{x\}} \cap \overline{\{y\}} = \varnothing$.

(2) $x, y \in X : \{x\}^{\wedge} \neq \{y\}^{\wedge} \Rightarrow \{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$.

Proof: If (X,τ) is a topological space then

(1) $x, y \in X : \overline{\{x\}} \neq \overline{\{y\}} \Rightarrow x \notin \overline{\{y\}} \vee y \notin \overline{\{x\}}$ and if (X,τ) is regular then

$$x \notin \overline{\{y\}} \Rightarrow x \in \overline{\{y\}}^{c} \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset \overline{\{y\}}^{c}$$

$$\Rightarrow \overline{\{x\}} \subset \overline{\{y\}}^{c} \Rightarrow \overline{\{x\}} \cap \overline{\{y\}} = \varnothing$$

and we gets the same result in the case $y \notin \overline{\{x\}}$.

(2) From Theorem 6.4.5 if (X,τ) is regular then $\{x\}^{\wedge} = \overline{\{x\}}$ which implies that $\{x\}^{\wedge} \in \tau_c$ and so $\{x\}^{\wedge c} \in \tau$ and so

$$x, y \in X : \{x\}^{\wedge} \neq \{y\}^{\wedge} \Rightarrow x \notin \{y\}^{\wedge} \vee y \notin \{x\}^{\wedge}$$

From which

$$x \notin \{y\}^{\wedge} \Rightarrow x \in \{y\}^{\wedge c} \Rightarrow \exists V \in \tau : x \in V \subset \overline{V} \subset \{y\}^{\wedge c} \Rightarrow$$

$$\Rightarrow \{x\}^{\wedge} \subset \{y\}^{\wedge c} \Rightarrow \{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$$

and we gets the same result in the case $y \notin \{x\}^{\wedge}$.

In fact one can directly proved (2) since by Theorem 6.4.5, $\{x\}^{\wedge} = \overline{\{x\}}$ for each point $x \in X$ then

$$x, y \in X : \{x\}^{\wedge} \neq \{y\}^{\wedge} \Rightarrow \overline{\{x\}} \neq \overline{\{y\}} \Rightarrow \overline{\{x\}} \cap \overline{\{y\}} = \varnothing$$

$$\Rightarrow \{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$$

Corollary 6.4.4. *If* (X, τ) *is a regular space then the family* $\{\overline{\{x\}} : x \in X\} = \{\{x\}^{\wedge} : x \in X\}$ *is a partition of* X *.*

Proof: this is clearly by Theorem 6.4.6 and since

$$X = \{\overline{\{x\}} : x \in X\} = \{\{x\}^{\wedge} : x \in X\}.$$

Corollary 6.4.5. *Any regular* T_o *−space is* T_1.

Proof: Let (X, τ) be a T_o −space. Then $x, y \in X : x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$ and so if (X, τ) is regular then by Theorem 6.4.6, $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ which implies by Theorem 6.2.1, that (X, τ) is T_1

Theorem 6.4.7. *Let* f *and* g *be two continuous functions from* (X, τ) *to* (Y, τ^*) *and* (Y, τ^*) *be a regular space. Then the set* $A = \{x \in X : f(x) \notin \overline{\{g(x)\}}\}$ *is open.*

Proof: Since (Y, τ^*) is regular then

$$x \in A \Rightarrow \exists U, V \in \tau^* : f(x) \in U, \overline{\{g(x)\}} \subset V \wedge U \cap V = \varnothing$$

$$\Rightarrow x \in f^{-1}(U) \cap g^{-1}(V)$$

If f and g are continuous then $f^{-1}(U) \cap g^{-1}(V) \in \tau$ and

$$t \in f^{-1}(U) \cap g^{-1}(V) \Rightarrow f(t) \in U \wedge g(t) \in V$$

but U^c is a closed set and then

$$U \cap V = \varnothing \Rightarrow g(t) \in V \subset U^c \Rightarrow \overline{\{g(t)\}} \subset U^c$$

$$\Rightarrow f(t) \notin \overline{\{g(t)\}} \Rightarrow t \in A$$

Therefore $x \in (f^{-1}(U) \cap g^{-1}(V)) \subset A \Rightarrow A \in \tau$.

Remark 6.4.3. *As in Theorem 6.1.7, one can show that for each point* $x \in A$ *there are* $U_x, V_x \in \tau^*$ *such that* $x \in (f^{-1}(U_x) \cap g^{-1}(V_x))$ *and*

$$A = \bigcup_{x \in A} (f^{-1}(U_x) \cap g^{-1}(V_x))$$

which implies that A *is open.*

Remark 6.4.4. *If* f *and* g *are two continuous functions from* (X, τ) *to* (Y, τ^*) *and* (Y, τ^*) *is a regular space then*

$$A = \{x \in X : f(x) \notin \overline{\{g(x)\}}\} = \{x \in X : \overline{\{f(x)\}} \neq \overline{\{g(x)\}}\}$$

$$= \{x \in X : \overline{\{f(x)\}} \cap \overline{\{g(x)\}} = \varnothing\} = \{x \in X : g(x) \notin \overline{\{f(x)\}}\}$$

Theorem 6.4.8. *If* (X, τ) *is a principal topological space then it is regular iff the minimal basis* $\beta_o = \{U_x : x \in X\}$ *for* τ *is a partition of* X *.*

Proof: If (X, τ) is a principal regular topological space then by Theorem 6.4.7, $\{x\}^{\wedge} = U_x$ for each point $x \in X$ and by Corollary 6.4.3, $\beta_o = \{U_x : x \in X\}$ is a partition of X.

Conversely if β_o is a partition of X and $x \in X$ is an arbitrary point then

$$y \in U_x^c \Rightarrow U_x \neq U_y \Rightarrow U_x \cap U_y = \varnothing$$

$$\Rightarrow U_y \subset U_x^c \Rightarrow U_x^c \in \tau \Rightarrow U_x \in \tau_c$$

i.e $U_x \in \tau_c$ for each point $x \in X$. Hence

$$G \in \tau \Rightarrow G = \bigcup \{U_x : x \in G\} \Rightarrow G \in \tau_c \Rightarrow \tau \subset \tau_c \Rightarrow \tau_c = \tau$$

Therefore by using Corollary 6.4.2, (X, τ) is a regular space.

Another proof for the second part, if β_o is a partition of X and $x \in X$ is an arbitrary point, $G \in \tau$ and $y \in G^c$ then

$$U_y \cap G \neq \varnothing \Rightarrow \exists x : x \in U_y \wedge x \in G \Rightarrow U_x \subset U_y \wedge U_x \subset G$$

$$\Rightarrow y \notin U_x \Rightarrow U_x \neq U_y \Rightarrow U_x \cap U_y = \varnothing \Rightarrow x \notin U_y$$

This contradiction means that such point x does not exists and so $U_y \cap G = \phi$ which implies that

$$y \in G^c \Rightarrow U_y \subset G^c \Rightarrow G^c \in \tau \Rightarrow G \in \tau_c \Rightarrow \tau \subset \tau_c \Rightarrow \tau = \tau_c$$

Hence (X,τ) is regular.

Third proof for the second part, if β_o is a partition of X and $x \in X$ is an arbitrary point then

$$y \notin U_x \Rightarrow U_x \neq U_y \Rightarrow U_x \cap U_y = \varnothing \Rightarrow \{x\} \cap U_y = \varnothing$$

$$\Rightarrow \overline{\{x\}} \cap U_y = \varnothing \Rightarrow y \notin \overline{\{x\}} \Rightarrow \overline{\{x\}} \subset U_x \quad (I)$$

Also

$$y \notin \overline{\{x\}} \Rightarrow x \notin U_y \Rightarrow U_x \neq U_y \Rightarrow U_x \cap U_y = \varnothing$$

$$\Rightarrow y \notin U_x \Rightarrow U_x \subset \overline{\{x\}} \quad (II)$$

From (I) and (II) $U_x = \overline{\{x\}}$ which implies by Corollary 6.4.2, that (X,τ) is regular.

Remark 6.4.6. *According to Theorem 6.4.7, if (X,τ) is a principal regular space then there is a correspondence between the families of all regular principal topologies and all equivalence relations on a set X, we means by a regular topology τ on X that (X,τ) is a regular topological space i.e., in each regular space (X,τ) the minimal basis $\beta_o = \{U_x : x \in X\}$ for τ is the quotient set of an equivalence relation S on X as well as the set of all equivalence classes of the points of X i.e. the quotient set $S \backslash X = \{[x] : x \in X\}$ of an equivalence relation S on X is the minimal basis for a regular topology on X.*

Example 6.4.6. Let $X = \{1,2,3,4\}$ and S be a relation on X such that

$$S = \{(1,1),(1,4),(4,1),(4,4),(2,2),(3,3)\}$$

Clearly S is an equivalence relation on X and $S \backslash E = \{\{1,4\},\{2\},\{3\}\}$ is the quotient set which is a partition of X. Then the topology which has the minimal basis $\beta_o = S \backslash E$ is

$$\tau=\{X,\varnothing,\{1,4\},\{2\},\{3\},\{1,2,4\},\{1,3,4\},\{2,3\}\}$$

and clearly by Corollary 6.4.2, that (X,τ) is regular.

Definition 6.4.2. The topological space (X,τ) is called a T_3- space if it is regular and at the same time is T_1-space that is

$$T_3=T_1+\text{ regular.}$$

Example 6.4.7. The discrete topological space (X,D) is a T_3-space also (X,τ) is a T_3- space where $\tau=E_p\cup C$ since it is T_2 by Example 6.1.3 and is regular by example 6.4.3.

Remark 6.4.7. *From the Definition 6.4.2, and Corollary 6.4.3, the topological space (X,τ) is T_3 iff it is T_o and regular that is*

$$T_3=T_o+\textit{ regular.}$$

Remark 6.4.8. *Of course there are topological spaces which are regular and not T_1 like for example the given in Example 6.4.2, which is regular because $\tau=\tau_c$ and not T_1 because $\tau\neq D$. Also if X is infinite the co-finite space (X,C) is T_1 and not regular since in this case the intersection of any two members of C is nonempty .*

Theorem 6.4.9. *The axiom T_3 is hereditary and invariant topological property.*

Proof: We proved that T_1 is hereditary Corollary 6.2.1 and is an invariant topological property Theorem 6.2.2, also proved before that the regularity property is hereditary Theorem 6.4.2 and is invariant topological property Theorem 6.4.3. Hence according to the Definition 6.4.2, of T_3, T_3 is both hereditary and invariant topological property.

Theorem 6.4.10. *Any T_3-topological space is T_2 i.e. $T_3\rightarrow T_2$.*

Proof: If (X,τ) is a T_3-space then

(1) it is T_1 and hence $\{y\} \in \tau_c$ for each point $y \in X$ from which

$$x, y \in X : x \neq y \Rightarrow \{y\} \in \tau_c, x \notin \{y\}$$

(2) it is regular which implies that there are two open sets $U, V \in \tau$ such that $x \in U$, $y \in \{y\} \subset V$ and $U \cap V = \varnothing$. Therefore (X, τ) is a T_2- space.

Remark 6.4.9. *The inverse of the implication $T_3 \to T_2$ is not correct generally for example consider the family $\sigma = \{(a,b), Q^c : a,b \in R\}$ where R is the set of the real numbers, Q is the set of the rational numbers and Q^c is the set of irrational numbers. Since $\beta = \{(a,b) : a,b \in R\}$ is a basis for the usual topology $\mho$ on R and*

$$\beta \subset \sigma \subset \beta(\sigma) \Rightarrow \mho \subset \tau(\beta(\sigma))$$

Then the topological space $(R, \tau(\beta(\sigma)))$ is T_2 since $(R, \mho)$ is T_2. But $Q \in (\tau(\beta(\sigma)))_c$ and any irrational number $p \in Q^c$ can not be separated by two nonempty open sets because the unique open set which contains Q is R this means that $(R, \tau(\beta(\sigma)))$ is not regular and so is not T_3. Therefore, $T_2 -/\to T_3$.

6.5. Normal and T_4-spaces:

Definition 6.5.1. The topological space (X, τ) is called a normal space if for each two disjoint closed sets $F_1, F_2 \in \tau_c$ there are two disjoint open sets $U, V \in \tau$ one of them contains F_1 and the other contains F_2 that is

$$F_1, F_2 \in \tau_c : F_1 \cap F_2 = \varnothing \Rightarrow \exists U, V \in \tau : F_1 \subset U, F_2 \subset V \wedge$$
$$U \cap V = \varnothing$$

Example 6.5.1. (X, D) is a normal space.

Example 6.5.2. Let $X = \{a,b,c\}$ and $\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{c\}, \{a,c\}\}$ then $\tau_c = \{\varnothing, X, \{b,c\}, \{c\}, \{a,b\}, \{b\}\}$. Clearly there are the following pairs of disjoint closed sets and the corresponding pairs of disjoint open sets different from X and $\varnothing$.

(1) $\{b\},\{c\} \in \tau_c - \{X,\varnothing\}$ and the corresponding open sets $\{a,b\},\{c\} \in \tau$ such that

$$\{b\} \subset \{a,b\}, \{c\} \subset \{c\} \wedge \{a,b\} \cap \{c\} = \varnothing .$$

(2) $\{a,b\},\{c\} \in \tau_c$ and $\{a,b\},\{c\} \in \tau$ such that

$$\{a,b\} \subset \{a,b\}, \{c\} \subset \{c\} \wedge \{a,b\} \cap \{c\} = \varnothing .$$

Therefore (X,τ) is a normal space.

Remark 6.5.1. *If $\tau \subset E_p$ then $\tau_c \subset E_{pc} = P_p$ and the topological space (X,τ) is normal where*

$$F_1, F_2 \in \tau_c - \{X,\varnothing\} \Rightarrow p \in F_1 \cap F_2 \Rightarrow F_1 \cap F_2 \neq \varnothing$$

and the unique disjoint closed sets are X and $\varnothing$.

Example 6.5.3. The topological space (X,τ) where $\tau = E_p \cup C$ is normal since

$$\tau = E_p \cup C \Rightarrow \tau_c = (E_p \cup C)_c = E_{pc} \cup C_c = P_p \cup C_c$$

this means that τ_c consists all subsets F of X such that $p \in F$ or F is finite together with X and $\varnothing$ and so $F_1, F_2 \in \tau_c$ such that $F_1 \cap F_2 = \varnothing$ implies that $F_1 \subset F_2{}^c$ and $F_2 \subset F_1^c$ and there are two possibilities $p \notin F_1$ or $p \notin F_2$ and we finds $p \notin F_1 \Rightarrow F_1 \in E_p, F_1^c \in C \Rightarrow F_1, F_1^c \in \tau$ and clearly $F_1 \subset F_1$, $F_2 \subset F_1{}^c$ and $F_1 \cap F_1^c = \varnothing$. The other case $p \notin F_2$ is similar.

Example 6.5.4. The lower limit point topological space (R,τ) where τ is the topology generated on the set of the real numbers R by the basis $\beta = \{[a,b) : a,b \in R\}$ is normal.

Proof: Suppose that $F_1, F_2 \in \tau_c$ such that $F_1 \cap F_2 = \varnothing$ then $F_1 = \overline{F_1}$ and $F_2 = \overline{F_2}$ which implies that $F_1 \cap \overline{F_2} = \varnothing$ and $\overline{F_1} \cap F_2 = \varnothing$. Hence

$$\begin{aligned} x \in F_1 &\Rightarrow x \notin \overline{F_2} \\ &\Rightarrow \exists\, a_x, b_x \in R : x \in [a_x, b_x) \wedge [a_x, b_x) \cap F_2 = \varnothing \end{aligned}$$

But

$$x \in [a_x, b_x) \Rightarrow [x, b_x) \subset [a_x, b_x) \Rightarrow [x, b_x) \cap F_2 = \varnothing .$$

Similarly for each $y \in F_2$ there is a real number $c_y \in R$ such that $[y, c_y) \cap F_1 = \varnothing$. Now Suppose that $U = \bigcup_{x \in F_1} [x, b_x)$ and $V = \bigcup_{y \in F_2} [y, c_y)$ then $F_1 \subset U$, $F_2 \subset V$ and $U, V \in \tau$ and we are going to prove that $U \cap V = \varnothing$. For let $U \cap V \neq \varnothing$. Then there exists $z \in U \cap V$ which implies that $z \in U$ and $z \in V$. Hence there are two points $x_o \in F_1$ and $y_o \in F_2$ such that $z \in [x_o, b_{x_o})$ and $z \in [y_o, c_{y_o})$. Clearly if $z = x_o$ then $x_o = z \in [y_o, c_{y_o})$ which implies that $[y_o, c_{y_o}) \cap F_1 \neq \varnothing$ a contradiction and so $x_o \neq z$. Similarly $y_o \neq z$. Now since $x_o \neq y_o$ because $F_1 \cap F_2 = \varnothing$ then we have the following two cases:

Case (1): $x_o < y_o \Rightarrow x_o < y_o < z < b_{x_o} \Rightarrow y_o \in [x_o, b_{x_o})$

$$\Rightarrow [x_o, b_{x_o}) \cap F_2 \neq \varnothing$$

Case (2): $y_o < x_o \Rightarrow y_o < x_o < z < c_{y_o} \Rightarrow x_o \in [y_o, c_{y_o})$

$$\Rightarrow [y_o, c_{y_o}) \cap F_1 \neq \varnothing$$

These two cases contradicts that $[x, b_x) \cap F_2 = \varnothing$ for each $x \in F_1$ and $[y, c_y) \cap F_1 = \varnothing$ for each $y \in F_2$. Therefore $U \cap V = \varnothing$ and so (R, τ) is a normal topological space.

Theorem 6.5.1. *Any principal regular topological space is normal*.

Proof: If (X, τ) is a principal regular space then by Corollary 6.4.2, $\tau = \tau_c$ and then

$$F_1, F_2 \in \tau_c : F_1 \cap F_2 = \varnothing \Rightarrow F_1, F_2 \in \tau \wedge F_1 \cap F_2 = \varnothing$$

Hence (X,τ) is normal.

Remark 6.5.2. *The topological space (X,τ) which is given by Example 6.5.2, is not regular since $\tau \neq \tau_c$ and not T_1 since $\tau \neq D$ i.e. the normal space may be not regular which implies that the converse of Theorem 6.5.1, is incorrect see Example 6.5.2, where (X,τ) is normal but not regular and the normal space need not be T_1. Also the topological space (X,C) is T_1 and if X is infinite then it is not normal since in this case the intersection of any two members of C is nonempty .*

Theorem 6.5.2. *The topological space (X,τ) is normal iff for each closed set $F \in \tau_c$ and each open set $U \in \tau$ such that $F \subset U$ there exists an open set $V \in \tau$ such that $F \subset V$ and $\overline{V} \subset U$ that is i.e.*

$$F \in \tau_c, U \in \tau : F \subset U \Rightarrow \exists V \in \tau : F \subset V \subset \overline{V} \subset U$$

Proof: If (X,τ) is a normal space, $F \in \tau_c$ and $U \in \tau$ then $(i)\ U \in \tau \Rightarrow U^c \in \tau_c$ and $(ii)\ F \subset U \Rightarrow F \cap U^c = \varnothing$ so by the Definition 6.5.1, there are two open sets $V, W \in \tau$ such that $F \subset V$, $U^c \subset W$ and $V \cap W = \varnothing$ and hence

(1) $U^c \subset W \Rightarrow W^c \subset U$ and

(2) $V \cap W = \varnothing \Rightarrow \overline{V} \cap W = \varnothing \Rightarrow \overline{V} \subset W^c$.

Hence

$$F \subset V \subset \overline{V} \subset W^c \subset U \Rightarrow F \subset V \subset \overline{V} \subset U$$

Conversely, If the condition is satisfied then

$$F_1, F_2 \in \tau_c : F_1 \cap F_2 = \varnothing \Rightarrow F_1 \in \tau_c, F_2^c \in \tau, F_1 \subset F_2^c$$

and from the given condition there exists $V \in \tau$ such that $F_1 \subset V \subset \overline{V} \subset F_2^c$ from which we finds that $F_1 \subset V$, $F_2 \subset \overline{V}^c$, $V \cap \overline{V}^c = \varnothing$ and $V, \overline{V}^c \in \tau$ which implies that (X,τ) is a normal space.

Theorem 6.5.3. *A topological space* (X,τ) *is normal iff for each two disjoint closed sets* $F_1,F_2\in\tau_c$ *there are two open sets* $U,V\in\tau$ *such that* $F_1\subset U$ *,* $F_2\subset V$ *and* $\overline{U}\cap\overline{V}=\varnothing$*.*

Proof: Let (X,τ) be a normal space and $F_1,F_2\in\tau_c$ such that $F_1\cap F_2=\varnothing$ then there are two open sets $U_1,V_1\in\tau$ such that $F_1\subset U_1$, $F_2\subset V_1$ and $U_1\cap V_1=\varnothing$ then by using Theorem 6.5.2, we gets

$$F_1\subset U_1\Rightarrow\exists U\in\tau:F_1\subset U\subset\overline{U}\subset U_1$$

and

$$F_2\subset V_1\Rightarrow\exists V\in\tau:F_2\subset V\subset\overline{V}\subset V_1$$

Which implies that $F_1\subset U$, $F_2\subset V$ and $\overline{U}\cap\overline{V}=\varnothing$ since clearly

$$\overline{U}\subset U_1,\overline{V}\subset V_1\Rightarrow\overline{U}\cap\overline{V}\subset U_1\cap V_1=\varnothing\Rightarrow\overline{U}\cap\overline{V}=\varnothing.$$

Conversely, if for each two disjoint closed sets $F_1,F_2\in\tau_c$ there are two open sets $U,V\in\tau$ such that $F_1\subset U$, $F_2\subset V$ and $\overline{U}\cap\overline{V}=\varnothing$ then

$$\overline{U}\cap\overline{V}=\varnothing\Rightarrow U\cap V\subset\overline{U}\cap\overline{V}=\varnothing\Rightarrow U\cap V=\varnothing.$$

Therefore (X,τ) is normal.

Theorem 6.5.4. *If* (X,τ) *is a normal space then* $F_1,F_2\in\tau_c$ *such that* $F_1\cap F_2=\varnothing$ *implies that* $F_1^{\wedge}\cap F_2^{\wedge}=\varnothing$*. If* (X,τ) *is a principal space then the converse of this implication is also true.*

Proof: Suppose that (X,τ) is a normal space and $F_1,F_2\in\tau_c$ such that $F_1\cap F_2=\varnothing$ then there are two open sets $U,V\in\tau$ such that $F_1\subset U$, $F_2\subset V$ and $U\cap V=\varnothing$ from which we gets $F_1\subset U\Rightarrow F_1^{\wedge}\subset U$ and $F_2\subset V\Rightarrow F_2^{\wedge}\subset V$ which implies that $F_1^{\wedge}\cap F_2^{\wedge}=\varnothing$.

If (X,τ) is a principal topological space then $F_1^{\wedge}, F_2^{\wedge} \in \tau$ for each subsets $F_1, F_2 \in P(X)$. But $F_1 \subset F_1^{\wedge}$ and $F_2 \subset F_2^{\wedge}$ and so if $F_1^{\wedge} \cap F_2^{\wedge} = \varnothing$ for each two closed sets $F_1, F_2 \in \tau_c$ such that $F_1 \cap F_2 = \varnothing$ then (X,τ) is normal.

Theorem 6.5.5. *If (X,τ) is a normal space then $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ implies that $\{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$ for each two distinct points $x, y \in X$. If (X,τ) is a principal topological space then the converse is also true.*

Proof: Let (X,τ) be a normal space and $x, y \in X$ be such that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$. Then there are two open sets $U, V \in \tau$ such that $\overline{\{x\}} \subset U$, $\overline{\{y\}} \subset V$ and $U \cap V = \varnothing$ from which $x \in U \Rightarrow \{x\}^{\wedge} \subset U$ and $y \in V \Rightarrow \{y\}^{\wedge} \subset V$ which implies that $\{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$. Let (X,τ) be a principal space such that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ implies that $\{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$ for each two distinct points $x, y \in X$ and $F_1, F_2 \in \tau_c$ such that $F_1 \cap F_2 = \varnothing$. Then $F_1 = \bigcup \{\overline{\{x\}} : x \in F_1\} = \bigcup_{x \in F_1} \{t \in X : x \in U_t\}\}$ and similarly $F_2 = \bigcup_{y \in F_2} \{z \in X : y \in U_z\}\}$ where $\{x\}^{\wedge} = U_x$ for each point $x \in X$. Then $U = \bigcup\{U_x : x \in F_1\}\}$ and $V = \bigcup\{U_y : y \in F_2\}\}$ implies that $U, V \in \tau$, $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \varnothing$. To prove that $U \cap V = \varnothing$ let $t \in U \cap V$ then $t \in U$ and $t \in V$ which implies that there exist $x \in F_1$ and $y \in F_2$ such that $t \in U_x$ and $t \in U_y$ this means that $U_x \cap U_y \neq \varnothing$ while $x \in F_1$ and $y \in F_2$ implies that $\overline{\{x\}} \subset F_1$ and $\overline{\{y\}} \subset F_2$ because F_1 and F_2 are closed sets and this implies that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ because $F_1 \cap F_2 = \varnothing$, Hence $U_x \cap U_y \neq \varnothing$ contradicts our assumption, this contradiction implies that $U \cap V \neq \varnothing$ is incorrect. Hence $U \cap V = \varnothing$ and (X, τ) is normal.

Remark 6.5.3. *Theorem 6.5.5, can be formulated to be in any topological space* (X, τ) *if we consider the following statements*

(1) (X, τ) *is normal,*

(2) $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ *implies that* $\{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$ *for each two points* $x, y \in X$ *and*

(3) $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$ *implies that* $\{t\}^{\wedge} \cap \{z\}^{\wedge} = \varnothing$ *for each two points* $t, z \in X$ *such that* $x \in \{t\}^{\wedge}$ *and* $y \in \{z\}^{\wedge}$. *Then* $(1) \Rightarrow (2) \Rightarrow (3)$ *and if* (X, τ) *is a normal principal space then* (1), (2) *and* (3) *are equivalent.*

Proof: we shall prove that (2) and (3) are equivalent as follows:

(2) $\Longrightarrow$ (3): Let $x, y \in X$ be such that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$. Then $t, z \in X$ implies that

$$x \in \{t\}^{\wedge},\ y \in \{z\}^{\wedge} \Rightarrow t \in \overline{\{x\}},\ z \in \overline{\{y\}}$$
$$\Rightarrow \overline{\{t\}} \subset \overline{\{x\}},\ \overline{\{z\}} \subset \overline{\{y\}} \Rightarrow \overline{\{t\}} \cap \overline{\{z\}} \subset \overline{\{x\}} \cap \overline{\{y\}} = \varnothing$$
$$\Rightarrow \overline{\{t\}} \cap \overline{\{z\}} = \varnothing$$

which implies by (2) that $\{t\}^{\wedge} \cap \{z\}^{\wedge} = \varnothing$.

(3) $\Longrightarrow$ (2): It is clearly since $x \in \{x\}^{\wedge}$ and $y \in \{y\}^{\wedge}$.

Theorem 6.5.6 *Let* $X \neq \varnothing$ *and* $T \subset P(X)$ *be a partition of* X *and* (Y, τ_Y) *be a topological space for each* $Y \in T$. *Then*

(1) $\beta = \bigcup \{\tau_Y : Y \in T\}$ *is a basis for a topology* τ *on* X *such that* (Y, τ_Y) *is a subspace of* (X, τ) *for each* $Y \in T$.

(2) (X,τ) *is normal iff* (Y,τ_Y) *is normal for each* $Y \in T$ *.*

Proof: Clearly $\bigcup\{Y : Y \in T\} = X$ and if $B_1, B_2 \in \beta$ then either there is a member $Y \in T$ such that $B_1, B_2 \in \tau_Y$ in this case $B_1 \cap B_2 \in \tau_Y \subset \beta$ or there are two members $Y, Z \in T$ such that $B_1 \in \tau_Y$ and $B_2 \in \tau_Z$ in this case $B_1 \cap B_2 = \varnothing$ since $Y \cap Z = \varnothing$. Hence β is a basis for a topology τ on X where $\tau = \tau(\beta)$. This topology is the same topology $\tau = \{G \subset X : G \cap Y \in \tau_Y ; \forall Y \in T\}$ on X. Clearly (Y,τ_Y) is a subspace of (X,τ) where $\{G \cap Y : G \in \tau\} = \tau_Y$ for each $Y \in T$.

(2) Let $F, M \in \tau_C$ be such that $F \cap M = \varnothing$ then

$$\begin{aligned} F = F \cap X = F \cap [\bigcup\{Y : Y \in T\}] &= \bigcup\{F \cap Y : Y \in T\} \\ &= \bigcup\{F_Y : Y \in T\} \end{aligned}$$

Where $F_Y = F \cap Y$ for each $Y \in T$. Similarly $M = \bigcup\{M_Y : Y \in T\}$ where $M_Y = M \cap Y$ for each $Y \in T$. Since (Y,τ_Y) is normal for each $Y \in T$ then there are $U_Y, V_Y \in \tau_Y$ such that $F_Y \subset U_Y$, $M_Y \subset V_Y$ and $U_Y \cap V_Y = \varnothing$. If $U = \bigcup\{U_Y : Y \in T\}$ and $V = \bigcup\{V_Y : Y \in T\}$ then $F \subset U$, $M \subset V$ and we are going to prove that $U \cap V = \varnothing$ for

$$\begin{aligned} y \in U \cap V &\Rightarrow y \in U, y \in V \\ &\Rightarrow \exists\, Y, Z \in T : y \in U_Y,\ y \in V_Z \Rightarrow U_Y \cap V_Z \neq \varnothing \end{aligned}$$

But we have the following two cases

(1) $Y = Z$ which implies that $V_Z = V_Y$ which implies that $U_Y \cap V_Z = U_Y \cap V_Y = \varnothing$ or

(2) $Y \neq Z$ in this case $Y \cap Z = \varnothing$, $U_Y \subset Y$ and $V_Z \subset Z$ which implies that $U_Y \cap V_Z = \varnothing$. Hence $U_Y \cap V_Z \neq \varnothing$ is a contradiction in

both cases this contradiction because of the incorrect assumption that $U \cap V \neq \varnothing$. Therefore $U \cap V = \varnothing$ and so (X, τ) is normal.

Conversely, $Y \in T$ implies that $X - Y = \bigcup\{Z \in T : Z \neq Y\} \in \tau$ which implies that $Y \in \tau_c$ which implies that $\tau_{Y_c} \subset \tau_c$. If (X, τ) is a normal space then $F_1, F_2 \in \tau_{Y_c} \Rightarrow F_1, F_2 \in \tau_c$ and if $F_1 \cap F_2 = \varnothing$ then there are two open sets $U, V \in \tau$ such that $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \varnothing$ and we finds

(1) $U \cap Y, V \cap Y \in \tau_Y$, (2) $F_1 \subset U \cap Y$, (3) $F_2 \subset V \cap Y$ and

(4) $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \varnothing$.

Therefore, (Y, τ_Y) is a normal space.

Example 6.5.5. Consider the set of the positive integers N and let Y be the set of the odd numbers and Z be the set of the even numbers. Then $(Y, E_1(Y))$ and $(Z, E_2(Z))$ are normal spaces where $E_1(Y)$ is the excluding point topology on Y with the excluding point 1 and $E_2(Z)$ is the excluding point topology on Z with the excluding point 2. Then $\beta = E_1(Y) \cup E_2(Z)$ is a basis for the topology on N,

$$\tau(\beta) = E_{\{1,2\}} \bigcup \{Y \cup G ; G \subset Z - \{2\}\} \bigcup \\ \bigcup \{Z \cup H ; H \subset Y - \{1\}\}$$

which implies that

$$\tau_c(\beta) = P_{\{1,2\}} \bigcup \{Z \subset F : 1 \in F\} \bigcup \{Y \subset M : 2 \in M\} \bigcup \{Y, Z\}$$

Hence if $F_1, F_2 \in \tau_c(\beta)$ such that $F_1 \cap F_2 = \varnothing$ then either $(i) F_1 \subset Y \wedge F_2 \subset Z$ or $(ii)\ F_1 \subset Z \wedge F_2 \subset Y$. Since $Y, Z \in \tau$ and $Y \cap Z = \varnothing$ then (N, τ) is a normal space.

Example 6.5.6. Consider the set of the positive integers N and let Y be the set of the odd numbers and Z be the set of the even numbers. Then

$(Y, E_1(Y) \cup C_Y)$ and $(Z, E_2(Z))$ are normal spaces where $E_1(Y)$ is the excluding point topology on Y with the excluding point 1, C_Y is the co-finite topology on Y and $E_2(Z)$ is the excluding point topology on Z with the excluding point 2. Then

$$\beta = (E_1(Y) \cup C_Y) \cup E_2(Z) = \{Y, Z, G \subset Y, H \subset Y, W \subset Z : \\ 1 \in H \wedge Y - H \text{ is finite } 2 \notin W\}$$

is a basis for a topology $\tau(\beta)$ on N where

$$\tau(\beta) = \{N, \varnothing, Y, Z, U, G, H, W \subset N : \\ U \cap \{1,2\} = \varnothing, 1 \notin G, 1 \in H, H \cup Z \text{ such that } Y - H \text{ is} \\ \text{finite and } 2 \notin W, G \cup Z, W \cup Y\}$$

Which implies that

$$\tau_c(\beta) = \{N, \varnothing, Y, Z, V \subset N, (Y - H) \cup Z, (Y - H) \cup (Z - W), \\ H, Y - G, Y - H, Z - W : \{1,2\} \subset V, 1 \notin G, 1 \in H \wedge Y - H \\ \text{is finite and } 2 \notin W\}$$

Then we have the following remarks:

(a) $1 \in H \Rightarrow 1 \notin Y - H \Rightarrow (Y - H) \in E_1(Y) \Rightarrow (Y - H) \in \tau(\beta)$

Equivalently

$$1, 2 \in H \cup Z \Rightarrow H \cup Z \in \tau_c(\beta) \Rightarrow N - (H \cup Z) = Y - H \in \tau(\beta)$$

(b) $Z \in \tau(\beta) \Rightarrow (Y - H) \cup Z \in \tau(\beta) \Rightarrow H \in \tau_c(\beta)$.

(c) If G is finite then $(Y - G) \in C(Y) \Rightarrow (Y - G) \in \tau(\beta)$.

Therefore there are the following cases of the pairwise disjoint closed sets and the corresponding pairs of open sets:

(1) $Y \cap Z = \varnothing$ and $Y \subset Y$, $Z \subset Z$,

(2) $H \cap Z = \varnothing$ and $H \subset H$, $Z \subset Z$,

(3) $H \cap (Z - W) = \varnothing$ and $H \subset H$, $Z - W \subset Z$,

(4) $H \cap (Y - H) = \varnothing$ and $H \subset H$, $Y - H \subset Y - H$,

(5) $H \cap [(Y - H) \cup Z] = \varnothing$ and $H \subset H$,

$(Y - H) \cup Z \subset (Y - H) \cup Z$,

(6) $H \cap [(Y - H) \cup (Z - W)] = \varnothing$ and $H \subset H$,

$(Y - H) \cup (Z - W) \subset (Y - H) \cup Z$,

(7) $Y \cap (Z - W) = \varnothing$ and $Y \subset Y$, $Z - W \subset Z$,

(8) $(Y - H_1) \cap (Y - H_2) = \varnothing$ and $(Y - H_2) \subset (Y - H_2)$,

$Y - H_1 \subset N - (Y - H_2) \subset H_2 \cup Z$,

(9) $(Y - G) \cap Z = \varnothing$ and $(Y - G) \subset Y$, $Z \subset Z$,

(10) $[(Y - H) \cup Z] \cap (Y - G) = \varnothing$ and $(Y - H) \cup Z \subset (Y - H) \cup Z$, $Y - G \subset N - [(Y - H) \cup Z] = H$,

(11) $[(Y - H) \cup (Z - W)] \cap (Y - G) = \varnothing$ and

$(Y - H) \cup (Z - W)] \subset (Y - H) \cup Z$ while $(Y - G) \subset H$,

(12) $(Y - G) \cap (Y - H) = \varnothing \Rightarrow (Y - G) \cap (Y \cap H^c) = \varnothing$

$\Rightarrow (Y - G) \cap H^c = \varnothing \Rightarrow Y - G \subset H$,

$Y - H \subset Y - H$ and

(13) $[(Y - H_1) \cup (Z - W)] \cap (Y - H_2) = \varnothing$ and

$Y - H_2 \subset Y - H_2$, $(Y - H_1) \cap (Y - H_2) = \varnothing$

$\Rightarrow Y - H_1 \subset N - (Y - H_2) = H_2 \cup Z \wedge Z - W \subset Z$

$$\Rightarrow (Y - H_1) \cup (Z - W) \subset (H_2 \cup Z).$$

Therefore $(N, \tau(\beta))$ is a normal topological space.

Example 6.5.7. Let $X = \{1,2,3,4,5\}$, $Y = \{1,2,3\}$, $Z = \{4,5\}$, $\tau_Y = \{Y, \varnothing, \{1,2\}\}$ and $\tau_Z = \{Z, \varnothing, \{4\}, \{5\}\}$. Then (Y, τ_Y) and (Z, τ_Z) are normal spaces and $\beta = \tau_Y \cup \tau_Z$ is a basis for the topology $\tau(\beta)$ on X where

$$\begin{aligned}&\beta = \{(1,2,3\}, \{4,5\}, \varnothing, \{1,2\}, \{4\}, \{5\}\} \Rightarrow \\ &\tau(\beta) = \{X, \varnothing, \{1,2\}, \{1,2,3\}, \{4\}, \{5\}, \{4,5\}, \{1,2,4\}, \{1,2,5\}, \\ &\qquad \{1,2,4,5\}, \{1,2,3,4\}, \{1,2,3,5\}\}\end{aligned}$$

Clearly $(X, \tau(\beta))$ is a normal space.

Remark 6.5.4. *The partition T of X is given and the topological space (X, τ) is normal if (Y, τ_Y) is normal for each $Y \in T$ and T is a subfamily of τ. The logical question is if (X, τ) is a topological space is there a subfamily T of τ such that if T is a partition of X then (X, τ) is normal?. At follows we try to answer this question when (X, τ) is a principal space.*

We defined before the set $U_A \in \tau$ and if (X, τ) is a principal space, β_o is the minimal basis for τ and $A \subset X$ then we consider the family

$$\begin{aligned}T &= \{U_A \in \tau : U_x \cap A = \varnothing; \forall x \in A^c\} \\ &= \{U_A \in \beta_o : A \cap U_B = \varnothing; \forall U_B \in \beta \text{ such that } U_A \neq U_B\} \\ &= \{U_x \in \beta_o : x \notin U_y;\ \forall y \in X \text{ such that } U_x \neq U_y\}.\end{aligned}$$

Clearly $U_A = U_x$ for each point $x \in A$ which implies that $T \subset \beta_o$ and so T can be written in the form

$$T = \{U_A \in \beta_o : U_x \cap A = \varnothing; \forall x \in A^c\}$$

From which we get

(1) $A \in \tau_c$ *for each* $U_A \in T$ *where*

$$U_A \in T \Rightarrow U_x \cap A = \varnothing; \forall x \in A^c \Rightarrow U_x \subset A^c$$
$$\Rightarrow A^c \in \tau \Rightarrow A \in \tau_c$$

(2) The family $T^* = \{A : U_A \in T\}$ *is a family of pairwise disjoint subsets of* X *and if* $U_A, U_B \in T$ *such that* $A \neq B$ *and* $x \in U_A - A$ *then* $x \in B \Rightarrow U_x = U_B \subset U_A$. *Now* $y \in A$ *implies that* $U_y = U_A$ *and* $A \cap B = \varnothing$ *implies that* $y \in B^c$ *but* $U_B \subset U_A$ *implies that* $B \subset U_y$ *i.e.* $B \cap U_y = B \neq \varnothing$ *which contradicts that* $U_B \in T$ *this contradiction because of the incorrect assumption that* $x \in B$ *and so* $x \in U_A - A \Rightarrow x \in B^c$ *for each* $U_B \in T$ *such that* $A \neq B$ *and accordingly if* T^* *is a partition of* X *i.e. if* $X = \bigcup\{A : A \in T^*\}$ *then* $U_A - A = \varnothing$ *for each* $U_A \in T$ *since* $x \in U_A - A$ *implies that* $x \in B^c$ *for each* $U_B \in T$ *such that* $A \neq B$ *from which*

$$x \in \bigcap\{\mathrm{B}^c : B \in T^* - \{A\}\} = [\bigcup\{\{\mathrm{B} : B \in T^* - \{A\}\}]^c = A$$

that is $x \in A$ *which contradicts the assumption* $x \in U_A - A$. *Then such point* x *does not exist. Hence* $U_A = A$ *for each* $U_A \in T$ *and this means that* $T^* = T$.

(3) $\bigcup\{U_A : U_A \in T\} = X$ *, for if* $x \in X$ *then either*

(i) For each proper subset A *of* X *such that* $U_A \in \beta_o$ *and* $x \in U_A$*, there is a point* $y \in A^c$ *such that* $U_y \cap A \neq \varnothing$. *In this case* X *is a minimal open set at a point* $p \in X$ *i. e.* $X = U_p$ *and so* $T = \{X\}$ *or*

(ii) There is a subset A *of* X *such that* $U_A \in \beta_o$*,* $x \in U_A$ *and* $U_y \cap A = \varnothing$ *for each* $y \in A^c$. *In this case* $U_A \in T$ *and* $x \in U_A$ *.*

(4) (U_A,τ_{U_A}) is a normal space in fact it is an $E-$space i.e. there is a point $x\in U_A$ such that $\tau_{U_A}\subset E_x$. For let $\beta_o(U_A)$ be the minimal basis for τ_{U_A} then $x\in U_A$ implies that either

(a) $x\in A\Rightarrow U_x\in\beta_o\Rightarrow U_x=U_A\in\beta_o\Rightarrow U_A\in\beta_o(U_A)$ or

(b) $x\in U_A-A\Rightarrow U_x\in\beta_o\wedge U_x\subset U_A\wedge U_x\neq U_A$

$$\Rightarrow U_x\in\beta_o-\{U_A\}\Rightarrow U_x\in\beta_o(U_A)-\{U_A\}$$

From (a) and (b) $\beta_o(U_A)=\{U_x,U_A:x\in U_A-\{x\}\}$ this means that $\tau_{U_A}\subset E_A=\{G\subset U_A:G\cap A=\varnothing\}\bigcup\{U_A\}\subset E_x$ where $x\in A$ is any point.

(5) How to obtain the family T: If $x\in X$ and $U_x\in\beta_o$ then there exists a subset $A\subset X$ such that $U_x=U_A$, may be $A=\{x\}$ or $A=X$ and we finds

(a) there is $y\in A^c$ such that $x\in U_y$ or equivalently $U_y\cap A\neq\varnothing$ in this case $U_x\notin T$.

(b) $x\notin U_y$ for each $U_y\in\beta_o-\{U_x\}$ or equivalently $U_y\cap A=\varnothing$ for each $y\in A^c$ in this case $U_x\in T$.

Example 6.5.8. (1) Let $X=\{1,2,3,4,5\}$ and

$$\begin{aligned}\tau=\{X,\varnothing,&\{1,2\},\{1,2,3\},\{4\},\{1,2,5\},\{1,2,4\},\{1,2,3,4\}\},\\&\{1,2,3,5\},\{1,2,4,5\}\}\end{aligned}$$

Clearly $\beta_o=\{\{1,2\},\{1,2,3\},\{4\},\{1,2,5\}\}$ where $U_1=U_2=U_{\{1,2\}}=\{1,2\}$, $U_3=\{1,2,3\}$, $U_4=\{4\}$ and $U_5=\{1,2,5\}$ from which we finds (a) $1,2\in U_3\Rightarrow U_{\{1,2\}}\notin T$,

(b) $3\notin U_y$ for each $U_y\in\beta_o-\{U_3\}$ which implies that $U_3\in T$,

(c) $4 \notin U_y$ for each $U_y \in \beta_o - \{U_4\}$ this implies that $U_4 \in T$ and

(d) $5 \notin U_y$ for each $U_y \in \beta_o - \{U_5\}$ which implies that $U_5 \in T$. Therefore $T = \{U_3, U_4, U_5\} = \{\{1,2,3\},\{4\},\{1,2,5\}\}$ and $T^* = \{\{3\},\{4\},\{5\}\}$.

(2) Consider the set of the positive integers N and let

$$\tau = \{N, \varnothing, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \ldots, \{1,3\}, \{1,3,5\}, \\ \{1,3,5,7\}, \ldots\}$$

Hence, $T = \{N\}$.

Theorem 6.5.7. *The principal topological space* (X, τ) *is normal iff the family* $T = \{U_A \in \tau : A \subset X, A \cap U_x = \varnothing; \forall x \in A^c\}$ *is a partition of* X.

Proof: Let (X, τ) be a normal space. Then by Remark 6.5.4, statement (3), $\bigcup\{U_A : U_A \in T\} = X$ and if $U_A, U_B \in T$ are arbitrary such that $U_A \neq U_B$ then by Remark 6.5.4, statements (2) and (3), $A, B \in \tau_c$ and $A \cap B = \varnothing$. If (X, τ) is normal then there are two open sets $G, H \in \tau$ such that $A \subset G$, $B \subset H$ and $G \cap H = \varnothing$ from which we gets

$$A \subset G, B \subset H \Rightarrow U_A \subset G, U_B \subset H \\ \Rightarrow U_A \cap U_B \subset G \cap H = \varnothing \Rightarrow U_A \cap U_B = \varnothing$$

Therefore T is a partition of X.

Conversely By statement (4) of Remark 6.5.4, (U_A, τ_{U_A}) is a normal space for each $U_A \in T$ and if T is a partition of X then according to Theorem 6.5.4, $\beta = \bigcup\{\tau_{U_A} : U_A \in T\}$ is a basis for a topology τ^* on X such that (X, τ^*) is normal. But $\tau_{U_A} \subset \tau$ for each $U_A \in \tau$ which implies that $\beta \subset \tau$ which implies that $\tau^* \subset \tau$ (I). If $G \in \tau$ then

$$G = G \cap X = G \cap [\bigcup\{U_A : U_A \in T\} = \bigcup\{G \cap U_A : U_A \in T\}$$

But $G \cap U_A \in \tau_{U_A}$ for each $U_A \in T$ which means that $G \in \tau^*$ which implies that $\tau^* \subset \tau$ (II). From (I) and (II), $\tau^* = \tau$ and hence (X, τ) is a normal space.

Example 6.5.9. In Example 6.5.6 (1) T is not a partition of X and so (X, τ) is not normal. (2) T is a partition of N and so (N, τ) is normal.

Example 6.5.10. Let $X = \{1,2,3,4,5,6\}$ and

$$\tau = \{X, \varnothing, \{1,2\}, \{3\}, \{4\}, \{3,5,6\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,5,6\}, \\ \{3,4\}, \{3,4,5,6\}, \{1,2,3,4\}\}$$

Clearly $\beta_o = \{\{1,2\}, \{3\}, \{4\}, \{3,5,6\}\}$and $T = \{\{1,2\}, \{4\}, \{3,5,6\}\}$ which is a partition of X and then (X, τ) is a normal space.

Example 6.5.11. Let $X \neq \varnothing$ and $p \in X \neq \{p\}$ then (a) (X, E_p) is a normal space since $T = \{X\}$ is a partition of X .

(b) (X, P_p) is not normal because $T = \{\{x, p\} : x \in X - \{p\}\}$is not a partition of X .

Corollary 6.5.1. *Let (X, τ) be a regular principal topological space. Then*

(1) (X, τ) is normal since in this case $T = \beta_o$ is a partition of X where β_o is the minimal basis for τ .

(2) If τ_{yz} is a strictly weaker topology than τ where y and z are any two distinct points of X then (X, τ_{yz}) is normal since in this case $T = \{U_x, U_y \cup U_z : U_x \in \beta_0 - \{U_z\}\}$ is a partition of X .

Remark 6.5.5. *The property that (X, τ) is normal is not hereditary since for example if $X = \{a,b,c,d\}$ and*

$$\tau = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$$

and

$$\tau_C = \{\varnothing, X, \{b,c,d\}, \{c,d\}, \{b,d\}, \{d\}\}$$

Then the unique disjoint two closed sets are X *and* $\varnothing$ *and accordingly* (X,τ) *is normal we must be remark that* $T = \{X\}$*. If* $Y = \{a,b,c\}$ *then* $\tau_Y = \{Y, \varnothing, \{a\}, \{a,b\}, \{a,c\}\}$ *and so* $T = \{\{a,b\}, \{a,c\}\}$ *which is not a partition of* Y *and then* (Y, τ_Y) *is not normal. Also* $\tau_{Y_C} = \{\varnothing, Y, \{b,c\}, \{c\}, \{b\}\}$ *clearly* $\{b\}, \{c\} \in \tau_{Y_C}$ *are two disjoint closed sets which can not be separated by two disjoint members of* τ_Y *.*

If $Y \in \tau_C - \{\varnothing\}$ *then* (Y, τ_Y) *is normal as it given by the following Theorem.*

Theorem 6.5.8. *If* (X,τ) *is a normal space and* $Y \in \tau_C - \{\varnothing\}$ *then* (Y, τ_Y) *is a normal space.*

Proof: See the proof of the second part of (2) of Theorem 6.5.5.

Theorem 6.5.9. *The property that the topological space* (X,τ) *is normal is hereditary.*

Proof: the proof is left to the reader.

Theorem 6.5.10. *The property that the topological space* (X,τ) *is normal is an invariant topological property.*

Proof: Let (X,τ) be a normal space, (Y,τ^*) be a topological space and $f: X \to Y$ be a homeomorphism, we need to prove that (Y,τ^*) is normal. For let $M_1, M_2 \in (\tau^*)_C$ such that $M_1 \cap M_2 = \varnothing$ then $f^{-1}(M_1), f^{-1}(M_2) \in \tau_C$ since f is continuous and $f^{-1}(M_1) \cap f^{-1}(M_2) = \varnothing$. Since (X,τ) is normal then there are

$U, V \in \tau$ such that $f^{-1}(M_1) \subset U$, $f^{-1}(M_2) \subset V$ and $U \cap V = \varnothing$ and since f is homeomorphism then

(a) $M_1 = f(f^{-1}(M_1)) \subset f(U)$, $M_2 = f(f^{-1}(M_2)) \subset f(V)$ because f is surjective .

(b) $f(U), f(V) \in \tau^*$ because f is open and

(c) $f(U) \cap f(V) = f(U \cap V) = f(\varnothing) = \varnothing$ because f is injective.

Therefore (Y, τ^*) is normal.

Definition 6.5.2. The topological space (X, τ) is called T_4-space if it is normal and T_1 i. e. T_4= normal $+T_1$.

Example 6.5.12. The discrete topological space is T_4 since it is T_1 and normal.

Example 6.5.13. The topological space (X, τ) where $\tau = E_p \cup C$ is normal by Example 6.5.3 and is T_2 and so is T_4.

Theorem 6.5.11. *Any* T_4-*space is* T_3 *i.e.* $T_4 \to T_3$.

Proof: If (X, τ) is a T_4-space then it is T_1 and hence

$F \in \tau_c, x \in F^c \Rightarrow F, \{x\} \in \tau_c, F \cap \{x\} = \varnothing$, $\{x\} \in \tau_c$ because (X, τ) is T_1 and since (X, τ) is normal then there are $U, V \in \tau$ such that $F \subset U$, $x \in \{x\} \subset V$ and $U \cap V = \varnothing$. Hence (X, τ) is a T_3-space.

6.6. Axioms of countability:

Definition 6.6.1. The topological space (X, τ) has or satisfies the first axiom of countability if for each point $x \in X$ there is a countable local basis β_x for x.

Example 6.6.1. Each topological space (X,τ) if X or τ is finite has the first axiom of countability.

Example 6.6.2. In the discrete topological space (X,D) the family $\beta_x=\{\{x\}\}$ is a local basis for the point x for each $x\in X$ and so it has the first axiom of countability.

Example 6.6.3. In the usual topological space $(R,\mho)$ we proved that $\beta_x=\{(x-\frac{1}{n},x+\frac{1}{n}):n\in N\}$ is a local basis at the point x for each $x\in R$ and hence $(R,\mho)$ has the first axiom of countability.

Remark 6.6.1. *If the topological space* (X,τ) *i.e. if it has a countable local basis* β_x *for each* $x\in X$ *where* $\beta_x=\{B_1,B_2,B_3,...\}$ *such that* $x\in B_n$ *for each* $B_n\in\beta_x$ *then we can construct a nested local basis for the point* x *as follows:*

Let $U_1=B_1$ *and* $U_{n+1}=U_n\cap B_{n+1}$ *such that* $n\geq 1$, *we obtains the family* $\beta_x^*=\{U_1,U_2,U_3,...\}$ *such that* $U_1\supset U_2\supset U_3\supset......$ *which is i.e.* β_x^* *a local basis for* x *since,*

(1) $\beta_x^*\subset\tau$ *because* $U_n\in\tau$ *for each* $U_n\in\beta_x^*$.

(2) If $G\in\tau$ *such that* $x\in G$ *and* $B_n\subset G$ *then* $U_n\subset B_n\subset G$.

Definition 6.6.2. The topological space (X,τ) has or satisfies the second axiom of countability if τ has a countable basis β.

Example 6.6.4. All finite topological spaces we means the spaces (X,τ) such that X or τ is finite each has the second axiom of countability.

Example 6.6.5. The usual topological space $(R,\mho)$ satisfies the second axiom of countability since $\beta_o=\{(a,b):a,b\in Q\}$ is a countable basis for u where Q is the set of the rational numbers. Remember that it satisfies also the first axiom.

Theorem 6.6.1. *Each second countable topological space* (X,τ) *is first countable.*

Proof: If (X,τ) satisfies the second axiom of countability then there exist a countable basis β for τ and for each point $x \in X$ the family $\beta_x = \{B \in \beta : x \in B\}$ is a local basis for the point x which means that (X,τ) has the first axiom of countability.

Remark 6.6.2. *The converse of Theorem 6.6.1, is not valid as illustrated by the following example.*

Example 6.6.6. The particular point topological space (R,P_p) where $p \in R$ satisfies the first axiom of countability where $\beta_x = \{\{p,x\}\}$ is a countable local basis for x for each $x \in R-\{p\}$ and $\beta_p = \{\{p\}\}$ is a local basis for the point p while it not satisfies the second axiom of countability since if β is a basis for the topology P_p,

$$\beta \supset \beta_o = \{\{p\},\{p,x\} : x \in R-\{p\}\}$$

and clearly β_o is uncountable which implies that β is also.

Theorem 6.6.2. *Each second countable space is separable.*

Proof: Suppose that (X,τ) satisfies the second axiom of countability then there exists a countable basis β for τ. By choosing one and only one point from each member of β we means that choose one point and don't choose more than one point from each member of β such that if $x \in X$ and $\{B_\alpha : \alpha \in \Delta\} \subset \beta$ are such that $\cap\{B_\alpha : \alpha \in \Delta\} = \{x\}$ and $B \cap [\cap\{B_\alpha : \alpha \in \Delta\}] = \emptyset$ for each $B \in \beta - \{B_\alpha : \alpha \in \Delta\}$ then the point x is considered to be the selected point from B_α for each $\alpha \in \Delta$. Then we obtains a subset A of X such that $B \cap A$ is singleton for each $B \in \beta$. Therefore $B \cap A \neq \emptyset$ for each $B \in \beta$ which implies that

$$G \in \tau - \{\emptyset\} \Rightarrow \exists B \in \beta : B \subset G \Rightarrow$$

$$\emptyset \neq B \cap A \subset G \cap A \Rightarrow G \cap A \neq \emptyset \Rightarrow \overline{A} = X$$

Now suppose that $f:\beta\to A$ is the function given by $f(B)=x$ for each $B\in\beta$ such that $B\cap A=\{x\}$ this function is surjective and so A is constable since β is constable. Therefore (X,τ) is separable.

Remark 6.6.3. *The converse of Theorem 6.6.2, is not valid since the space* (R,P_p) *is separable and not satisfies the second axiom of countability see Example 6.6.6. The following example also insures this remark.*

Example 6.6.7. The co-finite topological space (R,C) does not satisfy the second axiom of countability.

Solution: let $x\in R$ be any point and $\beta_x\subset C$ be a local basis for x. Then

$y\in\bigcap\{B:B\in\beta_x\}-\{x\}\Rightarrow y\in B-\{x\};\forall B\in\beta_x \quad (I)$.

But $\{y\}^c\in C$ and $x\in\{y\}^c$ from which there is $B\in\beta_x$ such that $B\subset\{y\}^c$ this means that $y\in\{y\}^c$ which is impossible this impossibility because of the incorrect assumption in (I) then the correct is $\bigcap\{B:B\in\beta_x\}=\{x\}$ from which we gets $\{x\}^c=\bigcup\{B^c:B\in\beta_x\}$. But B^c which is finite since $B\in C$ for each $B\in\beta_x$ which implies that $\{x\}^c$ where $\{x\}^c=R-\{x\}$ is countable since in fact it is a countable union of finite sets and this is impossible this impossibility because of the incorrect assumption that β_x is countable and so the correct is β_x is an uncountable family and since it is an arbitrary local basis of the arbitrary point $x\in R$. Therefore (R,C) does not satisfies the first axiom of countability and so does not satisfies the second (In both sentences arbitrary local basis and arbitrary point are need not it is sufficient to find one point with no countable local basis).

Theorem 6.6.3. *Each of the first and the second axiom of countability is hereditary.*

Proof: We shall prove the case of the second and left the first to the exercises for suppose that (Y,τ_Y) is a topological subspace of (X,τ) which has the second axiom of countability. Then there is a countable basis β for τ according of which $\beta_Y=\{B\cap Y:B\in\beta\}$ is a basis for τ_Y and clearly is countable since $\beta^*=\{B\in\beta:B\cap Y\neq\varnothing\}$ is a countable family since it is a subfamily of the countable family β. Hence the function $f:\beta^*\to\beta_Y-\{\varnothing\}$ given by $f(B)=B\cap Y$ for each $B\in\beta^*$ is surjective which implies that β_Y is a countable basis.

Theorem 6.6.4. *The first and the second axioms of countability are topological invariant.*

Proof: We shall prove the case of the first axiom and left the second to the exercises. Suppose that (X,τ) and (Y,τ^*) are two homeomorphic topological spaces by the homeomorphism $f:X\to Y$ and let (X,τ) be first countable space. If $y\in Y$ is an arbitrary point then there is a point $x\in X$ such that $y=f(x)$ and since (X,τ) satisfies the first axiom of countability then there exists a countable local basis β_X for the point x, we shall prove that $\beta_{y_Y}=\{f(B):B\in\beta_X\}$ is a countable local basis for the point y and this is enough to say that (Y,τ^*) satisfies the first axiom of countability. Since f is homeomorphism it is injective, surjective, continuous and open from which we finds

(1) $B\in\beta_X\Rightarrow B\in\tau_1\Rightarrow f(B)\in\tau_2$ because f is open which implies that $\beta_{y_Y}\subset\tau_2$.

(2) $x\in B$ for each $B\in\beta_X$ implies that $y=f(x)\in f(B)$ for each $B\in\beta_X$ which implies that $y\in U$ for each $U\in\beta_{y_Y}$.

(3) If $y\in Y$ and $V\in\tau^*$ such that $y\in V$ then there exists $x\in X$ such that $y=f(x)\in V$ and so

$$f^{-1}(V)\in\tau_1\wedge x\in f^{-1}(V)\Rightarrow \exists B\in\beta_x : B\subset f^{-1}(V)$$
$$\Rightarrow f(B)\subset f(f^{-1}(V))\subset V$$

where $f(B)\in\beta_{yY}$ for each $B\in\beta_x$.

Definition 6.6.3. A topological space (X,τ) is called Lindelof space if for each family of open sets $\Re=\{G_\alpha:\alpha\in\nabla\}\subset\tau$ such that $X=\bigcup\{G_\alpha:\alpha\in\nabla\}$ there exists a countable subfamily $\Re^*$ of $\Re$ such that $X=\bigcup\{G^*:G^*\in\Re^*\}$. Such family $\Re$ is called an open cover of the topological space X and $\Re^*$ is called a countable subcover of X of the cover $\Re$, the covers and subcovers will be discuss in Chapter (7).

Remark 6.6.4. *If $X\neq\varnothing$ and $\mathrm{T}\subset P(X)$ is a family of pair -wise disjoint nonempty members i.e. the intersection of any two members of T is empty then the cardinal number $|\mathrm{T}|$ of T less than or equals to the cardinal number $|X|$ of X i.e. $|\mathrm{T}|\leq|X|$ and $|\mathrm{T}|=|X|$ if $\mathrm{T}=\{\{x\}:x\in X\}$ but not conversely.*

Theorem 6.6.5. *Each second countable topological space is Lindelof.*

Proof: If (X,τ) satisfies the second axiom of countability then there exists a countable basis β for τ, to prove that (X,τ) is Lindelof let $\Re$ be an open cover of X and $G\in\Re-\{\phi\}$, set $G=G_1$ and define for each point $x\in G_1$ the family $\beta_x=\{B\in\beta:x\in B\subset G_1\}$ and attach G_1 to the family $\beta_1=\bigcup\{\beta_x:x\in G_1\}$. If $G_1=X$ then $\{G_1\}$ is the required subcover of $\Re$ and if $G_1\neq X$ then there is a member $G\in\Re-\{G_1\}$ such that $G-G_1\neq\varnothing$, set $G=G_2$ and define for each point $x\in G_2-G_1$ the family

$$\beta_x=\{B\in\beta-\bigcup\{\beta_x:x\in G_1\}:x\in B\subset G_2\}$$
$$=\{B\in\beta-\beta_1:x\in B\subset G_2\}$$

Then attach to G_2 the family $\beta_2=\bigcup\{\beta_x:x\in G_2-G_1\}$, if

$G_1\cup G_2=X$ then $\{G_1,G_2\}$ is the required subcover of $\Re$ and if $G_1\cup G_2\neq X$ then there exists $G\in\Re-\{G_1,G_2\}$ such that

$G-(G_1\cup G_2)\neq\varnothing$ then set $G=G_3$ and continue these processes tell obtain G_n and β_n where $n\geq 2$ and define the family

$$\beta_x=\{B\in\beta-\bigcup_{j=1}^{n-1}\{\beta_x : x\in(G_j-\bigcup_{i=1}^{j-1}G_i)\} : x\in B\subset G_n\}$$

$$=\{B\in\beta-(\bigcup_{i=1}^{n-1}\beta_i) : x\in B\subset G_n\}$$

For each point $x\in G_n-\bigcup_{i=1}^{n-1}G_i$ remarking that

$\bigcup_{j=1}^{0}\{\beta_x : x\in(G_1-\bigcup_{i=1}^{-1}G_i)\}=\varnothing$ at $n=1$ and $\bigcup_{i=1}^{0}G_i=\varnothing$ at $n=2$

Then attach G_n with the family $\beta_n=\bigcup\{\beta_x : x\in G_n-\bigcup_{i=1}^{n-1}G_i\}$. If $X=\bigcup_{i=1}^{n}G_i$ then $\{G_1,G_2,G_3,...,G_n\}\subset\Re$ is the required sub cover of $\Re$ otherwise there is $G\in\Re-\{G_1,G_2,G_3,...,G_n\}$ such that $G-\bigcup_{i=1}^{n}G_i\neq\varnothing$ in this case set $G=G_{n+1}$ and for each point $x\in G_{n+1}-\bigcup_{i=1}^{n}G_i$ define the family

$$\beta_x=\{B\in\beta-\bigcup_{j=1}^{n}\{\beta_x : x\in(G_j-\bigcup_{i=1}^{j-1}G_i)\} : x\in B\subset G_{n+1}\}$$

$$=\{B\in\beta-(\bigcup_{i=1}^{n}\beta_i) : x\in B\subset G_{n+1}\}$$

Then attach G_{n+1} with $\beta_{n+1} = \bigcup\{\beta_x : x \in G_{n+1} - \bigcup\limits_{i=1}^{n} G_i\}$ and so on we gets two families the first is $\Re^* = \{G_1, G_2, G_3, ..., G_n, ...\}$ which is a subfamily of $\Re$ which forms a cover of the topological space X this is clearly from the processes of the construction of $\Re^*$ since for each $x \in X$ there is $n \in N$ and $B \in \beta_n$ such that $x \in B \subset G_n$. The other family is $\beta^* = \{\beta_1, \beta_2, \beta_3, ..., \beta_n, ...\}$ and if $\beta_n, \beta_m \in \beta^*$ such that $n \neq m$ then either $n > m$ or $n < m$. In the case $n > m$ we finds

$$B \in \beta_n \Rightarrow \exists x \in G_n - \bigcup_{i=1}^{n-1} G_i : x \in B \subset G_n \frac{1}{2}$$

But $x \in G_n - \bigcup\limits_{i=1}^{n-1} G_i$ implies that $x \notin G_m$ from which $x \notin B^*$ for each $B^* \in \beta_m$ which implies that $B \notin \beta_m$ and therefore $B \in \beta_n \Rightarrow B \notin \beta_m$ from which $\beta_n \cap \beta_m = \emptyset$ and we gets the same result from the case $n < m$. Therefore $\beta^* \subset P(\beta)$ is a family of pairwise disjoint members then it is countable according to Remark 6.6.4. Consider the function $f : \beta^* \to \Re^*$ given by $f(\beta_n) = G_n$ for each $\beta_n \in \beta^*$, clearly f is a bijective function which implies that $\Re^*$ is a countable cover of X and is a subcover of $\Re$. Hence (X, τ) is a Lindelof topological space.

Another way to prove that $\Re^*$ is countable: Suppose that $\beta(\Re^*) = \bigcup\{\beta_n : n \in N\}$ then $\beta(\Re^*) \subset \beta$ and so is countable. Now consider the function $f : \beta(\Re^*) \to \Re^*$ for each $n \in N$ and each $B \in \beta_n$, $f(B) = G_n$ clearly f is surjective and hence $\Re^*$ is countable.

Theorem 6.6.6. *If the topological space (X, τ) has the second axiom of countability then each family of pairwise disjoint members of $\tau - \{\emptyset\}$ is countable.*

Proof: Suppose that (X, τ) has the second axiom of countability then there exists a countable basis β for the topology τ, if $\Re \subset \tau - \{\emptyset\}$ is a family of pairwise disjoint members and if $G \in \Re$ then there is $B \in \beta$ such that $B \subset G$ accordingly for each $G \in \Re$ we can define the family $\beta_G = \{B \in \beta : B \subset G\}$ according of which we gets the family $T = \{\beta_G : G \in \Re\}$ and if $G, H \in \Re$ are distinct then $G \cap H = \varnothing$ which implies that $\beta_G \cap \beta_H = \varnothing$ and so $T \subset P(\beta)$ is a family of pairwise disjoint members of $P(\beta)$ hence it is countable by using Remark 5.2.4. Now consider the function $f : \mathrm{T} \to \Re$ given by $f(\beta_G) = G$ for each $\beta_G \in \mathrm{T}$ this function clearly injective. Hence $\Re$ is a countable family.

Remark 6.6.5. *Any family of pairwise disjoint open intervals of real numbers is countable. In fact such family is a family of disjoint open sets in the usual topological space* $(R, \mho)$ *which satisfies the second axiom of countability and then by Theorem 6.6.5, such family is countable.*

Definition 6.6.4. In the topological space (X, τ), if $A \subset X$ is nonempty, the point $x \in X$ is called a condensation point of the set A if each open set $G \in \tau$ such that $x \in G$, $G \cap A$ is an uncountable set that is G consists an uncountable subset of A.

Theorem 6.6.7. *Each infinite uncountable subset of a second countable topological space has a condensation point.*

Proof: Let (X, τ) be a second countable space then there is a countable basis β for τ. Then if $A \subset X$ is an uncountable set which has no condensation points that is each $x \in A$ is not condensation point of A. So there is $G \in \tau$ such that $x \in G$, $G \cap A$ is a countable set and there is $B \in \beta$ such that $x \in B \subset G$ which implies that $B \cap A$ is countable then by choosing for each point $x \in A$ one and only one member B_x from the family

$$\beta_x = \{B \in \beta : \ x \in B \subset G, \ B \cap A \text{ is countable } \}$$

which implies that $A_x = B_x \cap A$ is countable for each $x \in A$ and the corresponding family $\beta_A = \{B_x : x \in A\}$ is countable . If we neglect the repetition of the members of β_A because we may have two distinct points $x, y \in A$ while $B_x = B_y$ and $x, y \in A_x \cap A_y \subset A_x \cup A_y$ or may be $A_x = A_y$ hence $A = \bigcup\{A_x : x \in A\}$ is a countable union of countable sets which implies that A is a countable set which contradicts the assumption that A is uncountable. Hence the set A has at least one condensation point from its points.

Theorem 6.6.8. *If (X, τ) is a second countable space then each isolated subset A of X is countable.*

Proof: Let (X, τ) be a second countable space. Then there is a countable basis β for τ and if A is an isolated subset of X i.e. $A = isd.(A)$ then

$$x \in A \Rightarrow x \in isd.(A) \Rightarrow \exists G \in \tau : G \cap A = \{x\}$$

Since $x \in G$ then there exists $B \in \beta$ such that $x \in B \subset G$ which implies that $B \cap A = \{x\}$. Now let $\beta_x = \{B \in \beta : B \cap A = \{x\}\}$ and consider the family $\beta(A) = \{\beta_x : x \in A\}$, then for each two distinct points $x, y \in A$ we find that

$$B \in \beta_x \Rightarrow B \cap A = \{x\} \Rightarrow y \notin B \Rightarrow B \notin \beta_y \Rightarrow \beta_x \cap \beta_y = \varnothing$$

Then by using Remark 6.6.4, $\beta(A)$ is a countable family and the function $f : A \to \beta(A)$ given by $f(x) = \beta_x$ for each point $x \in A$ is bijective. Hence A is countable.

Another proof: Consider the family β_x constructed in the forgoing proof and let $\beta(A) = \bigcup\{\beta_x : x \in A\}$ then $\beta(A) \subset \beta$ and so is countable and clearly for each $B \in \beta(A)$ there exists $x \in A$ such that $B \in \beta_x$ and $B \cap A = \{x\}$. Then we can define the function $f : \beta(A) \to A$ by $f(B) = x \in A$ for each $B \in \beta(A)$ such that $B \in \beta_x$, clearly this function is surjective and so A is countable

Exercise

1 – Prove that

(i) Any of the right or the left ray topological spaces is regular or normal.

(ii) The space (X,τ) where $\tau = E_{\{p,q\}} \cup C$ is T_1 and neither regular nor normal.

(iii) If (X,τ) is a regular topological space and $x \in X$ is any point show that $y \in X - \{x\} : \overline{\{x\}} \neq \overline{\{y\}} \Rightarrow \overline{\{x\}} \cap \{y\}^{\wedge} = \varnothing$.

(iv) The upper limit point topological space (R,τ) where τ is the topology generated by the basis $\beta = \{(a,b] : a,b \in R\}$ is a normal topological space.

(v) The usual topological space (R,τ) is a normal topological space.

2 – Show that a principal regular topological space (X,τ) is T_o iff $\tau = D$.

3 – Construct all possible regular topologies (they are 13) also construct all possible normal topologies (they are255) on the set $X = \{a,b,c,d\}$, we means that the topology τ on X is such that (X,τ) is regular or normal.

4 – Show that the space (X,τ) is a T_4 – space where τ is the lower limit topology generated on R by the family $\beta = \{[a,b) : a,b \in R\}$.

5 – Consider the topological space (X,τ) where $\tau = E_p \cup C$, prove that

(i) (Y,τ_Y) is a T_4 – space where $Y = X - \{p\}$.

(ii) (Y,τ_Y) is not T_4 – where $Y = X - \{q\}$; p ≠ q and .

6 – If $f : (X,\tau) \rightarrow (X,\tau)$ is continuous and (X,τ) is regular, prove that $A = \{x \in X : x \in \overline{\{f(x)\}}\}$ is open.

7 – If f and g are two continuous functions from the space (X,τ) to the regular space (Y,τ^*), show that the set $A=\{x\in X:\overline{\{f(x)\}}\cap\overline{\{g(x)\}}=\varnothing\}$ is a closed set.

8 – If $X=\{1,2,3,4,5,6,7,8\}$ write the minimal basis β_o of the topologies given by the given bases and show any of which is T_o, regular, normal or not?

(i) $\beta_1=\{X,\{1\},\{1,2\},\{1,2,3,4\},\{5,6\},\{1,5,6\},\{1,2,5,6\}\}$.

(ii) $\beta_2=\{X,\varnothing,\{1\},\{2\},\{1,2,3,4\},\{1,2,3,4,5,6\},$
$\{1,2,8\},\{1,2\},\{1,2,3,4,8\}\}$

(iii) $\beta_3=\{X,\{1,2,3\},\{4\},\{4,5\},\{6\},\{1,1,2,3,7,8\},$
$\{1,2,3,4\},\{4,5,6\},\{4,6\}\}$

(iv) $\beta_4=\{\{1,2,3\},\{4,6,8\},\{5,7\},1,2,3,5,6\},\{4,6,7,8\}\}$.

9 – Consider the set $X=\{1,2,3,4,5,6\}$ and show that

(i) (X,τ_1) is normal and not T_o and (X,τ_2) is a normal T_o – space and not regular where

$\tau_1=\{X,\varnothing,\{1\},\{1,2\},\{1,2,3,4\},\{5,6\},\{1,2,5,6\}\}$ and

$\tau_2=\{X,\varnothing,\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\}\}$.

(ii) (X,τ) is regular where

$\tau=\{X,\varnothing,\{1\},\{2,4\},\{3,5,6\},\{1,2,4\},\{1,3,5,6\},\{2,4,5,6\}\}$ and (i)

write all strictly weaker topologies than (i.e the topologies: τ_{12}, τ_{21}, τ_{15}, τ_{51}, τ_{45}, τ_{54}).

Chapter VII

Compactness and Connectedness

Topics:

- Compactness.
- Compactness and separation axioms:.
- Sequentially compact.
- Countable compact
- Locally compact.
- Connectedness.

7.1. Compactness:

Definition 7.1.1. Let (X,τ) be a topological space and $A\subset X$. Then the family $\Phi=\{G_\alpha:\alpha\in\Delta\}\subset P(X)$ is called

(1) a cover of X if $X=\bigcup_{\alpha\in\Delta} G_\alpha$ or equivalently if for each point $x\in X$ there exists $\alpha\in\Delta$ such that $x\in G_\alpha$.

(2) a cover of A if $A\subset\bigcup_{\alpha\in\Delta} G_\alpha$ or equivalently for each point $x\in A$ there exists $\alpha\in\Delta$ such that $x\in G_\alpha$.

and this cover is called

(a) an open cover if $\Phi=\{G_\alpha:\alpha\in\Delta\}\subset\tau$ i.e. Φ is a family of open sets.

(b) a closed cover if $\Phi=\{G_\alpha:\alpha\in\Delta\}\subset\tau_c$ i.e. Φ is a family of closed sets.

Definition 7.1.2. The topological space (X,τ) is called a compact topological space if each open cover Φ of X contains a finite subcover of X i.e. if for each $\Phi=\{G_\alpha:\alpha\in\Delta\}\subset\tau$ such that $X=\bigcup_{\alpha\in\Delta} G_\alpha$ there is a family $\Phi^*=\{G_{\alpha_1},G_{\alpha_2},...,G_{\alpha_n}\}\subset\Phi$ such that $X=\bigcup_{i=1}^{n} G_{\alpha_i}$ where $n\in N$ i.e. n is a positive integer.

At the following we give some examples of compact and incompact topological spaces:

(a) Each of the following topological spaces is compact:

Example 7.1.1. The topological space (X,τ) if X is finite or τ is finite.

Example 7.1.2. The excluding point topological space (X,E_p) since $G\in\tau$ such that $p\in G$ iff $G=X$. Then if $\Phi\subset\tau$ is an open cover of X then $X\in\Phi$ and so $\{X\}\subset\Phi$ is a finite subcover of X.

Example 7.1.3. The topological space (X,τ) where $\tau=E_p\cup C$ and $p\in X$. For let $\Phi=\{G_\alpha:\alpha\in\Delta\}\subset\tau$ be an open cover of X then there exists $\alpha_o\in\Delta$ such that $p\in G_{\alpha_o}$ and so

$$p\in G_{\alpha_o}\in\Phi\Rightarrow G_{\alpha_o}\in C\Rightarrow G^c_{\alpha_o}=\{x_1,x_2,\ldots,x_n\}\subset X$$

Since Φ is a cover of X then for each point $i\in\{1,2,\ldots,n\}$ there exists $\alpha_i\in\Delta$ such that $x_i\in G_{\alpha_i}$ which implies that $G^c_{\alpha_o}\subset\bigcup_{i=1}^{n}G_{\alpha_i}$ and so

$$X=G_{\alpha_o}\cup G^c_{\alpha_o}=G_{\alpha_o}\cup(\bigcup_{i=1}^{n}G_{\alpha_i})\Rightarrow$$
$$\Rightarrow\Phi^*=\{G_{\alpha_o},G_{\alpha_1},G_{\alpha_2},\ldots,G_{\alpha_n}\}\subset\Phi$$

That is Φ^* is a finite subcover of X. Therefore (X,τ) is a compact topological space.

(b) Each of the following topological spaces is not compact:

Example 7.1.4. The discrete topological space (X,D) if X is infinite since the family $\{\{x\}:x\in X\}\subset D$ is an open cover of X which not contains any finite subcover of X.

Example 7.1.5. The particular point topological space (X,P_p) if X is infinite since the family $\{\{x,p\}:x\in X-\{p\}\}\subset P_p$ is an open cover of X which not contains any finite subcover of X.

Example 7.1.6. The usual topological space $(R,\mho)$ since the family $\{(-n,n): n \in N\} \subset \mho$ is an open cover of R which not contains any finite subcover of R because

Firstly if $x \in R$ then either $x=0 \in (-1,1)$ or

$$x>0 \Rightarrow \exists n_o \in N: n_o > x \Rightarrow -n_o < x < n_o \Rightarrow x \in (-n_o, n_o) \text{ or}$$

$$x<0 \Rightarrow \exists n_o \in N: n_o > -x \Rightarrow -n_o < x < n_o \Rightarrow x \in (-n_o, n_o)$$

Therefore $R = \bigcup\{(-n,n): n \in N\}$.

Secondly consider the family

$$\{(-n_1,n_1),(-n_2,n_2),\ldots,(-n_m,n_m)\} \subset \{(-n,n): n \in N\}$$

Then

$$n_o = \max.(n_1,n_2,\ldots,n_m) \Rightarrow \bigcup_{k=1}^{m} (-n_k,n_k) = (-n_o,n_o) \neq R$$

This implies that there is no any finite subcover of the cover $\{(-n,n): n \in N\} \subset \mho$ of R. Hence $(R,\mho)$ is not compact.

Definition 7.1.3. If (X,τ) is a topological space then a subset A of X is said to be compact if each open cover of A contains a finite subcover of A, we may say that A is compact with respect to τ or in τ and we can easily prove the following theorem.

Theorem 7.1.1. *Let (X,τ) be a topological space and A be a subset of X. Then A is compact in τ iff (A,τ_A) is compact.*

Proof: it is an easy proof and we left it to the reader.

Definition 7.1.4. If $X \neq \varnothing$ then the family $\Psi = \{U_\alpha : \alpha \in \Delta\}$ satisfies or has the finite intersection property if the intersection of the members of any of its finite subfamilies is nonempty.

Example 7.1.7. The family $\Psi=\{[x,\infty):x\in R\}\subset P(R)$ satisfies the finite intersection property since if $\{[x_1,\infty),[x_2,\infty),...,[x_n,\infty)\}\subset\Psi$ and n_o is a positive integer such that $x_{n_o}=\max.\{x_1,x_2,...,x_n\}$ then

$$\bigcap_{i=1}^{n}[x_i,\infty)=[x_{n_o},\infty)\neq\varnothing.$$

Theorem 7.1.2. *The topological space (X,τ) is compact iff each family of closed sets satisfies the finite intersection property has nonempty intersection. Equivalently (X,τ) is compact iff each family of closed sets the intersection of their members is empty contains a finite subfamily the intersection of the members is empty i.e. iff $\Psi\subset\tau_c$ such that $\bigcap\{F:F\in\Psi\}=\varnothing$ implies that there is a finite subfamily $\{F_1,F_2,...,F_n\}\subset\Psi$ Such that $\bigcap_{i=1}^{n}F_i=\varnothing$.*

Proof: Suppose that $\Psi\subset\tau_c$ such that $\bigcap\{F:F\in\Psi\}=\varnothing$ then $\bigcup\{F^c:F\in\Psi\}=X$ which implies that $\Psi_c=\{F^c:F\in\Psi\}$ is an open cover of X. If (X,τ) is compact then there is a finite subfamily $\{F_1{}^c,F_2{}^c,...,F_n{}^c\}\subset\Psi_c$ such that $\bigcup_{i=1}^{n}F_i^c=X$ this implies that $\{F_1,F_2,...,F_n\}\subset\Psi$ and $\bigcap_{i=1}^{n}F_i=\varnothing$ where $n\in N$.

Conversely, suppose that each family of closed sets the intersection of their members is empty contains a finite subfamily the intersection of their members is empty and suppose that $\Phi\subset\tau$ is an open cover of X then $\bigcup\{G:G\in\Phi\}=X$. Then the family $\Phi_c=\{G^c:G\in\Phi\}$ is a family of closed sets and $\bigcap\{G^c:G\in\Phi\}=\varnothing$ which implies from the given condition that there is a finite subfamily $\{G_1{}^c,G_2{}^c,...,G_n{}^c\}$ of the family $\Phi_c=\{G^c:G\in\Phi\}$ such that $\bigcap_{i=1}^{n}G_i{}^c=\varnothing$ from which we gets

$\bigcup_{i=1}^{n} G_i = X$ which implies that $\{G_1, G_2, ..., G_n\} \subset \Phi$ is a finite subcover of X. Therefore (X, τ) is a compact space.

Theorem 7.1.3. *Any closed subset of a compact topological space is compact that is if* (X,τ) *a compact space and* $F \in \tau_c$ *then* F *is compact.*

Proof: Let (X,τ) be a compact space and $F \subset X$ such that $F \in \tau_c$ and $\Phi = \{G_\alpha : \alpha \in \Delta\} \subset \tau$ be an open cover of F. So $\Phi^* = \Phi \cup \{F^c\}$ is an open cover of X. Since (X,τ) is compact then there is a finite subfamily

$\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_n}, F^c\} \subset \Phi^*$ Where n is a positive integer such that $X = (\bigcup_{i=1}^{n} G_{\alpha_i}) \bigcup F^c$ which means that $F \subset \bigcup_{i=1}^{n} G_{\alpha_i}$ Which implies that $\{G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_n}\} \subset \Phi$ is a finite subcover of F. Therefore F is a compact set in τ.

Remark 7.1.1. *According to Theorem 7.1.3, a subset of a compact topological space which is not closed may be not compact an example the topological space* $(I, \mho_I)$ *is compact according to Hein-Borile's Theorem where* $I = [0,1]$ *while the subset* $(0,1)$ *of* I *is not compact for,* $\Phi = \{(\frac{1}{n}, 1) : n \in N\}$ *is an open cover of* $(0,1)$ *and by Archimedes's Theorem*

$$x \in (0,1) \Rightarrow x > 0 \Rightarrow \exists n_o \in N : x > \frac{1}{n_o} \Rightarrow 0 < \frac{1}{n_o} < x < 1$$

$$\Rightarrow x \in (\frac{1}{n_o}, 1) \Rightarrow x \in \bigcup\{(\frac{1}{n}, 1) : n \in N\} \Rightarrow$$

$$\Rightarrow (0,1) \subset \bigcup \{(\frac{1}{n}, 1) : n \in N\}$$

Now if $\{(\frac{1}{n_1},1),(\frac{1}{n_2},1),...,(\frac{1}{n_m},1)\}$ *is a finite subfamily of* Φ *and* $n_o = \max.\{n_1,n_2,...,n_m\}$ *then* $\frac{1}{n_o} \le \frac{1}{n_i}$ *for each* $i \in \{1,2,...,m\}$ *which implies that* $(\frac{1}{n_i},1) \subset (\frac{1}{n_o},1)$ *for each* $i \in \{1,2,...,m\}$. *Hence* $\bigcup_{i=1}^{m} (\frac{1}{n_i},1) \subset (\frac{1}{n_o},1)$. *Again from Archimedes's Theorem*

$$0 < \frac{1}{n_o} \Rightarrow \exists x \in (0,1) : 0 < x < \frac{1}{n_o} \Rightarrow x \notin (\frac{1}{n_o},1) \Rightarrow x \notin \bigcup_{i=1}^{m} (\frac{1}{n_i},1)$$

Therefore there is no any finite subcover from Φ *of* $(0,1)$ *which means that* $(0,1)$ *is not compact with respect to* $\mho_I$. *Clearly* $\Phi \subset \mho$ *and so* $(0,1)$ *is not compact with respect to* $\mho$.

Theorem 7.1.4. *The image of a compact subset* A *of a topological space* (X,τ) *by a continuous function* $f : X \to Y$ *is compact where* (Y,τ^*) *is any topological space.*

Proof: Suppose that (X,τ), (Y,τ^*) are two topological spaces, $f : X \to Y$ is a continuous function and A is a compact subset of X, we want to prove that $f(A) \subset Y$ is compact for let $\Phi = \{U_\alpha : \alpha \in \Delta\} \subset \tau^*$ be an open cover of $f(A)$ then $f(A) \subset \bigcup_{\alpha \in \Delta} U_\alpha$ and so

$$A \subset f^{-1}(f(A)) \subset f^{-1}(\bigcup_{\alpha \in \Delta} U_\alpha) = \bigcup_{\alpha \in \Delta} f^{-1}(U_\alpha)$$

Since f is continuous then $f^{-1}(\Phi) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\} \subset \tau$ is an open cover of A. Since A is compact then there is a subfamily

$$\{f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}),..., f^{-1}(U_{\alpha_n})\} \subset f^{-1}(\Phi)$$

such that $A\subset \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})=f^{-1}(\bigcup_{i=1}^{n} U_{\alpha_i})$ which implies that $f(A)\subset f(f^{-1}(\bigcup_{i=1}^{n} U_{\alpha_i}))\subset \bigcup_{i=1}^{n} U_{\alpha_i}$ which implies that $\{U_{\alpha_1},U_{\alpha_2},\ldots,U_{\alpha_n}\}\subset\Phi$ is a finite subcover of the set $f(A)$. Hence $f(A)$ is a compact subset of Y.

7.2. Compactness and separation axioms:

Theorem 7.2.1. *Let (X,τ) be a T_2-topological space, F be a compact subset of X and $p\in F^c$. Then there are two open sets $U,V\in\tau$ such that $F\subset U$, $p\in V$ and $U\cap V=\varnothing$.*

Proof: If (X,τ) is a T_2-topological space, $F\subset X$ and $p\in F^c$ then

$x\in F\Rightarrow p\neq x\Rightarrow\exists U_x ,V_x\in\tau : x\in U_x ,p\in V_x ,U_x\cap V_x=\varnothing$

and so $\Phi=\{U_x : x\in F\}\subset\tau$ is an open cover of F. If F is compact then there is a finite subcover $\Phi^*=\{U_{x_1},U_{x_2},\ldots,U_{x_n}\}\subset\Phi$ of F which means that $F\subset\bigcup_{i=1}^{n} U_{x_i}$. Let $U=\bigcup_{i=1}^{n} U_{x_i}$ and $V=\bigcap_{i=1}^{n} V_{x_i}$ then $F\subset U$, $p\in V$ and $U\cap V=\varnothing$ where $\{V_{x_1},V_{x_2},\ldots,V_{x_n}\}$ is the family corresponding to the family $\{U_{x_1},U_{x_2},\ldots,U_{x_n}\}$ we means that $x_i\in U_{x_i}$, $p\in V_{x_i}$ and $U_{x_i}\cap V_{x_i}=\varnothing$ for each $i\in\{1,2,\ldots,n\}$. To prove that $U\cap V=\varnothing$, if $U\cap V\neq\varnothing$ then there exists $j\in\{1,2,\ldots,n\}$ such that $U_{x_j}\cap V_{x_j}\neq\varnothing$ but V_{x_j} is the corresponding open set to U_{x_j} i.e. $x_j\in U_{x_j}$, $p\in V_{x_j}$ and $U_{x_j}\cap V_{x_j}=\varnothing$ which implies that $U_{x_j}\cap V_{x_j}\neq\varnothing$ is a contradiction because of the incorrect assumption that $U\cap V\neq\varnothing$. Hence, $U\cap V=\varnothing$ which completes the proof.

Theorem 7.2.2. *Any compact subset of a T_2-space is a closed set.*

Proof: If (X,τ) is a T_2 $-$space and $F \subset X$ is compact then by Theorem 7.2.1,

$$x \in F^c \Rightarrow \exists U, V \in \tau : x \in U, F \subset V, U \cap V = \varnothing$$
$$\Rightarrow x \in U \subset V^c \subset F^c \Rightarrow F^c \in \tau \Rightarrow F \in \tau_c$$

Theorem 7.2.3. *Let (X,τ) be a compact T_2 $-$topological space. Then it is T_4.*

Proof: Let (X,τ) be a compact space, if $F_1, F_2 \in \tau_c$ then by Theorem 7.1.1, F_1, F_2 are compact and if $F_1 \cap F_2 = \varnothing$ and (X,τ) is T_2 then by Theorem 7.2.1,

$$x \in F_1 \Rightarrow x \notin F_2 \Rightarrow \exists U_x, V_x \in \tau : x \in U_x, F_2 \subset V_x,$$
$$U_x \cap V_x = \varnothing$$

and so $\Phi = \{U_x : x \in F_1\} \subset \tau$ is an open cover of F_1 and since F_1 is compact then there is $\Phi^* = \{U_{x_1}, U_{x_2}, \dots, U_{x_n}\} \subset \Phi$ such that $F_1 \subset \bigcup\limits_{i=1}^{n} U_{x_i}$. Now set $U = \bigcup\limits_{i=1}^{n} U_{x_i}$ and $V = \bigcap\limits_{i=1}^{n} V_{x_i}$ then $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \varnothing$ where $\{V_{x_1}, V_{x_2}, \dots, V_{x_n}\}$ is the family which corresponding to the family $\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ which means that $F_2 \subset V_{x_i}$, $x_i \in U_{x_i}$ and $U_{x_i} \cap V_{x_i} = \phi$ for each point $i \in \{1,2,\dots,n\}$. The proof that $U \cap V = \varnothing$ is exactly similar to that which is given in Theorem 7.2.1.

Theorem 7.2.4. *Let (X,τ) be a regular topological space, F be a compact subset of X then*

$$U \in \tau : F \subset U \Rightarrow \exists V \in \tau : F \subset V \subset \overline{V} \subset U.$$

Proof: If (X,τ) is regular then

$$x \in F \Rightarrow x \in U \Rightarrow \exists V_x \in \tau : x \in V_x \subset \overline{V_x} \subset U$$

From which $\{V_x : x \in F\} \subset \tau$ is an open cover of F and since F is compact then there is $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\} \subset \{V_x : x \in F\}$ such that $F \subset \bigcup_{i=1}^{n} V_{x_i} = V$ and so

$$F \subset V \subset \overline{V} = \overline{\bigcup_{i=1}^{n} V_{x_i}} = \bigcup_{i=1}^{n} \overline{V_{x_i}} \subset U \Rightarrow F \subset V \subset \overline{V} \subset U$$

This completes the proof.

Theorem 7.2.5. *If (X,τ) is regular and compact then it is normal.*

Proof: Suppose that $F_1, F_2 \in \tau_c$ such that $F_1 \cap F_2 = \varnothing$ then by Theorem 7.1.2, we finds (1) F_1, F_2 are compact, (2) $F_2 \in \tau_c \Rightarrow F_2^{\,c} \in \tau$ and

(3) $F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subset F_2^{\,c}$ and if (X,τ) is regular then by Theorem 7.2.4, there exists $V \in \tau$ such that $F_1 \subset V \subset \overline{V} \subset F_2^{\,c}$ from which $\overline{V}^c \in \tau$, $F_1 \subset V$, $F_2 \subset \overline{V}^c$ and $V \cap \overline{V}^c = \varnothing$ which implies that (X,τ) is a normal topological space.

Theorem 7.2.6. *If (X,τ) is a T_o −topological space, regular and compact then it is a T_4 −space i.e.*

Regular + Normal + $T_o \to T_4$

Proof: Firstly by Corollary 6.4.5, any regular T_o −space is a T_1 −space. Secondly by Theorem 7.2.5, any regular compact space is normal. Then a regular and compact T_o −topological space is a T_4 −space.

Proposition 7.2.1. *A regular principal topological space (X,τ) is not compact if τ is infinite.*

Proof: If (X,τ) is a principal topological space then $\beta_O=\{U_X : x\in X\}$ is the minimal basis for τ and clearly it is an open cover of X. If (X,τ) is also regular then by Theorem 6.4.8, β_O is a partition of X and so β_O is an infinite open cover of X which not contains any finite subcover of X. Therefore (X,τ) is not compact.

Theorem 7.2.7. *If (X,τ) is a compact space and (Y,τ^*) is a T_2-space then the bijective function $f:X\to Y$ is homeomorphism if f is continuous.*

Proof: we need only to prove that f is a closed function, for

(1) By Theorem 7.1.3, $F\in\tau_C$ implies F is compact.

(2) By Theorem 7.1.4, $f(F)\subset Y$ is compact.

(3) By Theorem 7.1.3, $f(F)\in{\tau^*}_C$.

Therefore F is a closed set and so f is a closed function.

Theorem 7.2.8. *Let (X,τ) be a compact T_2 and (Y,τ^*) be any topological space, if $f:X\to Y$ is a surjective, closed and continuous then (Y,τ^*) is a T_2-space.*

Proof: Suppose that $r,z\in Y$ such that $r\neq z$, since f is surjective then there are two points $x,y\in X$ such that $f(x)=r$ and $f(y)=z$, clearly $x\neq y$. Since (X,τ) is T_2 then $\{x\},\{y\}\in\tau_C$ and $f(x),f(y)\in{\tau^*}_C$ because f is a closed function that is $\{r\},\{z\}\in{\tau^*}_C$ so $f^{-1}(r),f^{-1}(z)\in\tau_C$ because f is continuous and since $r\neq z$ then $f^{-1}(r)\cap f^{-1}(z)=\varnothing$. Since (X,τ) is compact T_2 then by Theorem 7.2.3, it is T_4 and so is normal accordingly there are two open sets $U,V\in\tau$ such that $f^{-1}(r)\subset U$, $f^{-1}(z)\subset V$ and $U\cap V=\varnothing$ and we finds

$$X=U\cup U^C\Rightarrow f(X)=Y=f(U)\cup f(U^C)$$

Since f is closed then $f(U^c) \in \tau^*_c$ which implies that $Y - f(U^c) \in \tau^*$ and so

$$x \in f^{-1}(r) \subset U \Rightarrow x \notin U^c \Rightarrow f(x) = r \notin f(U^c)$$
$$\Rightarrow r \in (Y - f(U^c))$$

Similarly $Y - f(V^c) \in \tau^*$ and $z \in (Y - f(V^c))$ and we gets

$$\begin{aligned}(Y - f(U^c)) \cap (Y - f(V^c)) &= Y \cap [(f(U^c))^c \cap (f(V^c))^c] \\ &= Y \cap [f(U^c) \cup f(V^c)]^c = Y \cap [f(U^c \cup V^c)]^c \\ &= Y \cap [f((U \cap V)^c)]^c = Y \cap [f(\varnothing^c)]^c \\ &= Y \cap [f(X)]^c = Y \cap Y^c = \varnothing\end{aligned}$$

Clearly $f(X) = Y$ because f is surjective. Therefore (Y, τ^*) is a $T_2 -$ space.

Theorem 7.2.9. *Suppose that (X, τ) is a compact $T_2 -$space and $\{x\} \notin \tau$ for each point $x \in X$ then X is uncountable.*

Proof: We are going to prove that there is no any surjective function from N to X, in such case X is uncountable. For let $f : N \to X$ be such that $f(n) = x_n \in X$ for each $n \in N$, if $\{x\} \notin \tau$ for each $x \in X$ then $\{x_1\} \notin \tau$ and if $G \in \tau - \{\varnothing\}$ then there exists $y_1 \in G - \{x_1\}$ and if (X, τ) is T_2 then there are $U_1, W_1 \in \tau$ such that $x_1 \in U_1$. $y_1 \in W$ and $U_1 \cap W_1 = \varnothing$ which implies that $U_1 \cap \overline{W}_1 = \varnothing$ from which $x_1 \notin \overline{W_1}$, set $V_1 = W_1 \cap G$ then $\overline{V_1} \subset \overline{W_1} \cap \overline{G}$ which implies that $x_1 \notin \overline{V_1}$ and $\overline{V_1} \subset \overline{G}$. Once again $\{x_2\} \notin \tau$ implies that there exists $y_2 \in V_1 - \{x_2\}$ and there are $U_2, W_2 \in \tau$ such that $x_2 \in U_2$, $y_2 \in W_2$ and $U_2 \cap W_2 = \varnothing$ and similarly if we set $V_2 = W_2 \cap V_1$ then $x_2 \notin \overline{V_2}$ and $\overline{V_1} \supset \overline{V_2}$. Continue these processes and by using mathematical induction we gets a sequence $\{\overline{V_n} : n \in N\}$ of a closed sets such that $\overline{V_1} \supset \overline{V_2} \supset \ldots\ldots \supset \overline{V_n} \supset \ldots\ldots$ and clearly $\bigcap_{n\in N} \overline{V_n} \neq \phi$ which means that there exists $x \in \bigcap_{n\in N} \overline{V_n}$ and clearly $f(n) = x_n \neq x$ for each $n \in N$ since $x_n \notin \overline{V_n}$ for each $n \in N$ this means that $f(N) \neq X$ from which f is not surjective. Since f is arbitrary then there is no any surjective function from N to X. Therefore X is an uncountable set.

Corollary 7.2.1. *Each real closed interval $I = [a,b]$ is uncountable.*

Proof: By Hein - Borile's Theorem each closed interval is compact, $(R, \mho)$ is a T_2 −space and T_2 is hereditary then $(I, I_{\mho})$ is a T_2 −space. Accordingly the closed interval $[a,b]$ is an uncountable set according to Theorem 7.2.9.

7.3. Sequentially compact:

Definition 7.3.1. A topological space (X,τ) is called Sequentially compact space if each sequence of points of X contains a subsequence converges to a point in X and a subset A of X is called sequentially compact if each sequence of points of A converges to a point in A.

Example 7.3.1. Consider the co-finite topological space (R, C) then any sequence of points of R converges to a point in R in fact to each point of R.

Example 7.3.2. Consider the usual topological space $(R, \mho)$ then the interval $A = (0,1)$ is not sequentially compact, the sequence $< \frac{1}{n} >$ of points of A i.e. the sequence $\{\frac{1}{n} : n \in N\}$ converges to zero and any of its subsequences converges to zero and $0 \notin A$.

Theorem 7.3.1. *Let (X,τ) and (Y,τ^*) be two topological spaces, if $f : X \to Y$ is continuous then the image of a sequentially compact by f is sequentially compact.*

Proof: Suppose that $A \subset X$ is sequentially compact and we want to prove that $f(A) \subset Y$ is sequentially compact. For let $< y_n >$ be a sequence of points of $f(A)$ then for each $n \in N$ there exists $x_n \in X$ such that $y_n = f(x_n)$ and we gets the sequence $< x_n >$ of points of A and since A is sequentially compact then the sequence converges to a point $x_o \in A$ which implies that $f(x_o) \in f(A)$. Since f is continuous then the sequence $< x_n >$ converges to x_o implies that the sequence $< y_n > = < f(x_n) >$ converges to $y_o = f(x_o)$ this means that $f(A)$ converges to a point in $f(A)$ and so it is sequentially compact.

7.4 . Countably compact.

Definition 7.4.1. A topological space (X, τ) is a countably compact space if each infinite subset Y of X has a limit point belongs to X and a subset A of X is countable compact if each infinite subset B of A has a limit point belongs to A.

Example 7.4.1. Consider the usual topological space $(R, \mho)$ then the closed interval $I = [a, b]$ where $a, b \in R$ is countably compact since if $B \subset I$ is an infinite set then it is bounded and from Bolzano-vairchtrass Theorem B has a limit point x and since $B \subset I$ then x is a limit point of I and since I is a closed set then $x \in I$ which implies that $I = [a, b]$ is countably compact.

Example 7.4.2. Consider the usual topological space $(R, \mho)$ then the open interval $A = (0,1)$ is not countably compact since the set $B = \{\frac{1}{n} : n \in N\}$ is an infinite subset of A to which has zero as the unique limit point and $0 \notin A$.

Theorem 7.4.1. *A compact topological space (X, τ) is countably compact.*

Proof: Suppose that Y be a subset of X which has no limit points belong to X then for each point $x \in X$ there exists $G_x \in \tau$ such that $x \in G_x$ and $G_x \cap Y = \varnothing$ or $G_x \cap Y = \{x\}$ i.e. G_x contain at most one point of

Y. Accordingly the family $\Phi=\{G_x : x\in X\}\subset\tau$ is an open cover of X since X is compact then there exists a finite subcover $\{G_{x_1},G_{x_2},\dots,G_{x_n}\}\subset\Phi$ of X such that $Y\subset X=\bigcup_{i=1}^{n}G_{x_i}$ but each G_{x_i} contains at most one point from Y and this means that the cardinal number of Y less than or equals to n i.e. $|Y|\le n$ which means that Y is finite . The contra positive of this result is the infinite subset of X has at least one limit point belongs to X .

Remark 7.4.1. *The inverse of Theorem 7.4.1, may be incorrect since for example the topological space* (Z,τ) *where* Z *is the set of the integers and* τ *is the topology on* Z *generated by the basis*

$$\beta_o=\{\{2n-1,2n\}:n\in Z\}=\{\dots,\{-3,-2\},\{-1,0\},\{1,2\},\dots\}$$

is not compact since β_o *is infinite and is a partition of* Z *while* (Z,τ) *is countably compact since if* $Y\subset Z$ *is an infinite set and if* $n\in Y$ *then either (1)* n *is odd negative or positive in this case* $n+1\in Z$ *is a limit point of* Y *or (2)* n *is even or* $n=0$ *in this case* $n-1\in Z$ *is a limit point of* Y*.*

Theorem 7.4.2. *A sequentially compact space* (X,τ) *is countably compact.*

Proof: Suppose that $Y\subset X$ is an infinite set then there exists an infinite sequence $<x_n>$ of distinct points of Y. If (X,τ) is sequentially compact then there exists a subsequence $<x_{i_n}>$ of $<x_n>$ which converges to a point $x\in X$ this means that there is $G\in\tau$ such that $x\in G$ and there exists $n_o\in N$ such that $x_{i_n}\in G$ whenever $i_n\ge n_o$ this means that $G\cap Y$ is an infinite set which implies that $\left(G-\{x\}\right)\cap Y\ne\varnothing$. Hence x is a limit point of Y belongs to X which implies that (X,τ) is countably compact.

Remark 7.4.2. *The inverse of Theorem 7.4.2, need not be valid. For consider the example given in Remark 7.4.1, where* (Z,τ) *is countably compact while it is not sequentially compact since the sequence* $<-n>$

i.e. $\{-1,-2,\ldots,-n,\ldots\}$ *is a sequence of points of* Z *which does not contains any convergent sub sequences.*

7.5. Locally compact:

Definition 7.5.1. A topological space (X,τ) is called locally compact if there exists a compact neighborhood about x for each point $x \in X$.

Remark 7.5.1. *The discrete topological space* (X,D) *is a locally compact space,* $\{x\} \in D$ *is a compact neighborhood about* x *for each point* $x \in X$. *If* X *is infinite then* (X,D) *is not compact and so a locally compact space may be not compact.*

Proposition 7.5.1. *Any compact topological space* (X,τ) *is locally compact since* X *is a compact neighborhood for each* $x \in X$.

Theorem 7.5.1. *A property that* (X,τ) *is a locally compact space is hereditary i.e.*

Proof: Suppose that $Y \subset X$ and $y \in Y$ then $y \in X$ and since (X,τ) is locally compact there is a compact neighborhood $F \subset X$ of the point y and if $M = F \cap Y$ then (M, τ_{FM}) is compact because F is compact. But $\tau_{FM} = \tau_{M}$ implies (M, τ_{M}) is compact and so M compact in Y. Now since F is a compact neighborhood of the point y in X then there exists $G \in \tau$ such that $y \in G \subset F$ from which $y \in G \cap Y \subset F \cap Y = M$ which implies that M is a compact neighborhood of the arbitrary point $y \in Y$. Therefore (Y, τ_{Y}) is a locally compact space.

Theorem 7.5.2. *If* (X,τ) *is a locally compact regular or* T_2*−space then the family of compact and closed neighborhoods of each point* $x \in X$ *forms a local basis for* x.

Proof: If (X,τ) is a locally compact and $x \in X$ then there exists a compact neighborhood $F \subset X$ about the point x and if U is another neighborhood of x then $M=F\cap U$ is also a neighborhood of x then there exists $G\in\tau$ such that $y\in G\subset M$ and so

(1) If (X,τ) is regular then there exists $V\in\tau$ such that $x\in V\subset\overline{V}\subset G\subset M$ which implies that $\overline{V}$ is a closed neighborhood of x and since $\overline{V}\subset M\subset F$ and F is compact then $\overline{V}$ is compact.

(2) If (X,τ) is a T_2-space since $F\subset X$ is compact then F is closed and so (F,τ_F) is T_2 and compact and so is T_4 and so is T_3 which implies that it is regular and the proof is as in (1).

From (1) and (2) the family of the closed neighborhoods of a point $x\in X$ forms a local basis for the point x.

Corollary 7.5.1. *Any locally compact T_2-space is T_3.*

7.6. Connectedness:

Definition 7.6.1. If (X,τ) is a topological space and $U,V\in\tau$ then the pair (U,V) is said to be a pair of disconnection of X if

(1) $U\cap V=\varnothing$ and

(2) $U\cup V=X$.

It well be written in the form $U|V$ and we say that U and V are a pair of disconnection.

Definition 7.6.2. If there exists for the topological space X a pair of disconnection $U|V$ such that $\varnothing\notin\{U,V\}$ then we say that the topological space (X,τ) is a disconnected space briefly X is disconnected other wise if the unique pair of disconnection of X is $X|\varnothing$, it is called a connected topological space that is if it is not disconnected then it is connected.

Remark 7.6.1. *From the definition $U|V$ is the pair of disconnection of X iff $U,V\in\tau\cap\tau_c$ i.e. U and V are co-open sets. Accordingly (X,τ) is* *connected iff* X *and* $\varnothing$ *are the only co-open sets, in this case* $G \in \tau - \{X, \varnothing\} \Rightarrow G^c \notin \tau$. *Accordingly* (X, τ) *is connected iff for each two open sets* $U, V \in \tau$,

(1) $U \cap V = \varnothing \Rightarrow U \cup V \neq X$ *or*

(2) $U \cup V = X \Rightarrow U \cap V \neq \varnothing$

and it should be remarked that if $U \cap V \neq \varnothing$ *for each* $U, V \in \tau - \{X, \varnothing\}$ *or* $U \cup V \neq X$ *for each* $U, V \in \tau - \{X, \varnothing\}$ *then* (X, τ) *is connected.*

Example 7.6.1. The following topological spaces are connected:

(1) The excluding point topological space (X, E_p) since

$$U, V \in E_p - \{X, \varnothing\} \Rightarrow U \cup V \neq X .$$

(2) The particular topological space (X, P_p) since

$$U, V \in P_p - \{X, \varnothing\} \Rightarrow U \cap V \neq \varnothing.$$

(3) The co-finite topological (X, C) if X is infinite since

$$U, V \in C - \{X, \varnothing\} \Rightarrow U \cap V \neq \varnothing$$

(4) The usual topological space $(R, \mho)$ since the unique co-open sets are R and $\varnothing$.

Example 7.6.2. The following topological spaces are disconnected:

(1) The discrete topological space (X, D) since $A, A^c \in \tau$ for each subset A of X.

(2) The principal regular topological space (X, τ) if $\tau \neq I$ since in this case, $\tau = \tau_c$.

Example 7.6.3. Let $X = \{a, b, c, d, e\}$ and

$$\tau = \{X, \varnothing \{a\}, \{a,b\}, \{a,c\}, \{d,e\}, \{a,d,e\}, \{a,b,c\} \{a,b,d,e\},$$
$$\{a,c,d,e\}\},$$

$$\tau^* = \{X, \varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,c,d\}, \{a,b,c\}, \{a,b,c,d\}\}.$$

Then we finds (a) (X,τ) is disconnected because $\{a,b,c\} | \{d,e\}$ is a disconnection pair of X and $\{a,b,c\}, \{d,e\} \in \tau \cap \tau_C$.

(b) (X,τ^*) is connected because

$$U, V \in \tau - \{X, \varnothing\} \Rightarrow U \cup V \neq X, U \cap V \neq \varnothing.$$

Since $\tau \subset E_e \cap P_a$.

Definition 7.6.3. Let (X,τ) be a topological space and A be a subset of X. Then A is connected if there are two open sets $G, H \in \tau$ such that $G \cap A | H \cap A$ is a disconnection of the topological space (A, τ_A) equivalently if G and H satisfies the conditions:

(a) $\varnothing \notin \{G \cap A, H \cap A\}$,

(b) $(G \cap A) \cap (H \cap A) = \varnothing$ Equivalently $G \cap H \subset A^c$ and

(c) $(G \cap A) \cup (H \cap A) = A$ Equivalently $A \subset G \cup H$.

If A is not disconnected then it is called connected.

Example 7.6.4. Let $X = \{a,b,c,d,e\}$ and

$$\tau = \{X, \varnothing, \{a,b,c\}, \{c\}, \{c,d,e\}\}$$

Then $A = \{a,d,e\}$ is a disconnected set since $G = \{a,b,c\}$ and $H = \{c,d,e\}$ are two open sets which satisfy the conditions of Definition 7.6.3, the space (X,τ) is connected and the set $A = \{b,c,e\}$ is connected.

Theorem 7.6.1. *If (X,τ) is a topological space and A is a subset of X then A is connected with respect to τ iff it is connected with respect to τ_A.*

Proof: If A is disconnected with respect to τ i.e. $\tau-$disconnected then there are two open sets $G,H\in\tau$ satisfy the conditions (a), (b) and (c) of definition 7.6.3. If $U=G\cap A$ and $V=H\cap A$ then $U,V\in\tau_A$, $\varnothing\notin\{U,V\}$, $U\cap V=\varnothing$ and $U\cup V=A$ that is (A,τ_A) is a disconnected topological space i.e. A is disconnected with respect to τ_A equivalently is τ_A-disconnected.

Conversely Suppose that (A,τ_A) is disconnected i.e. A is τ_A-disconnected then there are $U,V\in\tau_A$ such that $U|V$ is a pair of disconnection of A and we finds

$$U,V\in\tau_A\Rightarrow\exists G,H\in\tau:U=G\cap A,V=H\cap A$$

And one can easily show that G and H satisfy the conditions of Definition 7.6.3 from which A is $\tau-$disconnected. Therefore the contra positive of the result gives us the required aim of the theorem.

Theorem 7.6.2. *If (X,τ) is a topological space then the following statements are equivalent*

(1) (X,τ) is connected,

(2) Each continuous function $f:(X,\tau)\to(\{Y,D)$ is constant i.e. $f(X)$ is a singleton subset of Y where Y is non singleton and D is the discrete topology on Y.

(3) $b(A)\neq\varnothing$ for each $A\subset X$ such that $A\in P(X)-\{X,\varnothing\}$.

(4) The unique cl-open sets are X and $\varnothing$.

(5) There is no any continuous and surjective function $f:(X,\tau)\to(\{a,b\},D)$ where a and b are two distinct points and D is the discrete topology on $\{a,b\}$.

Proof: We shall prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2): Suppose that Y is non singleton and the function $f:(X,\tau)\to(\{Y,D)$ is continuous then we finds if $f(X)$ is non singleton Then there exists a point $y\in f(X)$ such that $f(X)-\{y\}\neq\varnothing$. Hence,

(a) $f^{-1}(\{y\})\neq\varnothing$ since $y\in f(X)$ and $f^{-1}(f(X)-\{y\})\neq\varnothing$ since $f(X)-\{y\}\neq\varnothing$,

(b) $f^{-1}(\{y\}), f^{-1}(f(X)-\{y\})\in\tau$ since f is continuous,

(c) $f^{-1}(\{y\})\cap f^{-1}(f(X)-\{y\})=f^{-1}((f(X)-\{y\})\cap\{y\})$

$$=f^{-1}(\varnothing)=\varnothing \text{ and}$$

(d) $f^{-1}(\{y\})\cup f^{-1}(f(X)-\{y\})=f^{-1}((f(X)-\{y\})\cup\{y\})$

$$=f^{-1}(f(X))=X.$$

Therefore $f^{-1}(\{y\})\Big| f^{-1}(f(X)-\{y\})$ is a disconnection of X which means that (X,τ) is disconnected. Then the result is

if $f:(X,\tau)\to(\{Y,D)$ is continuous and not constant then (X,τ) is disconnected and the contra positive of this result is if

(X,τ) is connected then each continuous function $f:(X,\tau)\to(\{Y,D)$ is constant.

(2) $\Rightarrow$ (3): Suppose that $A\in P(X)-\{X,\varnothing\}$ such that $b(A)=\varnothing$ then $\overline{A}\cap\overline{A^c}=\varnothing$ from which $A, A^c\in\tau$ and if Y is non singleton then there are two distinct points $a,b\in Y$. Now define the function $f:(X,\tau)\to(\{Y,D)$ by

$$f(x)=\begin{cases} a; & x\in A \\ b; & x\in A^c \end{cases}$$

We find that if $V \in D-\{\varnothing\}$ then either (i) $a \in V$ and $b \notin V$ or (ii) $b \in V$ and $a \notin V$ or (iii) $\{a,b\} \subset V$ or (iv) $\{a,b\} \cap V=\varnothing$. Therefore (i) $f^{-1}(V)=A$ or (ii) $f^{-1}(V)=A^{c}$ or

(iii) $f^{-1}(V)=X$ or (iv) $f^{-1}(V)=\varnothing$ which implies that f is continuous and clearly is not constant. Then the final result is if there is a set $A \in P(X)-\{X,\varnothing\}$ such that $b(A)=\varnothing$ then there is a continuous function $f:(X,\tau)\to(\{Y,D)$ which is not constant and the contra positive of this result is (3) i.e. if the continuous function $f:(X,\tau)\to(\{Y,D)$ is constant then $b(A)\neq\varnothing$ for each $A \in P(X)-\{X,\varnothing\}$.

(3) $\Longrightarrow$ (4): If $A \in P(X)-\{X,\varnothing\}$ such that $A\in\tau\cap\tau_{c}$ i.e. A is a cl-open set i.e. $A, A^{c}\in\tau_{c}$ then $\overline{A}=A$ and $\overline{A^{c}}=A^{c}$ which implies that $b(A)=\overline{A}\cap\overline{A^{c}}=\varnothing$ and the contra positive of this result is $b(A)\neq\varnothing$ for each $A \in P(X)-\{X,\varnothing\}$ means that the unique cl-open sets are X and $\varnothing$.

(4) $\Longrightarrow$ (5): Suppose that $f:(X,\tau)\to(\{a,b\},D)$ is a surjective and continuous function

(a) $\phi\notin\{f^{-1}(\{a\}),f^{-1}(\{b\})\}$ since f is surjective,

(b) $f^{-1}(\{a\}),f^{-1}(\{b\})\in\tau$ since f is continuous,

(c) $X=f^{-1}(\{a\})\cup f^{-1}(\{b\})$ and

(d) $f^{-1}(\{a\})\cap f^{-1}(\{b\})=f^{-1}(\{a\}\cap\{b\})=f^{-1}(\phi)=\phi$. Hence

$f^{-1}(\{a\})\in\tau\cap\tau_{c}-\{X,\varnothing\}$ is a cl-open set in fact $f^{-1}(\{b\})$ is also. The final result is if there is a continuous and surjective function $f:(X,\tau)\to(\{a,b\},D)$ then there exists a cl-open set $A \in P(X)-\{X,\varnothing\}$ here $A=f^{-1}(\{a\})$ and $A^{c}=f^{-1}(\{b\})$ and the contra positive of this result is (5) i.e. if the unique cl-open sets are X and $\varnothing$ then the function $f:(X,\tau)\to(\{a,b\},D)$ is either not continuous or not surjective.

(5) $\Rightarrow$ (1): Suppose that (X,τ) is a disconnected space then there are two open sets U and V such that $U|V$ is a non trivial separation of X i.e. $U,V \notin \{X,\varnothing\}$. Now consider the function $f:(X,\tau)\to(\{a,b\},D)$ given by

$$f(x)=\begin{cases} a; & x\in U \\ b; & x\in V \end{cases}$$

Clearly f is surjective and continuous. Hence (X,τ) is disconnected implies that there is continuous and surjective function $f:(X,\tau)\to(\{a,b\},D)$, the contra positive of this result is if there is no surjective and continuous functions $f:(X,\tau)\to(\{a,b\},D)$ then (X,τ) is connected.

Example 7.6.5. If $X=\{1,2,3\}$, $Y=\{a,b\}$, $\tau=\{X,\varnothing,\{1\},\{2,3\}\}$ and $f=\{(1,a),(2,b),(3,b)\}$ then (X,τ) is disconnected and the function $f:(X,\tau)\to(\{Y,D)$ is continuous, not constant and surjective we remarks also that $\{2,3\}$ is a cl-open set and $b\left(\{1\}\right)=\varnothing$.

Proposition 7.6.1. *A topological space (X,τ) is connected if for each two distinct points $x,y\in X$ (1) $\{x\}^\wedge \cap \{y\}^\wedge \neq \varnothing$ or*

(2) $\overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$ but not conversely.

Proof: Suppose that (X,τ) is disconnected then there exists a cl-open proper subset A of X i.e. $A\in P(X)-\{X,\varnothing\}$ such that $A\in\tau\cap\tau_C$ from which $A^C\in\tau\cap\tau_C$ and there are two points $x\in A$ and $y\in A^C$. Hence

(1) $\{x\}^\wedge \subset A$ and $\{y\}^\wedge \subset A^C$ since $A,A^C\in\tau$ which implies that $\{x\}^\wedge \cap \{y\}^\wedge = \varnothing$.

(2) $\overline{\{x\}} \subset A$ and $\overline{\{y\}} \subset A^C$ since $A, A^C \in \tau_C$ which implies that $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$.

The contra positive in both (1) and (2) gives us the required result.

To prove that the inverse of this result may be not valid, the co-finite space (X, C) is connected where X is infinite while not only for specific two points but for any two distinct points $x, y \in X$, $\{x\}^{\wedge} \cap \{y\}^{\wedge} = \varnothing$ and $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$.

Theorem 7.6.3. *A topological space* (X, τ) *is connected if* $b^{\wedge}(A) = A^{\wedge} \cap A^{c\wedge} \neq \varnothing$ *for each set* $A \in P(X) - \{X, \varnothing\}$ *. If* (X, τ) *is a principal space then the reverse direction is also true.*

Proof: Suppose that (X, τ) is disconnected then there is a set $A \in P(X) - \{X, \varnothing\}$ such that $A, A^C \in \tau$ which implies that $A^{\wedge} = A$ and $A^{C\wedge} = A^C$ which implies that $b^{\wedge}(A) = A^{\wedge} \cap A^{c\wedge} = A \cap A^c = \varnothing$ and the contra positive is $b^{\wedge}(A) = A^{\wedge} \cap A^{c\wedge} \neq \varnothing$ for each set $A \in P(X) - \{X, \varnothing\}$ implies that (X, τ) is connected.

Conversely let $A \in P(X) - \{X, \varnothing\}$ be such that $A^{\wedge} \cap A^{c\wedge} = \varnothing$ then remembering that $A^C \subset A^{C\wedge}$ we finds

$$\begin{aligned} x \in A^{\wedge} \Rightarrow x \notin A^{C\wedge} \Rightarrow x \notin A^C \Rightarrow x \in A \Rightarrow A^{\wedge} \subset A \\ \Rightarrow A^{\wedge} = A \end{aligned}$$

Similarly one can prove that $A^{C\wedge} = A^C$. If (X, τ) is a principal space then $A, A^C \in \tau$ and therefore it is disconnected, the contra positive of this result is $b^{\wedge}(A) = A^{\wedge} \cap A^{c\wedge} \neq \varnothing$ for each set $A \in P(X) - \{X, \varnothing\}$ implies that (X, τ) is connected.

Theorem 7.6.4. *A principal topological space* (X,τ) *is connected if the intersection of any two members of the family* T *is nonempty but not conversely where* $T=\{U_A\in\tau : A\subset X,\ A\cap U_x=\varnothing;\forall x\in A^c\}$.

Proof: Suppose that (X,τ) is disconnected then there exists a set $G\in P(X)-\{X,\varnothing\}$ such that $G,G^c\in\tau$ which implies that there are two points $x,y\in X$ such that $x\in G$ and $y\in G^c$, since $\bigcup\{U_A : U_A\in T\}=X$ then there are $U_A,U_B\in T$ such that $x\in U_A$ and $y\in U_B$ from which $A\subset G$ since $A\cap G^c\neq\varnothing$ implies that $U_A\subset G^c$ from which $x\in G^c$ which contradicts that $x\in G$ and so $U_A\subset G$. Similarly we can prove that $U_B\subset G^c$. Therefore $U_A\cap U_B=\varnothing$ and the contra positive of this result is $U_A\cap U_B\neq\varnothing$ for each $U_A,U_B\in T$ implies that (X,τ) is connected.

To show that the inverse is incorrect, consider the following example:

If $X=\{1,2,3,4,5,6\}$ and $\beta_O=\{\{1,2\},\{1,2,3\},\{4\},\{4,5\},\{1,2,4,6\}\}$ is the minimal basis for a topology τ on X then (X,τ) is not connected while $U_{\{3\}}\cap U_{\{5\}}=\{1,2,3\}\cap\{4,5\}=\varnothing$ where $T=\{\{1,2,3\},\{4,5\},\{1,2,4,6\}\}$.

Proposition 7.6.2. *If* (X,τ) *is a principal connected topological space then for each* $U_A\in T$ *there exists* $U_B\in T$ *such that* $U_A\cap U_B\neq\varnothing$ *but not conversely.*

Proof: Suppose that (X,τ) is a principal space and $U_A\in T$ such that $U_A\cap U_B=\varnothing$ for each $U_B\in T-\{U_A\}$ then $U_A, X-U_A\in\tau$ where $X-U_A=\bigcup\{U_B : U_B\in T-\{U_A\}\}$ which implies that (X,τ) is not connected and the contra positive of this result is the required aim. To show that the inverse need not be valid let $X=\{1,2,3,4,5,6\}$ and the minimal basis $\beta_O=\{\{1\},\{1,2\},\{1,3\},\{4\},\{4,5\},\{4,6\}\}$ for a topology τ on X then $T=\{\{1,2\},\{1,3\},\{4,5\},\{4,6\}\}$ from which we remarks that for each $U_A \in T$ there exists $U_B \in T$ such that $U_A \cap U_B \neq \varnothing$ while (X,τ) is not connected.

Theorem 7.6.5. *If (X,τ) is a topological space and $A \subset X$ is connected then $A \subset B \subset \overline{A}$ implies that B is connected.*

Proof: Suppose that $U,V \in \tau_B$ are such that $U|V$ is a pair of disconnection of B then $U \cap A, V \cap A \in (\tau_A)_B = \tau_A$ and $U \cap A|V \cap A$ is a disconnection of A and since A is connected with respect to τ then by Theorem 7.6.1, either $U \cap A = \varnothing$ or $V \cap A = \varnothing$ and we finds

$$V \cap A = \phi \Rightarrow U \cap A = A \Rightarrow A \subset U \Rightarrow \overline{A} \subset \overline{U}$$

$$\Rightarrow A \subset B \subset \overline{A} \subset \overline{U} \Rightarrow B \cap \overline{U} = \overline{U}_B = B$$

where $B \cap \overline{U} = \overline{U}B$ but $U|V$ is a disconnection of B from which $U \in \tau_{Bc}$ and so $\overline{U}B = U$ and so $U = B$ which implies that $V = \varnothing$, Similarly if $U \cap A = \varnothing$ then $V = B$ and $U = \varnothing$. Therefore B is connected.

Theorem 7.6.6. *The family of connected sets the intersection of the members is nonempty their union is connected.*

Proof: If (X,τ) is a topological space and $\{A_\alpha : \alpha \in \Delta\} \subset P(X)$ is a family of subsets of X such that $\bigcap_{\alpha \in \Delta} A_\alpha \neq \varnothing$ then there is a point $p \in \bigcap_{\alpha \in \Delta} A_\alpha$. If $Y = \bigcup_{\alpha \in \Delta} A_\alpha$ and (Y,τ_Y) is not connected then there are $U,V \in \tau_Y$ such that $U|V$ is a disconnection of Y and so p belongs to U or V if $p \in U$ then there is a point $x \in V$ because $\varnothing \notin \{U,V\}$ and since $U \cap V = \varnothing$ then there is $\alpha_o \in \Delta$ such that $x \in A_{\alpha_o}$ and $p \in A_\alpha$ for each $\alpha \in \Delta$ and so (i) $p \in U, p \in A_{\alpha_o} \Rightarrow p \in U \cap A_{\alpha_o}$.

(ii) $x \in V, x \in A_{\alpha_o} \Rightarrow x \in V \cap A_{\alpha_o}$.

Which implies that $U \cap A_{\alpha_o} \Big| V \cap A_{\alpha_o}$ is a non trivial disconnection of A_{α_o} which contradicts that A_{α_o} is not connected. The contra positive is $\{A_\alpha : \alpha \in \Delta\} \subset P(X)$ a family of connected sets imply that (Y, τ_Y) is connected.

Definition 7.6.4. If (X, τ) is a topological space then the points $x, y \in X$ are said to be connected if there is a connected subset M of X such that $x, y \in M$.

Theorem 7.6.7. *If in a topological space* (X, τ)*, each two points* $x, y \in X$ *are connected then* X *is connected.*

Proof: Suppose that each two points in the space (X, τ) are connected and let $U, V \in \tau$ be such that $U|V$ is non trivial disconnection of X then there are two points $x, y \in X$ such that $x \in U$ and $y \in V$. Since x and y are connected then there is a connected set $M \subset X$ such that $x, y \in M$ and so $x \in U \Rightarrow x \in U \cap M$, $y \in V \Rightarrow y \in V \cap M$ and $U \cap M, V \cap M \in \tau_M$. Hence $U \cap M | V \cap M$ is a non trivial disconnection of M which contradicts that M is connected. This contradiction because of the incorrect assumption that X is disconnected, the correction is X is connected.

Theorem 7.6.8. *Let* (X, τ) *be a topological space and* S *be a relation on* X *"* $(x, y) \in S$ *for each two points* $x, y \in X$ *iff* x *and* y *are connected" then* S *is an equivalence relation.*

Proof: Clearly for each $x \in X$, $\{x\}$ is connected and $x \in \{x\}$ which implies that $(x, x) \in S$ and so S is reflexive. If $x, y \in X$ such that $(x, y) \in S$ then there exists a connected set $M \subset X$ such that $x, y \in M$ which means that $(y, x) \in S$. Hence S is symmetric.

Thirdly let $x, y, z \in X$ be such that $(x, y) \in S$ and $(y, z) \in S$. Then there are two connected sets $M_1, M_2 \subset X$ such that $x, y \in M_1$ and $y, z \in M_2$ from which $y \in M_1 \cap M_2$ and so by Theorem 7.6.6, $M_1 \cup M_2$ is connected and clearly $x,z \in M_1 \cup M_2$ which implies that $(x,z) \in S$ and so S is a transitive relation. Therefore, S is an equivalence relation.

Definition 7.6.5. If (X,τ) is a topological space and S is the equivalence relation on X defined by Theorem 7.6.8, then the equivalence class $[x]$ or M_x of a point $x \in X$ is called a component of x and is given by

$$M_x = [x] = \{y \in X : (x,y) \in S\}.$$

Theorem 7.6.9. *If* (X,τ) *is a topological space and* $x \in X$ *then* M_x *is the union of all connected subsets of* X *which contains* x *i.e.* $M_x = \bigcup\{A \subset X :$ *A is connected and* $x \in A\}$, M_x *is connected since it is a union of connected sets with nonempty intersection.*

Proof: It is a direct consequence of Theorem 7.6.6.

Remark 7.6.2. *As a consequence of Theorem 7.6.9,* M_x *is the greatest connected set which contains* x *i.e if* $A \subset X$ *is a connected set and* $x \in A$ *then* $A \subset M_x$*.*

Corollary 7.6.1. *In any topological space* (X,τ), M_x *is a closed set i.e* $M_x \in \tau_c$ *for each point* $x \in X$ *.*

Proof: $\overline{M_x}$ is connected by Theorem 7.6.5, and by Theorem 7.6.9, $\overline{M_x} \subset M_x$ but $M_x \subset \overline{M_x}$ implies that $\overline{M_x} = M_x$. Hence M_x is a closed set.

Theorem 7.6.10. *If* (X,τ) *is a topological space then the family* $\{M_x : x \in X\}$ *is a partition of* X *.*

Proof: Clearly $\bigcup\{M_x : x \in X\} = X$ and if $x,y \in X$ such that $M_x \cap M_y \neq \varnothing$. Then by Theorem 7.6.6, $M_x \cup M_y$ is connected and

$$\begin{aligned} z \in M_x \cap M_y &\Rightarrow (x,z),(z,y) \in S \\ &\Rightarrow (x,y) \in S \Rightarrow x,y \in M_x \cup M_y \end{aligned}$$

Which implies by Theorem 7.6.9, that $M_x = M_x \cup M_y = M_y$. The contra positive of this result is $M_x \neq M_y$ implies that $M_x \cap M_y = \phi$. Hence $\{M_x : x \in X\}$ is a partition of X.

Theorem 7.6.11. *If $f:(X,\tau)\to(Y,\tau^*)$ is continuous then the image by f of a connected subset A of X is connected.*

Proof: Let A be a subset of X and $f(A)$ be disconnected. Then by the Definition 7.6.3, there is a separation $U\cap A \big| V\cap A$ of $f(A)$ where $U, V \in \tau^*$ and

(a) $\varnothing \notin \{U \cap f(A), V \cap f(A)\}$,

(b) $U \cap V \subset Y - f(A)$ and

(c) $f(A) \subset U \cup V$.

Hence $f^{-1}(U), f^{-1}(V) \in \tau$ since f is continuous and

(1) From (a)

$$\varnothing \notin \{U \cap f(A), V \cap f(A)\}$$
$$\Rightarrow \varnothing \notin \{f^{-1}(U) \cap A, f^{-1}(V) \cap A\}$$

Since $y \in U \cap f(A) \Rightarrow y \in U \wedge y \in f(A)$ and there exists $x \in X$ such that $y = f(x)$ but $y = f(x) \in U \Rightarrow x \in f^{-1}(U)$ and $y = f(x) \in f(A) \Rightarrow x \in A$ which implies that $x \in f^{-1}(U) \cap A \neq \varnothing$ which implies that $f^{-1}(U) \cap A \neq \varnothing$ Similarly $f^{-1}(V) \cap A \neq \varnothing$.

(2) From (b) we get

$$f^{-1}(U \cap V) \subset f^{-1}(Y - f(A)) = X - f^{-1}(f(A))$$
$$\Rightarrow f^{-1}(U) \cap f^{-1}(V) \subset X - A$$

Since $A \subset f^{-1}(f(A)) \Rightarrow X - f^{-1}(f(A)) \subset X - A$.

(3) From (c)

$$A \subset f^{-1}(f(A)) \subset f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$
$$\Rightarrow A \subset f^{-1}(U) \cup f^{-1}(V)$$

Therefore $f^{-1}(U) \cap A | f^{-1}(V) \cap A$ is a separation of A which contradicts that A is connected, this contradiction because of the incorrect assumption that $f(A)$ is disconnected and so $f(A)$ is connected.

Another proof: Let A be a subset of X, and $f(A)$ be disconnected. Then by Theorem 7.6.2 (5), there is a surjective and continuous function $h:(f(A), \tau^*_{f(A)}) \rightarrow (\{a,b\}, D)$ where a and b are any two points, D is the discrete topology on $\{a,b\}$ and the function $f:(X,\tau) \rightarrow (Y,\tau^*)$ is a continuous function. The function $f/A:(A,\tau_A) \rightarrow (Y,\tau^*)$ is a continuous function and hence $g:(A,\tau_A) \rightarrow (f(A), \tau^*_{f(A)})$ given by $g(x) = f/A(x)$ for each point $x \in A$ is continuous and surjective. Therefore $f \odot g:(A,\tau_A) \rightarrow (\{a,b\}, D)$ is a continuous and surjective function which implies that A is disconnected. This contradiction means that $f(A)$ is connected.

At the following theorem we generalize the mean value theorem for the continuous real functions which is valid for the function defined on a connected topological space into the usual topological space.

Theorem 7.6.12. *(Mean value Theorem) Let* $f:(X,\tau) \rightarrow (R,\mho)$ *be a continuous function,* (X,τ) *be connected,* $a,b \in f(X)$ *such that* $a<b$ *and* $c \in R$ *such that* $a<c<b$*. Then there is a point* $x_o \in X$ *such that* $f(x_o) = c$.

Proof: Suppose that such point $x_o \in X$ which satisfies $f(x_o) = c$ does not exist i.e. $f(x) \neq c$ for each $x \in X$, then $U = (-\infty, c)$ and $V = (c, \infty)$ are two open subsets of R i.e. $U, V \in \mho$, $a \in U$ and $b \in V$. Since f is continuous then

(1) $f^{-1}(U), f^{-1}(V) \in \tau$, (2) $\varnothing \notin \{f^{-1}(U), f^{-1}(V)\}$,

(3) $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varnothing) = \varnothing$ and

(4) $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(R - \{c\}) = X$.

Hence $f^{-1}(U) \big| f^{-1}(V)$ is a separation for the space X but this contradicts that X is connected, this contradiction because of the incorrect assumption that the point $x_o \in X$ which satisfies $f(x_o) = c$ does not exist and hence it must be exists.

Remark 7.6.3. *Theorem 7.6.12, help us to obtain the roots of the equation* $f(x) = 0$ *where* $f : R \to R$ *and if* $f(x_1) < 0$ *and* $f(x_2) > 0$ *Theorem 7.6.12, says that there is a point* $x_o \in R$ *such that* $f(x_o) = 0$, x_o *is called the root of the equation* $f(x) = 0$. *Theorem 7.6.12, also leads to an interesting result about the real numbers that is for each continuous function* $f : [a, b] \to [a, b]$ *there is a point* $x_o \in [a, b]$ *such that* $f(x_o) = x_o$. *Generally if* $f : X \to X$ *with the property that there is a point* $x_o \in X$ *such that* $f(x_o) = x_o$ *such space* X *is said to be satisfies the fixed point property.*

Theorem 7.6.13. *The closed interval* $[a, b]$ *has the fixed point property.*

Proof: If $a = b$ then the theorem is valid then let $a \neq b$ and the function $f : [a, b] \to [a, b]$ be continuous and $f(x) \neq x$ for each $x \in [a, b]$. Consider the function $g : [a, b] \to R$ given by $g(x) = \dfrac{1}{x - f(x)}$ for each $x \in [a, b]$ this function is well defined and is continuous and clearly $g(a) < 0$ since $f(a) \in [a, b]$ and $f(a) \neq a$ that is $a < f(a) \leq b$ which implies that $a-f(a)<0$ i.e. $g(a)=\dfrac{1}{a-f(a)}<0$. Similarly one can show that $g(b)>0$. Hence by Theorem 7.6.12, there is a point $t\in[a,b]$ such that $g(t)=0$ i.e. $g(t)=\dfrac{1}{t-f(t)}=0$ which is impossible since $f(x)\neq x$ for each $x\in[a,b]$. Then the correction is there is a point $x_o\in[a,b]$ such that $f(x_o)=x_o$. Therefore the closed interval has the fixed point property.

Theorem 7.6.14. *In the usual topological space* $(R,\mho)$*, the subset* E *of* R *is connected iff it is an interval.*

Proof: If $E\subset R$ is not an interval then there are the points $a,b\in E$ and $p\notin E$ such that $a<p<b$. Then $G=(-\infty,p)$ and $H=(p,\infty)$ are open and $E\cap G|E\cap H$ is a separation of E which implies that E is disconnected. The contra positive is if E is connected then it is an interval.

Conversely let E be an interval then it contains at least two points and if it is not connected then there are two open sets $G,H\in u$ such that $E\cap G|E\cap H$ is a separation of E. Set $A=E\cap G$ and $B=E\cap H$ then there are two points $a\in A$ and $b\in B$, suppose that $a<b$ and $p=\inf .B\cap[a,b]$ then $a$is a lower bound of $B\cap[a,b]$and $b\in B\cap[a,b]$ implies that $a\leq p\leq b$ implies that $p\in[a,b]$ and since E is an interval then $p\in E$and we have the following two cases:

Case (1): $p\in B\Rightarrow p\in H$and $a\in A$, $A\cap B=\varnothing$ implies that $a\neq p$ implies that $a<p\leq b$. Hence $p\in H\in u$implies that there are $c,d\in R$ such that $p\in(c,d)\subset H$. Now $\phi\neq(a,p)\cap(c,p)\subset H$implies by Archimedes's low that there is a real number $x_o\in R$ such that $a<x_o<p$ and $c<x_o<p$ since either $c<a<p<d$ or $a\leq c<p<d$. Then

$$a<x_o<p\Rightarrow a<x_o<p\leq b\Rightarrow x_o\in[a,b]\Rightarrow x_o\in E--(I)\text{ and}$$
$$c<x_o<p\Rightarrow c<x_o<p<d\Rightarrow x_o\in H--(II)$$

From (I) and (II), $x_o \in E \cap H = B \Rightarrow x_o \in B \cap [a,b]$. Therefore $x_o < p$ contradicts that p is a lower bound of $B \cap [a,b]$.

Case (2): $p \in A \Rightarrow p \in E$, $p \in G$ and $p \in A$, $A \cap B = \varnothing$ implies that $b \neq p$ implies that $a \leq p < b$ which implies that $p \in [a,b]$. Hence $p \in G \in u$ implies that there are $c,d \in R$ such that $p \in (c,d) \subset G$. Now $\varnothing \neq (p,d) \cap (p,b) \subset G$ implies by Archimedes's axiom that there is a real number $y_o \in R$ such that $p < y_o < b < d$ and $p < y_o < d < b$ since either $c < p < b \leq d$ or $c < p < d \leq b$ and the intersection is either

(p,b) in this case $y_o \in (p,b)$ or (p,d) and $y_o \in (p,d)$. Hence $p < y_o < b \Rightarrow y_o \in [a,b] \subset E$ and

$$p < y_o < d \Rightarrow y_o \in G \Rightarrow y_o \in E \cap G = A$$
$$\Rightarrow y_o \in A \cap [a,b]$$

Then y_o is a lower bound of $B \cap [a,b]$ and so $y_o > p$ contradicts that $p = \inf . B \cap [a,b]$.

From case (1) $p \notin B$ and from case (2) $p \notin A$ which implies that $p \notin E$ and this contradicts that $p \in E$ this contradiction because of the incorrect assumption that E is disconnected and therefore E is connected.

Theorem 7.6.15. *If* $f : (R, \mho) \to (R, \mho)$ *is an injective and continuous then it is an open function.*

Proof: If $G \in \mho$ then there is a pair wise disjoint family $\{(a_\alpha, b_\alpha) \subset R : \alpha \in \Delta\}$ such that $G = \bigcup_{\alpha \in \Delta} (a_\alpha, b_\alpha)$. So enough to prove that $f : (R, \mho) \to (R, \mho)$ is an open function is the proof of $f((a,b)) \in u$ for each $a,b \in R$. By Theorem 7.6.11, $f((a,b))$ is connected since f is continuous and so it is interval by Theorem 7.6.14. Since f is injective then $f^{-1}(f(a,b)) = (a,b)$ from which $f((a,b))$ can not be closed since f is continuous. Clearly $f(a), f(b) \notin f((a,b))$ and we have two cases:

Case (1): $f(a) < f(x)$ for each $x \in (a,b)$ in this case if there is a point $y \in R - f(a,b)$ such that $f(a) < y < f(x)$ for each point $x \in (a,b)$ then $f(a) \in (f(a)-1, y)$ and $(f(a)-1, y) \cap f(a,b) = \varnothing$ which implies that $f^{-1}((f(a)-1, y)) \cap (a,b) = \varnothing$, $a \in f^{-1}(f(a)-1, y))$ and $f^{-1}(f(a)-1, y)) \in u$ which implies that $a \notin (a,b)'$ and this is a contradiction which means that such point $y \in R - f(a,b)$ does not exists. In this case $f(b) > f(x)$ for each $x \in (a,b)$ since other $f(b) < f(a) < f(x)$ for each $x \in (a,b)$ which implies that $b \notin (a,b)'$ a contradiction and in a similar way we can prove that there is no any point $y \in R - f(a,b)$ such that $f(b) > y > f(x)$ for each $x \in (a,b)$. Therefore $f((a,b)) = (f(a), f(b))$.

Case (2): $f(b) < f(x)$ for each $x \in (a,b)$ and $f(b) > f(x)$ for each $x \in (a,b)$ and similar to the way which used in case (1) one can show that $f((a,b)) = (f(b), f(a))$.

Exercise

(1) Prove that:

1 – $E, F \subset X$ are such that $F \in \tau_c$ and E is compact then $E \cap F$ is compact.

2 – The union of a finite number of compact sets is compact.

3 – The compactness is a hereditary property.

4 – The sequentially compactness is an invariant topological property.

5 – A closed subset of a sequentially compact space is sequentially compact.

6 – A locally compact space is an invariant topological property.

7 – If (X,τ) is a locally compact space and $\tau^* \subset \tau$ then (X,τ^*) is locally compact.

8 – A T_1 –topological space is locally compact iff each open countable cover contains a finite subcover.

9 – If (X,τ) is T_1 and has the second axiom of countability then it is compact iff it is locally compact.

10 – A closed subset A of a locally compact space is locally compact.

11 – If X is an infinite set and $\tau = E_p \cup C$ is the topology defined on X and if τ^* a topology is on X such that $\tau \subset \tau^*$ then (X,τ^*) is not compact.

(12) Consider the usual topological space (R,u) and prove that Q and Q^c are connected while Z is not connected.

(13) If (X,τ) is a not connected topological space and τ^* is a topology on X $\tau \subset \tau^*$, prove that (X,τ^*) is disconnected.

(14) Prove that the indiscrete topological space (X,I) is connected.

(15) Prove that (R,τ) is connected where τ is the write rays topology.

(16) Show by an example that the connectedness is not hereditary.

(17) Show that if τ and τ^* are two topologies on a nonempty set X such that $\tau \subset \tau^*$, if $A \subset X$ is $\tau-$connected then it is τ^*- connected.

(18) If (X,τ) is a topological space and $A,E \subset X$ such that E is connected, $A \cap E \neq \phi$ and $A \cap E^{c} \neq \varnothing$ show that $A \cap b(E) \neq \varnothing$.

(19) If (X,τ) is a topological space and $\{A, A_{\alpha} : \alpha \in \Delta\} \subset P(X)$ is a family of connected sets such that $A \cap A_{\alpha} \neq \varnothing$ for each $\alpha \in \Delta$, prove that $A \cup (\bigcup_{\alpha \in \Delta} A_{\alpha})$ is connected.

(20) Determine the component of

(a) The discrete space (X,D).

(b) The cofinite space (X,C) if X is infinite.

(21) If the family of the components of a topological space (X,τ) is finite, prove that each component is a clopen set.

(22) If (X,τ) is a compact space and the family of the components is a family of open sets, prove that it is finite.

Chapter VIII

Product Space and metric spaces

Topics:

- The Product spaces
- Metric Spaces
- The metric spaces and separation axioms

8.1. The Product Spaces:

Definition 8.1.1. Let (X,τ) and (Y,τ^*) be two topological spaces and $\beta=\{U\times V: U\in\tau, V\in\tau^*\}\subset X\times Y$. Then

(1) $X\in\tau$ and $Y\in\tau^*$ implies that β satisfies condition (a_1).

(2) $U_1,U_2\in\tau$ and $V_1,V_2\in\tau^*$ imply that $U_1\cap U_2\in\tau$ and $V_1\cap V_2\in\tau^*$ which implies that

$(U_1\times V_1)\cap(U_2\times V_2)=(U_1\cap U_2)\times(V_1\cap V_2)\in\beta$ which means that β satisfies condition (b_2).

From (1) and (2) β forms a a basis for a topology on $X\times Y$ denoted $\tau\times\tau^*$ and is called the Cartesian product topology on $X\times Y$. The ordered pair $(X\times Y,\tau\times\tau^*)$ is called the Cartesian product topological space and we may write and say that $X\times Y$ is a Cartesian product topological space sees the figure.

Example 8.1.1. If $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\},\{b,c\}\}$, $Y=\{1,2,3\}$ and $\tau^*=\{Y,\varnothing,\{1\}\}$ then the family β is a basis for the topology $\tau\times\tau^*$ where

$$\beta=\{X\times Y, X\times\varnothing, X\times\{1\},\{a\}\times Y,\{a\}\times\varnothing,\{a\}\times\{1\},$$
$$\{b,c\}\times Y,\{b,c\}\times\varnothing,\{b,c\}\times\{1\}\}\Rightarrow$$

$$\beta = \{X \times Y, \varnothing, \{(a,1),(b,1),(c,1)\}, \{(a,1),(a,2),(a,3)\},$$
$$\{(a,1)\}, \{(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\},$$
$$\{(b,1),(c,1)\}\}$$

The reader can write the elements of the topology $\tau \times \tau^*$.

Theorem 8.1.1. *Let* (X,τ) *and* (Y,τ^*) *be two topological spaces,* β *and* β^* *be the bases of* τ *and* τ^* *respectively. Then the family* $\beta \times \beta^* = \{A \times B : A \in \beta, B \in \beta^*\}$ *is a basis for* $\tau \times \tau^*$*.*

Proof: Clearly

$$A \in \beta, B \in \beta^* \Rightarrow A \in \tau, B \in \tau^* \Rightarrow A \times B \in \tau \times \tau^*$$
$$\Rightarrow \beta \times \beta^* \subset \tau \times \tau^*$$

Secondly $H \in \tau \times \tau^*$ and $(x,y) \in H$ implies that there are $U \in \tau$ and $V \in \tau^*$ such that $(x,y) \in U \times V \subset H$. But $(x,y) \in U \times V \subset H$ and $x \in U$ implies that there is $A \in \beta$ such that $x \in A \subset U$ and $y \in V$ implies that there is $B \in \beta$ such that $y \in B \subset V$ from which $(x,y) \in A \times B \subset U \times V \subset H$. Hence for each $H \in \tau \times \tau^*$ and each $(x,y) \in H$ there exists $W \in \beta \times \beta^*$ such that $(x,y) \in W \subset H$ where $W = A \times B$. Therefore $\beta \times \beta^*$ is a basis for $\tau \times \tau^*$.

Example 8.1.2. Consider the usual topological space $(R, \mho)$ where $\beta = \{(a,b) : a,b \in R\}$ is a basis for $\mho$, the family $\beta^2 = \{(a,b) \times (c,d) : a,b,c,d \in R\}$ is a basis for the topology $\mho \times \mho$ on R^2 denoted $\mho^2$. Clearly β^2 is the family of all open rectangles with sides parallel to the co-ordinates axes i.e. $\mho^2$ is the usual topology on R^2.

Definition 8.1.2. At follows two interesting continuous functions from $(X \times Y, \tau \times \tau^*)$ to (X,τ) and (Y,τ^*) called projections functions:

(1) $\pi_1 : X \times Y \to X$ defined by $\pi_1(x,y) = x$ for each point $(x,y) \in X \times Y$ which is called the first projection function.

(2) $\pi_2 : X\times Y \to Y$ defined by $\pi_2(x,y)=y$ for each point $(x,y)\in X\times Y$ which is called the second projection function.

Remark 8.1.1. *(1) Clearly* π_1 *and* π_2 *are surjective since* $\pi_1(X\times Y)=X$ *and* $\pi_2(X\times Y)=Y$ *and are not injective since if* $y_1, y_2 \in Y$ *such that* $y_1 \neq y_2$ *then* $\pi_1(x,y_2)=\pi_1(x,y_1)=x$ *while* $(x,y_1)\neq(x,y_2)$ *where* $x\in X$ *is any point which means that* π_1 *is not injective and similarly* π_2 *is not injective.*

(2) π_1 *and* π_2 *are continuous where*

$(i)\ U\in\tau \Rightarrow \pi_1^{-1}(U)=U\times Y\in\tau\times\tau^*$ *and*

$(ii)\ V\in\tau^* \Rightarrow \pi_2^{-1}(V)=X\times V\in\tau\times\tau^*$

(3) π_1 *and* π_2 *are open where*

$$(i)\ \pi_1\Big(\bigcup_{\alpha\in\Delta,\lambda\in\Lambda}(U_\alpha\times V_\lambda)\Big)=\Big(\bigcup_{\alpha\in\Delta,\lambda\in\Lambda}\pi_1(U_\alpha\times V_\lambda)\Big)=\bigcup_{\alpha\in\Delta}U_\alpha\in\tau$$

and

$$(ii)\ \pi_2\Big(\bigcup_{\alpha\in\Delta,\lambda\in\Lambda}(U_\alpha\times V_\lambda)\Big)=\Big(\bigcup_{\alpha\in\Delta,\lambda\in\Lambda}\pi_2(U_\alpha\times V_\lambda)\Big)=\bigcup_{\lambda\in\Lambda}V_\lambda\in\tau$$

Theorem 8.1.2. *If* (X,τ) *and* (Y,τ^*) *are two topological spaces then the family* $\sigma=\{\pi_1^{-1}(U),\pi_2^{-1}(V):U\in\tau,V\in\tau^*\}$ *is a sub- basis for the topology* $\tau\times\tau^*$ *on* $X\times Y$*.*

Proof: From Remark 8.1.1, we find that $(i)\ U\in\tau\Rightarrow\pi_1^{-1}(U)\in\tau\times\tau^*$ and $(ii)\ V\in\tau^*\Rightarrow\pi_2^{-1}(V)\in\tau\times\tau^*$

Which implies that $\sigma\subset\tau\times\tau^*$ and hence $\beta(\sigma)\subset\tau\times\tau^*\ (I)$ where $\beta(\sigma)$ is the family of all finite intersections of the elements of σ. But $\beta=\{U\times V:U\in\tau,V\in\tau^*\}$ is a basis for $\tau\times\tau^*$ which implies that

$$\begin{aligned} U\in\tau, V\in\tau^* \Rightarrow U\times V &= (U\cap X)\times(Y\cap V)\\ &= (U\times Y)\cap(X\times V) = \pi_1^{-1}(U)\cap\pi_2^{-1}(V)\\ &\Rightarrow U\times V\in\beta(\sigma)\Rightarrow \beta\subset\beta(\sigma) \qquad (II) \end{aligned}$$

From (I) and (II), $\beta(\sigma)$ is a basis for $\tau\times\tau^*$ and so σ is a subbasis for $\tau\times\tau^*$.

Theorem 8.1.3. *If* (X,τ_1), (Y,τ_2) *and* (Z,τ_3) *are topological spaces then the function* $f:Z\to X\times Y$ *is continuous iff the functions* $\pi_1\circ f:Z\to X$ *and* $\pi_2\circ f:Z\to Y$ *are continuous.*

Proof: If $\pi_1\circ f$ and $\pi_2\circ f$ are continuous then

(i) $f^{-1}(\pi_1^{-1}(U) = (\pi_1\circ f)^{-1}(U)\in\tau_3$ for each $U\in\tau_1$ and

(ii) $f^{-1}(\pi_2^{-1}(V) = (\pi_2\circ f)^{-1}(V)\in\tau_3$ for each $V\in\tau_2$.

This means that the inverse of any element of the subbasis σ of $\tau_1\times\tau_2$ by the function f see Theorem 8.1.2, is an element of τ_3 and so f is continuous.

Conversely if f is continuous and since π_1 and π_2 are continuous then $\pi_1\circ f$ and $\pi_2\circ f$ are continuous.

Theorem 8.1.4. *If* (X,τ_1), (Y,τ_2) *and* (Z,τ_3) *are topological spaces and* $f_1:Z\to X$ *and* $f_2:Z\to Y$ *then* $f_1\times f_2:Z\to X\times Y$ *given by* $(f_1\times f_2)(x) = (f_1(x), f_2(x))$ *for each point* $x\in Z$ *is continuous iff* f_1 *and* f_2 *are continuous.*

Proof: If $f_1\times f_2$ is continuous then since π_1 and π_2 are continuous so $\pi_1\circ(f_1\times f_2)=f_1$ and $\pi_2\circ(f_1\times f_2)=f_2$ are continuous where

$(\pi_i\circ(f_1\times f_2))(x)=\pi_i(f_1(x),f_2(x))=f_i(x)$ for each $x\in X$ and each $i\in\{1,2\}$.

Conversely the family $\sigma=\{\pi_1^{-1}(U),\pi_2^{-1}(V):U\in\tau,V\in\tau^*\}$ is a subbasis for the topology $\tau_1\times\tau_2$ on $X\times Y$ and if f_1 and f_2 are continuous and $S\in\sigma$ then either there exists $U\in\tau_1$ such that $S=\pi_1^{-1}(U)$ or $V\in\tau_2$ such that $S=\pi_2^{-1}(V)$. Then we find

$$(f_1\times f_2)^{-1}(S)=(f_1\times f_2)^{-1}(\pi_1^{-1}(U)$$
$$=(\pi_1\circ(f_1\times f_2))^{-1}(U)=f_1^{-1}(U)$$

or

$$(f_1\times f_2)^{-1}(S)=(f_1\times f_2)^{-1}(\pi_2^{-1}(V)$$
$$=(\pi_1\circ(f_1\times f_2))^{-1}(V)=f_2^{-1}(V)$$

In both cases $(f_1\times f_2)^{-1}(S)\in\tau_1\times\tau_2$ which means that $f_1\times f_2$ is a continuous function.

Theorem 8.1.5. *If (X,τ) and (Y,τ^*) are two topological spaces, $A\subset X$ and $B\subset Y$ then*

(1) $\overline{A\times B}=\overline{A}\times\overline{B}$,

(2) $(A\times B)^{\wedge}=A^{\wedge}\times B^{\wedge}$

(3) $(A\times B)^{o}=A^{o}\times B^{o}$,

(4) $(A\times B)'=(A'\times\overline{B})\cup(\overline{A}\times B')$,

(5) $b(A\times B)=(b(A)\times\overline{B})\cup(\overline{A}\times b(B))$,

(6) $isd(A\times B)=isd(A)\times isd(B)$ *and*

(7) $(\tau\times\tau^*)_{A\times B}=\tau_A\times\tau^*_B$.

Proof: (1) $(x,y)\in\overline{A\times B}$ iff $H\cap A\times B\neq\emptyset$ for each $H\in\tau\times\tau^*$ such that $(x,y)\in H$ iff $U\times V\cap A\times B=(U\cap A)\times(V\cap B)\neq\emptyset$ for each $U\in\tau$ such that $x\in U$ and each $V\in\tau^*$ such that $y\in V$ iff $(U\cap A)\neq\emptyset$ and $(V\cap B)\neq\emptyset$ for each $U\in\tau$ such that $x\in U$ and each for $V\in\tau^*$ such that $y\in V$ iff $x\in\overline{A}$ and $y\in\overline{B}$ iff $(x,y)\in\overline{A}\times\overline{B}$ iff $\overline{A\times B}=\overline{A}\times\overline{B}$.

(2) (a) $(x,y)\notin(A\times B)^{\wedge}$ implies that there exists $H\in\tau\times\tau^*$ such that $A\times B\subset H$ and $(x,y)\notin H$ which implies that there are $U\in\tau$ and $V\in\tau^*$ such that $A\times B\subset U\times V$ and $(x,y)\notin U\times V$ implies that either $x\notin U$ or $y\notin V$ which implies that $x\notin A^{\wedge}$ or $y\notin B^{\wedge}$ which implies that $(x,y)\notin A^{\wedge}\times B^{\wedge}$ which implies that $(A\times B)^{\wedge}\subset A^{\wedge}\times B^{\wedge}$.

(b) Conversely

$$\begin{aligned}
(x,y)\notin A^{\wedge}\times B^{\wedge} &\Rightarrow x\notin A^{\wedge}\vee y\notin B^{\wedge}\\
&\Rightarrow \exists U\in\tau : A\subset U,\, x\notin U \vee \exists V\in\tau^*: B\subset V,\, y\notin V\\
&\Rightarrow (x,y)\notin U\times Y,\ A\times B\subset U\times Y\\
&\qquad\qquad \vee (x,y)\notin X\times V,\ A\times B\subset X\times V\\
&\Rightarrow (x,y)\notin(A\times B)^{\wedge}
\end{aligned}$$

which implies that $A^{\wedge}\times B^{\wedge}\subset(A\times B)^{\wedge}$. From (a) and (b)

$$(A\times B)^{\wedge}=A^{\wedge}\times B^{\wedge}.$$

(3) $(x,y)\in(A\times B)^{o}$ iff there are $U\in\tau$ and $V\in\tau^*$ such that $(x,y)\in(U\times V)\subset A\times B$ iff $x\in U\subset A$ and $y\in V\subset B$ iff $x\in A^{o}$ and $y\in B^{o}$ iff $(x,y)\in A^{o}\times B^{o}$ Therefore $(A\times B)^{o}=A^{o}\times B^{o}$.

(4) The proof of (4) depends on (1) of this Theorem also it depends on the properties of both the Cartesian product of the sets and the closure

of the sets and we find that $(x,y)\notin(A\times B)'$ implies that there is an open set $H\in\tau\times\tau^*$ such that $(x,y)\in H$ and $H-\{(x,y)\}\cap A\times B=\varnothing$ accordingly there are $U\in\tau$ and $V\in\tau^*$ such that $(x,y)\in U\times V\subset H$ and $(U\times V)-\{(x,y)\}\cap A\times B=\varnothing$. But

$$\begin{aligned}(U\times V)-\{(x,y)\}&=(U\times V)-\{x\}\times\{y\}\\&=[(U-\{x\})\times V]\bigcup[U\times(V-\{y\})]\end{aligned}$$

Therefore

$$\begin{aligned}&(U\times V)-\{(x,y)\}\cap A\times B\\&=[(U-\{x\})\cap A)\times(V\cap B)]\bigcup[(A\cap U)\times(V-\{y\}\cap B)]\\&=[((U-\{x\})\times V)\bigcup(U\times(V-\{y\}))]\bigcap(A\times B)=\varnothing\end{aligned}$$

iff (a) $(U-\{x\})\cap A)\times(V\cap B)=\varnothing$ iff $(U-\{x\})\cap A=\varnothing$ or

$V\cap B=\varnothing$ iff $x\notin A'$ or $y\notin\overline{B}$ iff $(x,y)\notin A'\times\overline{B}$ and

(b) $(A\cap U)\times(V-\{y\}\cap B)=\varnothing$ iff $A\cap U=\varnothing$ or $V-\{y\}\cap B=\varnothing$ iff $x\notin\overline{A}$ or $y\notin B'$ iff $(x,y)\notin\overline{A}\times B'$.

From (a) and (b), $(x,y)\notin(A'\times\overline{B})\bigcup(\overline{A}\times B')$.

Hence,

$$(A'\times\overline{B})\bigcup(\overline{A}\times B')\subset(A\times B)'\quad(I)$$

Conversely

$$\begin{aligned}&(x,y)\notin(A'\times\overline{B})\bigcup(\overline{A}\times B')\\&\qquad\Rightarrow(x,y)\notin A'\times\overline{B}\ \wedge\ (x,y)\notin\overline{A}\times B'\\&\qquad\Rightarrow(x\notin A'\vee y\notin\overline{B})\wedge(x\notin\overline{A}\vee y\notin B')\end{aligned}$$

From which we have the following cases::

Case (1): $x\notin A'$ and $y\notin B'$ which implies that there are

$U \in \tau$ and $V \in \tau^*$ such that $x \in U$ and $y \in V$ such that $(U-\{x\})\cap A=\varnothing$ and $V-\{y\}\cap B=\varnothing$ which implies that

$$\begin{aligned}
&[(U\times V)-\{(x,y)\}]\bigcap(A\times B)=\\
&=[((U-\{x\})\times V)\bigcup(U\times(V-\{y\}))]\bigcap(A\times B)\\
&=[(U-\{x\}\cap A)\times(V\cap B)]\bigcup[(U\cap A)\times(V-\{y\}\cap B))]\\
&=\varnothing\cup\varnothing=\varnothing
\end{aligned}$$

Case (2): $y\notin\overline{B}$ which implies that there is an open set $V\in\tau^*$ such that $y\in V$ and $V\cap B=\varnothing$ which implies that

$$\begin{aligned}
&[(X\times V)-\{(x,y)\}]\bigcap(A\times B)=\\
&=[((X-\{x\})\times V)\bigcup(X\times(V-\{y\}))]\bigcap(A\times B)\\
&=[A-\{x\}\times(V\cap B)]\bigcup[A\times(V-\{y\}\cap B)]\\
&=\varnothing\cup\varnothing=\varnothing
\end{aligned}$$

Case (3): $x\notin\overline{A}$ which implies that there is an open set $U\in\tau$ such that $x\in U$ and $U\cap A=\varnothing$ which implies that

$$\begin{aligned}
&[(U\times Y)-\{(x,y)\}]\bigcap(A\times B)=\\
&=[((U-\{x\})\times Y)\bigcup(U\times(Y-\{y\}))]\bigcap(A\times B)\\
&=[(U-\{x\}\cap A)\times B]\bigcup[(U\cap A)\times B-\{y\}]\\
&=\varnothing\cup\varnothing=\varnothing
\end{aligned}$$

Hence case (1), case (2) and case (3) implies that

$(x,y)\notin(A'\times\overline{B})\bigcup(\overline{A}\times B')\Rightarrow(x,y)\notin(A\times B)'$ which implies that

$$(A\times B)'\subset(A'\times\overline{B})\bigcup(\overline{A}\times B')\quad(II)$$

From (I) and (II)

$$(A\times B)'=(A'\times\overline{B})\bigcup(\overline{A}\times B').$$

Another way for the proof:

$$(x,y)\in(A\times B)'\Leftrightarrow(x,y)\in\overline{A\times B-\{(x,y)\}}$$
$$=\overline{(A-\{x\})\times B\bigcup A\times(B-\{y\})}$$
$$=\overline{(A-\{x\})\times B}\bigcup\overline{A\times B-\{y\}}$$

Iff

$(x,y)\in\overline{(A-\{x\})\times B}=\overline{A-\{x\}}\times\overline{B}\Leftrightarrow x\in\overline{A-\{x\}},y\in\overline{B}\quad(I)$ or

$\Leftrightarrow x\in A',y\in\overline{B}\Leftrightarrow(x,y)\in A'\times\overline{B}$

$(x,y)\in\overline{A\times B-\{y\}}=\overline{A}\times\overline{B-\{y\}}\Leftrightarrow x\in\overline{A},y\in\overline{B-\{y\}}\quad(II)$ iff

$\Leftrightarrow x\in\overline{A},y\in B'\Leftrightarrow(x,y)\in\overline{A}\times B'$

$(x,y)\in(A'\times\overline{B})\bigcup(\overline{A}\times B')$ iff

$$(A\times B)'=(A'\times\overline{B})\cup(\overline{A}\times B').$$

(5)

$$b(A\times B)=\overline{A\times B}\bigcap\overline{(A\times B)^c}$$
$$=(\overline{A}\times\overline{B})\bigcap[\overline{(A^c\times Y)}\bigcup\overline{X\times B^c}]$$
$$=(\overline{A}\times\overline{B})\bigcap[(\overline{A^c}\times Y)\bigcup(X\times\overline{B^c})]\Rightarrow$$

$$b(A\times B)=[(\overline{A}\times\overline{B})\bigcap(\overline{A^c}\times Y)]\bigcup[(\overline{A}\times\overline{B})\bigcap(X\times\overline{B^c})]$$
$$=[(\overline{A}\bigcap\overline{A^c})\times\overline{B}]\bigcup[\overline{A}\times(\overline{B}\bigcap\overline{B^c})]$$
$$=(b(A)\times\overline{B})\bigcup(\overline{A}\times b(B))$$

(6) Suppose that $(x,y)\in isd(A\times B)$ then there exist an open set $H\in\tau\times\tau^*$ such that $H\cap(A\times B)=\{(x,y)\}$. But

$$
\begin{aligned}
(x,y)\in H &\Rightarrow \exists U\in\tau, V\in\tau^{*}:(x,y)\in U\times V\subset H\\
&\Rightarrow (U\times V)\cap(A\times B)=\{(x,y)\}\\
&\Rightarrow (U\cap A)\times(V\cap B)=\{(x,y)\}\\
&\Rightarrow (U\cap A)=\{x\},\ (V\cap B)=\{y\}\\
&\Rightarrow x\in isd(A),\ y\in isd(B)\\
&\Rightarrow (x,y)\in isd(A)\times isd(B)\\
&\Rightarrow isd(A\times B)\subset isd(A)\times isd(B)----(I)
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
(x,y)\in isd(A)\times isd(B) &\Rightarrow x\in isd(A),\ y\in isd(B)\\
&\Rightarrow \exists U\in\tau, V\in\tau^{*}:(U\cap A)=\{x\},\in(V\cap B)=\{y\}\\
&\Rightarrow (U\times V)\cap(A\times B)=(U\cap A)\times(V\cap B)=\{(x,y)\}\\
&\Rightarrow (x,y)\in isd(A\times B)\\
&\Rightarrow isd(A)\times isd(B)\subset isd(A)\times isd(B)----(II)
\end{aligned}
$$

From (I) and (II) we gets

$$isd(A\times B)=isd(A)\times isd(B)$$

Remark 8.1.2. *$A\times B$ is an isolated subset of $X\times Y$ iff A is an isolated subset of X and B is an isolated subset of Y i.e.*

$isd(A\times B)=A\times B$ iff $isd(A)=A$ and $isd(B)=B$. For, if

$isd(A)=A$, $isd(B)=B$, and $y\in B$ is any point then $x\in A$ implies that

$$
\begin{aligned}
(x,y)\in A\times B &\Rightarrow x\in A\wedge y\in B\\
&\Rightarrow x\in isd(A)\wedge y\in isd(B)\\
&\Rightarrow (x,y)\in isd(A)\times isd(B)\\
&\Rightarrow (x,y)\in isd(A\times B)\Rightarrow isd(A\times B)=A\times B
\end{aligned}
$$

Conversely, if $isd(A\times B)=A\times B$ and $y\in B$ is any point then

$$\begin{aligned} x \in A &\Rightarrow (x,y) \in A \times B \Rightarrow (x,y) \in isd(A \times B) \\ &\Rightarrow (x,y) \in isd(A) \times isd(B) \Rightarrow x \in isd(A) \\ &\Rightarrow A \subset isd(A) \Rightarrow isd(A) = A \end{aligned}$$

Similarly one can prove that $isd(B) = B$.

(7) Since $\beta = \{U \times V : U \in \tau, V \in \tau^*\}$ is a basis for the topology $\tau \times \tau^*$ and the family

$$\begin{aligned} \beta_{A \times B} &= \{(U \times V) \cap (A \times B) : U \in \tau, V \in \tau^*\} \\ &= \{(U \cap A) \times (V \cap B) : U \in \tau, V \in \tau^*\} \end{aligned}$$

is a basis for the topology $(\tau \times \tau^*)_{A \times B}$. Also the family $\beta_A = \{(U \cap A) : U \in \tau\}$ is a basis for τ_A and the family $\beta_B = \{(V \cap B) : V \in \tau^*\}$ is a basis for τ_B from which $\beta_A \times \beta_B == \{(U \cap A) \times (V \cap B) : U \in \tau, V \in \tau^*\}$ is a basis for the topology $\tau_A \times \tau^*_B$ and clearly $\beta_A \times \beta_B = \beta_{A \times B}$ according to which $\tau_A \times \tau^*_B = (\tau \times \tau^*)_{A \times B}$.

Theorem 8.1.6. *If* (X, τ) *and* (Y, τ^*) *are two topological spaces and* $(a,b) \in X \times Y$ *then the following two topologies are homeomorphic in both cases.*

(a) $(X \times \{b\}, (\tau \times \tau^*)_{X \times \{b\}})$ *and* (X, τ).

(b) $(\{a\} \times Y, (\tau \times \tau^*)_{\{a\} \times Y})$ *and* (Y, τ^*).

Proof: Since $\pi_1 : X \times \{b\} \to X$ and $\pi_2 : \{a\} \times Y \to Y$ are surjective, continuous and open functions and to show that they are injective if $(x_1, b), (x_2, b) \in X \times \{b\}$ then

$$\pi_1(x_1, b) = \pi_1(x_2, b) \Rightarrow x_1 = x_2 \Rightarrow (x_1, b) = (x_2, b)$$

Which means that π_1 is injective, similarly π_2 is injective. Hence π_1 and π_2 are homeomorphisms.

Remark 8.1.3. *The functions* $\pi_1^{-1}:X \to X \times \{b\}$ *and* $\pi_2^{-1}:Y \to \{a\}\times Y$ *are continuous since if* $H \in \tau_{\{a\}} \times \tau^*$ *then there exists* $V \in \tau^*$ *such that* $H = \{a\} \times V$ *and then* $(\pi_2^{-1})^{-1}(H) = V \in \tau^*$ *which means that* π_2^{-1} *is continuous and Similarly* π_1^{-1} *is also where* $\tau_{\{a\}} \times \tau^* = (\tau \times \tau^*)_{\{a\}\times Y}$.

Theorem 8.1.7. *If* (X,τ) *and* (Y,τ^*) *are two topological spaces,* $f:X \to Y$ *is a continuous function and* (Y,τ^*) *is a* T_2-*space then* $f \subset X \times Y$ *is a closed set.*

Proof: Suppose that $(x,y) \in X \times Y - f$ then $f(x) \neq y$ and since (Y,τ^*) is a T_2-space, there are two open sets $W, V \in \tau^*$ such that $f(x) \in V$, $y \in W$ and $V \cap W = \varnothing$. Since f is continuous then $f^{-1}(V) \in \tau$ which implies that $(f^{-1}(V) \times W) \in \tau \times \tau^*$ and $(x,y) \in f^{-1}(V) \times W$. But

$$(t, f(t)) \in f^{-1}(V) \times W \Rightarrow t \in f^{-1}(V), f(t) \in W$$
$$\Rightarrow f(t)) \in V \cap W \Rightarrow V \cap W \neq \varnothing$$

Which contradicts that $V \cap W = \varnothing$. Then $(f^{-1}(V) \times W) \cap f = \varnothing$ which implies that $(x,y) \notin \overline{f}$ which implies that $\overline{f} = f$ which implies that f is a closed subset of $X \times Y$.

Theorem 8.1.8. *If* $f:(X,\tau) \to (Y,\tau^*)$ *is any function such that* $f \subset X \times Y$ *is a closed set then*

(a) f *is injective implies that* (X,τ) *is a* T_1-*space.*

(b) f *is surjective implies that* (Y,τ^*) *is a* T_1-*space.*

Proof: (a) Let $x \in X$ be an arbitrary point and $t \in \overline{\{x\}} - \{x\}$. Then $f(x) \neq f(t)$ because f is injective and so $(t, f(x)) \in X \times Y - f$. But $f \subset X \times Y$ is a closed set implies that there are two open sets $U \in \tau$ and $V \in \tau^*$ such that $(t, f(x)) \in U \times V \subset X \times Y - f$ which implies that $t \in U$ and $t \in \overline{\{x\}}$ implies that $U \cap \{x\} \neq \varnothing$ implies that $x \in U$ which implies that $(x, f(x)) \in U \times V \subset X \times Y - f$ and this is impossible which means that such point $t \in \overline{\{x\}} - \{x\}$ does not exists. Then $\overline{\{x\}} = \{x\}$ and hence (X, τ) is a T_1-space.

Another proof: Let $x, t \in X$ be such that $x \neq t$. Then $f(x) \neq f(t)$ because f is injective. Hence $(x, f(t)) \in X \times Y - f$ and $X \times Y - f$ is open because $f \subset X \times Y$ is a closed set which implies that there are two open sets $U \in \tau$ and $V \in \tau^*$ such that $(x, f(t)) \in U \times V \subset X \times Y - f$ which implies that $x \in U$ and $t \notin U$ since $t \in U$ implies that $(t, f(t)) \in U \times V \subset X \times Y - f$ which is impossible. In a similar way one can prove that there is an open set $V \in \tau$ such that $t \in V$ and $x \notin V$. Therefore (X, τ) is a T_1-space.

(b) Let $y \in Y$ be an arbitrary point. Then $z \in \{y\}^{\wedge} - \{y\} \Rightarrow y \neq z$ implies that there are two distinct points $x, t \in X$ such that $y = f(x)$ and $z = f(t)$ since f is surjective and clearly $f(x) \neq f(t)$. But if f is a closed set then $X \times Y - f$ is open and $(t, y) = (t, f(x)) \in X \times Y - f$ implies that there are two open sets $U \in \tau$ and $V \in \tau^*$ such that $(t, y) \in U \times V \subset f^c$ which implies that $y \in V$ but $z \in \{y\}^{\wedge}$ implies that $z \in V$ which implies that $(t, z) = (t, f(t)) \in f^c$ which is impossible this impossibility because of the incorrect assumption that there is a point $z \in \{y\}^{\wedge} - \{y\}$. Therefore such point z does not exist and so $\{y\}^{\wedge} = \{y\}$ which implies that (Y, τ^*) is a T_1-space.

Another proof: Let $y, z \in Y$ be two distinct points then there are two distinct points $x, t \in X$ such that $y = f(x)$ and $z = f(t)$ this because f is surjective and clearly $f(x) \neq f(t)$. Then $(x, f(t)) \in X \times Y - f$ and $f \subset X \times Y$ is a closed set implies that there are two open sets $U \in \tau$ and $V \in \tau^*$ such that $(x, z) = (x, f(t)) \in U \times V \subset f^c$ which implies that $z \in V$ and $y \notin V$ because $y \in V \Rightarrow (x, f(x)) = (x, y) \in f^c$. Similarly one can prove that there is an open set $H \in \tau^*$ such that $y \in H$ and $z \notin H$. Therefore (Y, τ^*) is a $T_1 -$space.

Theorem 8.1.9. *Let* $f : (X, \tau) \to (Y, \tau^*)$ *be any function such that* $f \subset X \times Y$ *is a closed set and* $B \subset Y$ *be a compact set. Then* $f^{-1}(B)$ *is a closed set.*

Proof: Suppose that $B \subset Y$ is compact and consider the point $x \in X - f^{-1}(B)$ then $x \notin f^{-1}(B)$ and we finds

$$y \in B \Rightarrow y \neq f(x) \Rightarrow (x, y) \notin f \Rightarrow (x, y) \in X \times Y - f$$

If f is a closed set then $X \times Y - f$ is open and so there are two open sets $U_x \in \tau$ and $V_x \in \tau^*$ such that $(x, y) \in U_x \times V_x \subset X \times Y - f$ which implies that $(U_x \times V_x) \cap f = \varnothing$ for each $y \in B$. Then $\{V_y : y \in B\} \subset \tau^*$ is an open cover of B, if B is compact then it contains a finite subcover $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\} \subset \tau^*$ of B. Suppose that $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\} \subset \tau$ is the family of the members of τ which corresponding to the members of the finite subcover of B which satisfy the condition $U_{x_i} \times V_{y_i} \cap f = \varnothing$ for each $i \in \{1, 2, ..., n\}$. If $U = \bigcap_{i=1}^{n} U_{x_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$ then $x \in U$, $B \subset V$ and $(U \times V_{y_i}) \cap f = \varnothing$ for each $i \in \{1, 2, ..., n\}$ which implies that

$$\begin{aligned} x \in U &\Rightarrow f(x) \notin V \Rightarrow f(x) \notin B \Rightarrow x \notin f^{-1}(B) \\ &\Rightarrow x \in X - f^{-1}(B) \Rightarrow x \in U \subset X - f^{-1}(B) \\ &\Rightarrow X - f^{-1}(B) \in \tau \Rightarrow f^{-1}(B) \in \tau_c \end{aligned}$$

Corollary 8.1.1. *If (X,τ) and (Y,τ^*) are two topological spaces, $f : X \to Y$ such that $f \subset X \times Y$ is a closed set then (Y,τ^*) is compact implies that f is continuous.*

Proof: Suppose that $F \in \tau^*_c$, since Y is compact then F is a closed set and $f^{-1}(F) \in \tau_c$ which means that f is continuous.

Theorem 8.1.10. *The topological space (X,τ) is T_2 iff the set $\Delta = \{(x,x) : x \in X\}$ is a closed set i.e. $\Delta \in (\tau \times \tau)_c$.*

Proof: Suppose that (X,τ) is a T_2 – space then

$$x, y \in X : x \neq y \Rightarrow \exists U, V \in \tau : x \in U, y \in V, U \cap V = \varnothing$$

But $x \neq y \Rightarrow (x,y) \in X \times Y - \Delta$ and

$x \in U, y \in V \Rightarrow (x,y) \in U \times V$ from which we finds that

$$(U \times V) \cap \Delta \neq \varnothing \Rightarrow \exists z \in X : (z,z) \in U \times V \Rightarrow U \cap V \neq \varnothing$$

Which contradicts that $U \cap V = \varnothing$. Therefore $(U \times V) \cap \Delta = \varnothing$ according to which $(x,y) \in U \times V \subset X \times X - \Delta$ which implies that $(X \times X - \Delta) \in \tau$ which implies that $\Delta \in (\tau \times \tau)_c$.

Conversely let $\Delta \in (\tau \times \tau)_c$ equivalently $(X \times X - \Delta) \in \tau$. Then

$$\begin{gathered} x, y \in X : x \neq y \Rightarrow (x,y) \in X \times X - \Delta \Rightarrow \exists U, V \in \tau : \\ (x,y) \in U \times V \subset X \times X - \Delta \Rightarrow x \in U, y \in V, (U \times V) \cap \Delta = \varnothing \\ \Rightarrow x \in U, y \in V, U \cap V = \varnothing \end{gathered}$$

Therefore (X,τ) is a T_2 – space, $U \cap V = \varnothing$ since

$$\begin{aligned} U \cap V \neq \varnothing &\Rightarrow \exists z \in U \cap V \Rightarrow (z,z) \in (U \times V) \cap \Delta \\ &\Rightarrow (U \times V) \cap \Delta \neq \varnothing \end{aligned}$$

Contradicts the result $(U \times V) \cap \Delta = \varnothing$.

Theorem 8.1.11. *The topological space* $(X \times Y, \tau \times \tau^*)$ *is* T_i *iff* (X, τ) *and* (Y, τ^*) *are both* T_i *−spaces where* $i \in \{0,1,2,3\}$.

Proof: We are going to prove this theorem for $i = 2$ and $i = 3$. Firstly when $i = 2$ i.e. we are going to prove that $(X \times Y, \tau \times \tau^*)$ is a T_2 −space iff X and Y are T_2 −spaces. For suppose that (X, τ) and (Y, τ^*) are both T_2 −spaces then

$$\begin{aligned} (x_1, y_1), (x_2, y_2) \in X \times Y &: (x_1, y_1) \neq (x_2, y_2) \\ &\Rightarrow x_1 \neq x_2 \vee y_1 \neq y_2 \end{aligned}$$

And since (X, τ) is T_2 then

$$\begin{aligned} &x_1, x_2 \in X, x_1 \neq x_2 \\ &\quad \Rightarrow \exists U, V \in \tau : x_1 \in U, x_2 \in V, U \cap V = \varnothing \ (I) \end{aligned}.$$

Hence,

(1) $(x_1, y_1) \in U \times Y$ and $(x_2, y_2) \in V \times Y$,

(2) $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \varnothing$ and

(3) $U \times Y, V \times Y \in \tau \times \tau^*$.

Similarly we can obtained the same result in the case $y_1, y_2 \in Y$, $y_1 \neq y_2$. Therefore $(X \times Y, \tau \times \tau^*)$ is a T_2 −space.

Conversely suppose that $(X \times Y, \tau \times \tau^*)$ is a T_2 −space then

$x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow (x_1, y), (x_2, y) \in X \times Y$ and $(x_1, y) \neq (x_2, y)$ where $y \in Y$ is any point which implies that there are two open sets

$H, W \in \tau \times \tau^*$ such that $(x_1, y) \in H$, $(x_2, y) \in W$ and $H \cap W = \varnothing$ accordingly there are $U_1, U_2 \in \tau$ and $V_1, V_2 \in \tau^*$ such that $(x_1, y) \in (U_1 \times V_1) \subset H$ --(I) and $(x_2, y) \in (U_2 \times V_2) \subset W$ --(II) which implies that $x_1 \in U_1$, $x_2 \in U_2$ $y \in V_1 \cap V_2$ and $U_1 \cap U_2 = \varnothing$. Therefore (X, τ) is a T_2 −space. Now to show that $U_1 \cap U_2 = \varnothing$ from (I) and (II) and since $V_1 \cap V_2 \neq \varnothing$ then

$$(U_1 \cap U_2) \times (V_1 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2) \subset H \cap W = \varnothing$$
$$\Rightarrow (U_1 \cap U_2) \times (V_1 \cap V_2) = \varnothing \Rightarrow (U_1 \cap U_2) = \varnothing$$

In a similar way one can prove that (Y, τ^*) is a T_2 −space. This is the end of the proof of the case when $i = 2$.

Secondly for the case $i = 3$ we know that the topological space is T_3 if it is regular and T_1 accordingly this case consists two cases the first when $i = 1$ which we left it to the reader and the second case is $(X \times Y, \tau \times \tau^*)$ is regular iff (X, τ) and (Y, τ^*) are both regular which we shall prove as follows:

Suppose that $(X \times Y, \tau \times \tau^*)$ is a regular space then $U \in \tau$ and $x \in U$ implies that $U \times Y \in \tau \times \tau^*$ and $(x, y) \in U \times Y$ where $y \in Y$ is any point and since $(X \times Y, \tau \times \tau^*)$ is regular then there is an open set $H \in \tau \times \tau^*$ such that $(x, y) \in H \subset \overline{H} \subset U \times Y$ and we finds that

$$H \in \tau \times \tau^* \Rightarrow \exists U_1 \in \tau, V_1 \in \tau^* : (x, y) \in U_1 \times V_1 \subset H$$
$$\subset \overline{H} \subset U \times Y \Rightarrow (x, y) \in U_1 \times V_1 \subset \overline{U_1 \times V_1} \subset \overline{H} \subset U \times Y$$
$$\Rightarrow (x, y) \in U_1 \times V_1 \subset \overline{U_1} \times \overline{V_1} \subset U \times Y$$
$$\Rightarrow x \in U_1 \subset \overline{U_1} \subset U$$

Therefore (X, τ) is a regular space.

Conversely Let (X, τ) and (Y, τ^*) be regular spaces. Then

$$H \in \tau\times\tau^*,(x,y)\in H \Rightarrow \exists U_1\in\tau,V_1\in\tau^*:(x,y)\in U_1\times V_1\subset H$$

$$\Rightarrow x\in U_1,y\in V_1\Rightarrow \exists U_2\in\tau,V_2\in\tau^*:x\in U_2\subset\overline{U_2}\subset U_1\wedge$$

$$y\in V_2\subset\overline{V_2}\subset V_1\Rightarrow(x,y)\in U_2\times V_2\subset\overline{U_2}\times\overline{V_2}$$

$$=\overline{U_2\times V_2}\subset H\Rightarrow (x,y)\in U_2\times V_2\subset\overline{U_2\times V_2}\subset H$$

Clearly $U_2\times V_2\in\tau\times\tau^*$ and hence $(X\times Y,\tau\times\tau^*)$ is a regular space.

Remark 8.1.3. *The topological spaces* (X,τ) *and* (Y,τ^*) *may be normal and* $(X\times Y,\tau\times\tau^*)$ *need not be normal as explained by the following example:*

Consider the lower limit topological space (R,τ) *where* τ *is the topology on* R *generated by the family* $\beta=\{[a,b):a,b\in R\}$ *then* (R,τ) *is a normal topological space while* (R^2,τ^2) *is not normal where* $\tau^2=\tau\times\tau$ *since*

$$F=\{(x,y)\in R^2:x,y\in Q,x+y=1\}$$

and

$$M=\{(x,y)\in R^2:x,y\in Q^c,x+y=1\}$$

are nonempty closed disjoint subsets of R^2 *and if* U *and* V *are two open subsets of* R^2 *such that* U *contains* F *and* V *contains* M *then* $U\cap V\neq\varnothing$ *which means that* (R^2,τ^2) *is not normal space.*

Theorem 8.1.12. *If* $(X\times Y,\tau\times\tau^*)$ *is a normal topological space then the topological spaces* (X,τ) *and* (Y,τ^*) *are normal.*

Proof: Clearly

$F\in\tau_c,U\in\tau:F\subset U\Rightarrow F\times Y\subset U\times Y$ and $U\times Y\in\tau\times\tau^*$, $F\times Y\in(\tau\times\tau^*)_c$. So if $(X\times Y,\tau\times\tau^*)$ is normal then there is an open set $H\in\tau\times\tau^*$ such that $F\times Y\subset H\subset\overline{H}\subset U\times Y$ but

$$H \in \tau\times\tau^*, F\times Y \subset H \Rightarrow \exists V \in \tau : H = V\times Y$$

which implies

$$F\times Y \subset V\times Y \subset \overline{V\times Y} = \overline{V}\times Y \subset U\times Y \Rightarrow F \subset V \subset \overline{V} \subset U$$

Therefore (X,τ) is a normal space.

Similarly one can prove that (Y,τ^*) is normal.

Remark 8.1.4. *The topological space (R,τ) which is given in Remark 8.1.3, is a T_3-space since it is T_1 because $C \subset u \subset \tau$ and we can show that (R,τ) is regular as follows: $F \in \tau_c$ implies that there is a family $\{[a_\alpha,b_\alpha):\alpha\in\Delta\}\subset P(R)$ such that $F^c = \bigcup_{\alpha\in\Delta}[a_\alpha,b_\alpha)$. Hence $x \in F^c \Rightarrow \exists\alpha\in\Delta : x \in [a_\alpha,b_\alpha)$ but*

$$[a_\alpha,b_\alpha) \subset F^c \Rightarrow F \subset [a_\alpha,b_\alpha)^c = (-\infty,a_\alpha)\cup[b_\alpha,\infty)\in\tau\,.$$

Therefore (R,τ) is a regular space and from Theorem 8.1.12, (R^2,τ^2) is T_1 and regular and so is a T_3-space while by Remark 8.1.3, it is not T_4-space that is

$$T_3 -/\rightarrow T_4$$

Theorem 8.1.13. *The topological space $(X\times Y,\tau\times\tau^*)$ has the first (second) axiom of countability iff the topological spaces (X,τ) and (Y,τ^*) have the first (second) axiom.*

Proof: We shall gives the proof for the first axiom of countability and left the second axiom to the reader.

Firstly suppose that $(X\times Y,\tau\times\tau^*)$ has the first axiom of countability and $(x,y)\in X\times Y$ then $(X\times\{y\},(\tau\times\tau^*)_{X\times\{y\}})$ is a subspace of $(X\times Y,\tau\times\tau^*)$ and it has the first axiom of countability and

$(X\times\{y\},(\tau\times\tau^*)_{X\times\{y\}})$ is homeomorphic to (X,τ) and (X,τ) has the first axiom of countability.

Conversely suppose that each of (X,τ) and (Y,τ^*) has the first axiom of countability and $(x,y)\in X\times Y$ then $x\in X$ and $y\in Y$ accordingly there are a countable local bases $\beta_x\subset\tau$ for the point $x\in X$ and $\beta_y\subset\tau^*$ for the point $y\in Y$ such that $\beta_x=\{U_1,U_2,\dots\dots\}$ and $\beta_y=\{V_1,V_2,\dots\dots\}$ and then

$$\beta_x\times\beta_y=\{U_n\times V_m:n,m\in N\}=\bigcup\{A_n:n\in N\}$$

Where $A_n=\bigcup\{U_n\times V_m:m\in N\}$ for each $n\in N$ and A_n is

a countable set according to which $\beta_x\times\beta_y$ is a countable family which we prove that it a local basis for the point (x,y),

$$H\in\tau\times\tau^*:(x,y)\in H\Rightarrow\exists U\in\tau,V\in\tau^*:(x,y)\in U\times V\subset H$$
$$\Rightarrow x\in U,y\in V\Rightarrow\exists n,m\in N:x\in U_n\subset U,y\in V_m\subset V\text{ But}$$
$$\Rightarrow(x,y)\in U_n\times V_m\subset U\times V\subset H$$

$U_n\in\beta_x,V_m\in\beta_y\Rightarrow U_n\times V_m\in\beta_x\times\beta_y$ which implies that $(X\times Y,\tau\times\tau^*)$ has the first axiom of countability.

Theorem 8.1.14. *The topological space* $(X\times Y,\tau\times\tau^*)$ *is compact iff* (X,τ) *and* (Y,τ^*) *are compact topological spaces.*

Proof: Suppose that $(X\times Y,\tau\times\tau^*)$ is a compact space then the projection functions π_1 and π_2 are continuous and so (X,τ) is compact because $\pi_1:(X\times Y)=X$ is continuous and similarly (Y,τ^*) is compact.

Conversely if (X,τ) and (Y,τ^*) are compact and $\Psi\subset\tau\times\tau^*$ is an open cover of $X\times Y$ then if $H\in\Psi$ then there are two families $\{U_\alpha:\alpha\in\Delta\}\subset\tau$ and $\{V_\alpha:\alpha\in\Delta\}\subset\tau^*$ such that $H=\bigcup_{\alpha\in\Delta}U_\alpha\times V_\alpha$ this by the definition of $\tau\times\tau^*$ and so we gets $\Psi^*=\{U_\lambda\times V_\lambda:\lambda\in\Lambda\}$ is a cover of $X\times Y$. If we obtain a finite subcover of $X\times Y$ from Ψ^{*} then we can obtain a finite subcover from Ψ. Now if $x\in X$ is an arbitrary point then $\{x\}\times Y$ is homeomorphic to Y and since Y is a compact space then by $\{x\}\times Y$ is compact. Since Ψ^{*} is an open cover $\{x\}\times Y$ then it has a finite subcover $\Im=\{U_{x1}\times V_1, U_{x2}\times V_2,\dots,U_{xn}\times V_n\}\subset\Psi^{*}$ and if we assume that $G_x=\bigcap_{i=1}^{n}U_{xi}$ then $\Im$ is also is a cover of $G_x\times Y$ where $x\in X$ and hence $\{G_x : x\in X\}$ is an open cover of X and since X is compact then it contains a finite sub cover of X $\{G_{x_1},G_{x_2},\dots,G_{x_m}\}$. But for each G_{x_j} the set $G_{x_j}\times Y$ has the finite subcover $\Im_j=\{U_{x_{j1}}\times V_{j1}, U_{x_{j2}}\times V_{j2},\dots,U_{x_{jn}}\times V_{jn}\}$. Therefore $\bigcup_{j=1}^{m}\Im_j\subset\Psi^{*}$ is a finite subcover of $X\times Y$ and hence $(X\times Y,\tau\times\tau^{*})$ is a compact space.

Theorem 8.1.15. *The topological product space $(X\times Y,\tau\times\tau^{*})$ is a connected space iff (X,τ) and (Y,τ^{*}) are both connected.*

Proof: Suppose that $(X\times Y,\tau\times\tau^{*})$ is a connected space then

(i) $b(A\times Y)=(b(A)\times\overline{Y})\cup(\overline{A}\times b(Y))\neq\varnothing$ for each subset $A\in P(X)-\{X,\varnothing\}$ which implies that $(b(A)\times Y)\cup\varnothing\neq\varnothing$ which implies that $b(A)\neq\varnothing$ which implies that (X,τ) is connected.

(ii) $b(X\times B)=(b(X)\times\overline{B})\cup(\overline{X}\times b(B))\neq\varnothing$ for each subset $B\in P(Y)-\{Y,\varnothing\}$ which implies that $\varnothing\cup(X\times b(B))\neq\varnothing$ which implies that $b(B)\neq\varnothing$ which implies that (Y,τ^{*}) is connected.

Conversely suppose that (X,τ) and (Y,τ^{*}) are both connected and $M\in P(X\times Y)-\{X\times Y,\varnothing\}$ then there are three cases:

(i) $M=A\times B$ where $A\in P(X)-\{X,\varnothing\}$ and $B\in P(Y)-\{Y,\varnothing\}$ in this case (X,τ) is connected implies that $b(A)\neq\varnothing$ from which $\overline{A}\neq\varnothing$ and

(Y, τ^*) is connected implies that $b(B) \neq \varnothing$ from which $\overline{B} \neq \varnothing$ and then $b(A) \times \overline{B} \neq \varnothing$ and $\overline{A} \times b(B) \neq \varnothing$ which implies that

$$b(M) = b(A \times B) = (b(A) \times \overline{B}) \cup (\overline{A} \times b(B)) \neq \varnothing .$$

(ii) $M = A \times Y$ where $A \in P(X) - \{X, \varnothing\}$ in this case (X, τ) is connected implies that $b(A) \neq \varnothing$ implies that $b(A) \times Y \neq \varnothing$ and $b(Y) = \varnothing$ which implies that

$$b(M) = b(A \times Y) = (b(A) \times \overline{Y}) \cup (\overline{A} \times b(Y)) = (b(A) \times Y) \cup \varnothing \neq \varnothing$$

(iii) $M = X \times B$ where $B \in P(Y) - \{Y, \varnothing\}$ in this case (Y, τ^*) is connected implies that $b(B) \neq \varnothing$ which implies that $X \times b(B) \neq \varnothing$ and $b(X) = \varnothing$ which implies that

$$b(M) = b(X \times B) = (b(X) \times \overline{B}) \cup (\overline{X} \times b(B)) = \varnothing \cup (X \times b(B)) \neq \varnothing$$

Therefore $(X \times Y, \tau \times \tau^*)$ is connected if both (X, τ) and (Y, τ^*) are connected.

Theorem 8.1.16. *If* (X, τ) *and* (Y, τ^*) *are two topological spaces and* $(x, y) \in X \times Y$ *then*

(a) $x \notin X', y \notin Y' \Rightarrow (x, y) \notin (X \times Y)'$ *and*

(b) $(x, y) \notin (X \times Y)' \Rightarrow x \notin X' \vee y \notin Y'$*.*

Proof:

$$\begin{aligned} x \notin X', y \notin Y' &\Rightarrow \{x\} \in \tau, \{y\} \in \tau^* \Rightarrow \{x\} \times \{y\} \\ &= \{(x, y)\} \in \tau \times \tau^* \Rightarrow (x, y) \notin (X \times Y)' \end{aligned}$$

(b) If $(x, y) \in (X \times Y)$ then for each $H \in \tau \times \tau^*$ such that $(x, y) \in H$ there exists $U \in \tau$ and $V \in \tau^*$ such that $(x, y) \in U \times V \subset H$ which implies that $x \in U$ and $y \in V$ and hence

$$x \in X' \Rightarrow U - \{x\} \neq \varnothing \Rightarrow \exists p \in U - \{x\} - -(I)$$

$$y \in Y' \Rightarrow V - \{y\} \neq \varnothing \Rightarrow \exists q \in V - \{y\} - -(II)$$

From which

1. (1) From (I), $(p,y)\in H-\{(x,y)\}$ which implies that $(x,y)\in(X\times Y)'$.
2. (2) From (II), $(x,q)\in H-\{(x,y)\}$ which implies $(x,y)\in(X\times Y)'$

From (1) and (2) $(x,y)\notin(X\times Y)'\Rightarrow x\notin X'\vee y\notin Y'$.

Corollary 8.1.2. *If (X,τ) and (Y,τ^*) are two topological spaces then $X\times Y$ is dense in itself if either X or Y is dense in itself and if any of X or Y is dense in itself then $X\times Y$ is also.*

Proof: According to the definition, $X\times Y$ is dense in itself if $X\times Y\subset(X\times Y)'$ and it is not dense in itself iff there is a point $(x,y)\in X\times Y$ such that $(x,y)\notin(X\times Y)'$ from which we gets

$$x\notin X',y\notin Y'\Rightarrow(x,y)\notin(X\times Y)'\Rightarrow x\notin X'\vee y\notin Y'$$

Or equivalently

$$x\in X',y\in Y'\Rightarrow(x,y)\in(X\times Y)'\Rightarrow x\in X'\vee y\in Y'$$

8.2. Metric spaces.

Any book of the general topology start by an introduction of metric spaces since we consider that the general topology is a generalization of idea of the concepts of the metric spaces. In this article in the last chapter of this book we give a glance about the main concepts of the metric spaces and some of the interesting properties.

Definition 8.2.1. If X is a non empty set i.e. $X\neq\varnothing$ then the function $d:X\times X\to R$ is called the distance function on X if it satisfies the following conditions:

(D_1**)**$d(x,y)\geq 0$ and $d(x,y)=0\Leftrightarrow x=y$ for each $x,y\in X$,

(D_2**)** $d(x,y)=d(y,x)$ for each $x,y\in X$ and

(D_3**)**$d(x,z)\leq d(x,y)+d(y,z)$ for each $x,y,z\in X$.

The property D_3 is called the triangle inequality and if d is a distance on X where $X \neq \varnothing$ then the ordered pair (X, d) is called a metric space and we write that X is a metric space if essentially there is a distance d defined on X .

Example 8.2.1. Consider the set of the real numbers R then the function $d : R \times R \to R$ defined by $d(x, y) = |x - y|$ for each $x, y \in R$ is a distance on R since it satisfies the three conditions D_1, D_2 and D_3 this is directly by using the properties of the absolute value of the real numbers in this case d is called the usual metric on R and (R, d) is called the usual metric space.

To define the usual metric on R^n where $n \in N$ we start by the following useful two inequalities:

Theorem 8.2.1. *(Cauchy-Minkowiski inequality):*

Let $\{a_1, a_2, \ldots, a_n\}$ *and* $\{b_1, b_2, \ldots, b_n\}$ *then*

$$\boxed{\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}\ .$$

Proof: Clearly

$$0 \leq \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \Rightarrow$$

$$\Rightarrow 0 \le \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i^2 b_j^2 - 2a_i b_j\, a_j b_i + a_j^2 b_i^2)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i^2 b_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_j^2 b_i^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j\, a_j\, b_i$$

$$= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2 \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j\, b_j$$

$$= \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i^2 - 2 \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} a_i\, b_i$$

$$= 2 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - 2 (\sum_{i=1}^{n} a_i b_i)^2$$

$$\Rightarrow (\sum_{i=1}^{n} a_i b_i)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

Then by taking the square root of both sides we gets

$$\boxed{\left| \sum_{i=1}^{n} a_i\, b_i \right| \le \sqrt{\sum_{i=1}^{n} a_i^2}\ \sqrt{\sum_{i=1}^{n} b_i^2}}$$

Corollary 8.2.1. *(Minkowisky inequality)*

Consider the two sets of numbers $\{a_1, a_2, \ldots, a_n\}$ *and* $\{b_1, b_2, \ldots, b_n\}$ *mentioned in Theorem 8.2.1, then*

$$\boxed{\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}}\,.$$

Proof:

$$\sum_{i=1}^{n}(a_i+b_i)^2=\sum_{i=1}^{n}(a_i^2+b_i^2+2a_i b_i)=$$

$$\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2+2\sum_{i=1}^{n}a_i b_i\le\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2+2\left|\sum_{i=1}^{n}a_i b_i\right|\le$$

$$\le\sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2+2\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}=(\sqrt{\sum_{i=1}^{n}a_i^2}+\sqrt{\sum_{i=1}^{n}b_i^2})^2$$

$$\Rightarrow\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2}\le\sqrt{\sum_{i=1}^{n}a_i^2}+\sqrt{\sum_{i=1}^{n}b_i^2}$$

This inequality which is called Minkowiski inequality is useful in the study of some metric functions on R^n or on C^n where C^n is the set of the complex numbers.

Example 8.2.2. Consider the set $R^n=\{(x_1,x_2,...,x_n):x_1,x_2,...,x_n\in R\}$ where R is the set of the real numbers and $n\in N$, define the function $d:R^n\times R^n\to R$ by the rule $d(x,y)=\sqrt{\sum_{i=1}^{n}(y_i-x_i)^2}$ for each $x,y\in R^n$ where $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$ one can prove that d is a distance on R^n by proving that it satisfies the conditions D_1, D_2 and D_3 as follows:

D_1: $(y_i-x_i)^2\ge 0$ for each $i\in\{1,2,..,n\}$ and so $d(x,y)\ge 0$. Also $(y_i-x_i)^2=0\Leftrightarrow x_i=y_i$ for each $i\in\{1,2,..,n\}$ iff $x=y$ i.e. $d(x,y)=0\Leftrightarrow x=y$.

D_2: $(y_i-x_i)^2=(x_i-y_i)^2$ for each $i\in\{1,2,..,n\}$ iff $d(x,y)=d(y,x)$.

D_3: By using Minkowiski inequality we find that if $x,y,z\in R^n$ then

$$d(x,z)=\sqrt{\sum_{i=1}^{n}(x_i-z_i)^2}=\sqrt{\sum_{i=1}^{n}(x_i-y_i+y_i-z_i)^2}$$
$$\leq\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}=d(x,y)+d(y,z)$$

Hence d is a distance on R^n called the usual metric and one can define another distances on R^n like for example the distance $\rho:R^n\times R^n\to R$ where $\rho(x,y)=\max.\{|x_i-y_i|:i\in\{1,2,..,n\}\}$, the proof that ρ is a distance on R^n left to the reader.

If $n=2$ then $x,y\in R^2\Rightarrow x=x=(x_1,x_2),y=(y_1,y_2)$ and the usual distance d on R^n is given by

$$d(x,y)=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\ .$$

Example 8.2.3. Let $X\neq\varnothing$ then distance d on X define to be the function $d:X\times X\to R$ given by

$$d(x,y)=\begin{cases}1; & x=y\\ 0; & x\neq y\end{cases}\ ;\ \forall x,y\in R$$

Which is clearly satisfies the conditions D_1, D_2 and D_3, the proof left to the reader. This distance is called the trivial metric.

Example 8.2.4. If d is a metric on a non empty set X then the function ρ defined on X given by $\rho(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$ for each $x,y\in X$ is also a distance on X . To prove that ρ is a distance on X . Clearly it satisfies the conditions D_1 and D_2 and we prove that it satisfies D_3 for let $x,y,z\in X$ be arbitrary three points and let $d(x,y)=\varepsilon_1$, $d(y,z)=\varepsilon_2$ and $d(x,z)=\varepsilon_3$. Then

$$\rho(x,y)+\rho(y,z)=\frac{\varepsilon_1}{1+\varepsilon_1}+\frac{\varepsilon_2}{1+\varepsilon_2}\ge\frac{\varepsilon_1+\varepsilon_2}{1+\varepsilon_1+\varepsilon_2}$$

$$\Rightarrow\rho(x,y)+\rho(y,z)-\rho(x,z)=\frac{\varepsilon_1}{1+\varepsilon_1}+\frac{\varepsilon_2}{1+\varepsilon_2}-\frac{\varepsilon_3}{1+\varepsilon_3}\ge$$

$$\frac{\varepsilon_1+\varepsilon_2}{1+\varepsilon_1+\varepsilon_2}-\frac{\varepsilon_3}{1+\varepsilon_3}=\frac{(\varepsilon_1+\varepsilon_2)(1+\varepsilon_3)-\varepsilon_3(1+\varepsilon_1+\varepsilon_2)}{(1+\varepsilon_1+\varepsilon_2)(1+\varepsilon_3)}$$

$$=\frac{\varepsilon_1+\varepsilon_2-\varepsilon_3}{(1+\varepsilon_1+\varepsilon_2)(1+\varepsilon_3)}$$

And so

$$d(x,y)+d(y,z)\ge d(x,z)\Rightarrow\varepsilon_1+\varepsilon_2\ge\varepsilon_3$$

$$\Rightarrow\frac{\varepsilon_1+\varepsilon_2-\varepsilon_3}{(1+\varepsilon_1+\varepsilon_2)(1+\varepsilon_3)}\ge 0\Rightarrow\rho(x,y)+\rho(y,z)\ge\rho(x,z)$$

which implies that ρ satisfies the condition D_3 and therefore it is a distance on X .

Definition 8.2.2. If (X,d) is a metric space and r is a positive real number i.e. $r\in R^+$ where $R^+=(0,\infty)$ and $x\in X$ then $S(x;r)=\{y\in X:d(x,y)<r\}$ is called an open sphere with center at the point x and radius r .

Example 8.2.5. Consider the usual distance d on R where $d(x,y)=|x-y|$ then if $x\in X$ and $r\in R^+$ implies that $S_d(x;r)=\{y\in R:|x-y|<r\}=(x-r,x+r)$ which is an open interval.

Example 8.2.6. If d is a distance on R^2 such that for each $x,y\in R^2$, $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ where $x=(x_1,x_2)$ and $y=(y_1,y_2)$ then for each $x\in R^2$ and each $r\in R^+$ the open sphere with center at the point x and with radius r is

$S_d(x;r)=\{y\in R^2:\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}<r\}$, the set of all interior points of the sphere $(x_1-y_1)^2+(x_2-y_2)^2=r^2$.

Example 8.2.7. By considering the trivial distance given by Example 8.2.3, the open sphere centered at the point x and with radius r is

$$S_d(x;r)=\begin{cases}\{x\}; & 0<r<1\\ X; & r>1\end{cases}.$$

Remark 8.2.1. *If* (X,d) *is a metric space,* $x\in X$ *and* $S_d(x;r)$ *is an open sphere with center at* x *and with radius* r *then* $x\in S_d(x;r)$.

Lemma 8.2.1. *If* (X,d) *is a metric space,* $x\in X$ *and* $S_d(x;\varepsilon)$ *is an open sphere with* x *where* ε *is any positive real number then*

$$y\in S_d(x;\varepsilon)\Rightarrow\exists\delta>0:S_d(y;\delta)\subset S_d(x;\varepsilon).$$

Proof: Suppose that $\delta=\varepsilon-d(x,y)$ then

$$\begin{aligned}z\in S_d(y;\delta)&\Rightarrow d(y,z)<\delta=\varepsilon-d(x,y)\\&\Rightarrow d(x,y)+d(y,z)<\varepsilon\end{aligned}$$

and by using D_3 we finds

$$\begin{aligned}d(x,z)\le d(x,y)+d(y,z)<\varepsilon&\Rightarrow d(x,z)<\varepsilon\\&\Rightarrow z\in S_d(x;\varepsilon)\Rightarrow S_d(y;\delta)\subset S_d(x;\varepsilon)\end{aligned}$$

Remark 8.2.2. *If* (X,d) *is a metric space,* $x\in X$, ε_1 *and* ε_2 *are two real numbers then*

$$0<\varepsilon_1<\varepsilon_2\Rightarrow S_d(x;\varepsilon_1)\subset S_d(x;\varepsilon_2)$$

which is clearly from the definition where

$$y\in S_d(x;\varepsilon_1)\Rightarrow d(x,y)<\varepsilon_1<\varepsilon_2\Rightarrow y\in S_d(x;\varepsilon_2)$$

Lemma 8.2.2. *If* (X, d) *is a metric space then for each two spheres* S_1 *and* S_2 *in a metric space* X *and for each point* $x \in S_1 \cap S_2$ *there exists a positive real number* ε *such that* $S_d(x; \varepsilon) \subset S_1 \cap S_2$.

Proof: Suppose that the two radii are r_1 and r_2 and if $S_1 \subset S_2$ then $S_1 \cap S_2 = S_1$, so $x \in S_1 \subset S_1 \cap S_2 = S_1$ and there exists $\delta > 0$ such that $S_d(x; \varepsilon) \subset S_1 \subset S_1 \cap S_2$ this by Lemma 8.2.1, and we get a similar result if $S_2 \subset S_1$. If $S_1 \cap S_2 \neq S_1$, $S_1 \cap S_2 \neq S_2$ and $x \in S_1 \cap S_2$ then again by Lemma 8.2.1, there are ε_1 and ε_2 such that $S_d(x; \varepsilon_1) \subset S_1$ and $S_d(x; \varepsilon_2) \subset S_1$, if $\varepsilon = \min.\{\varepsilon_1, \varepsilon_2\}$ then

$$\begin{gathered} S_d(x; \varepsilon) \subset S_d(x; \varepsilon_1) \cap S_d(x; \varepsilon_2) \subset S_1 \cap S_2 \\ \Rightarrow S_d(x; \varepsilon) \subset S_1 \cap S_2 \end{gathered}.$$

The following interesting theorem depends on the definition of the open sphere and the two lemmas 8.2.1 and 8.2.2.

Theorem 8.2.2. *If* (X, d) *is a metric space then the family of all open spheres* $\beta = \{S_d(x; r) : x \in R, r \in R^+\}$ *form a bases for a topology on* X *denoted* τ_d .

Proof: It is just the proof of the lemmas 8.2.1 and 8.2.2.

Definition 8.2.3. According to Theorem 8.2.2, a subset $G \subset X$ where (X, d) is a metric space is an open set i.e. $G \in \tau_d$ if for each point $x \in G$ there exist a positive real number ε such that $S_d(x; \varepsilon) \subset G$ i.e .

$$x \in G \Rightarrow \exists \varepsilon > 0 : S_d(x; \varepsilon) \subset G .$$

Example 8.2.8. To describe the metric space which is generated by the distance $d = |x - y|$ for each two points $x, y \in R$ Example 8.2.5, we finds that each open sphere is open interval where

$S_d(x;\varepsilon)=(x-\varepsilon,x+\varepsilon)$ as well as for each $a,b\in R$ if we set $x=\frac{a+b}{2}$ and $\varepsilon=\frac{b-a}{2}$, $S_d(x;\varepsilon)=(a,b)$ which implies that

$$\beta=\{(a,b):a,b\in R\}=\{S_d(x;r):x\in R,r\in R^+\}.$$

Then the metric topology τ_d in this case coincides with the usual topology u on R i.e. $\tau_d=u$.

Example 8.2.9. Return to Example 8.2.6, the metric topology τ_d generated by the distance $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ on R^2 for each two points $x,y\in R^2$ where $x=(x_1,x_2)$ and $y=(y_1,y_2)$ and we remark that the topology τ_d generated in this case on R^2 is the usual topology u^2. The topology τ_d in this case on R^2 is the usual topology u^2 since the open spheres is the open circles by the distance d.

Example 8.2.10. To describe the metric topology τ_d on a nonempty set X generated by the trivial metric consider Example 8.2.7, we easily remark that it is the discrete topology D on X.

Definition 8.2.4. A topological space (X,τ) is said to be metrizable if there is a distance d on X such that the metric topology τ_d equivalent or equals to τ.

Example 8.2.11. The usual topological space $(R,\mho)$, the usual topological space $(R^2,\mho^2)$ and the discrete space (X,D) where $X\neq\varnothing$ see Examples 8.2.8-19, are metrizable.

Definition 8.2.5. Consider the nonempty set X, if d_1 and d_2 are two defined distances on X then they are equivalent if the metric topologies

τ_{d_1} and τ_{d_2} generated by d_1 and d_2 on X are equivalent i.e. if $\tau_{d_1} = \tau_{d_2}$.

Theorem 8.2.3. *if* d_1 *and* d_2 *are two defined distances on* X *and* τ_{d_1} *and* τ_{d_2} *are the metric topologies generated by the distances* d_1 *and* d_2 *on* X *respectively then* $\tau_{d_1} \subset \tau_{d_2}$ *iff for each point* $x \in X$ *and each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $S_{d_2}(x;\delta) \subset S_{d_1}(x;\varepsilon)$ *i. e.*

$$\tau_{d_1} \subset \tau_{d_2} \Leftrightarrow \forall x \in X, \forall \varepsilon > 0; \exists \delta > 0 : S_{d_2}(x;\delta) \subset S_{d_1}(x;\varepsilon).$$

Proof: Suppose that $\tau_{d_1} \subset \tau_{d_2}$ and $x \in X$, $\varepsilon > 0$ then $S_{d_1}(x;\varepsilon)$ is a basic element for the topology τ_{d_1} and by Theorem 8.2.2, there exist a basic element S for τ_{d_2} such that

$$x \in S \subset S_{d_2}(x;\varepsilon) -- (I).$$

But $S \in \tau_{d_2}$ which implies that there is a positive number $\delta > 0$ such that $S_{d_2}(x;\delta) \subset S$ and from (I), $S_{d_2}(x;\delta) \subset S_{d_1}(x;\varepsilon)$.

Secondly suppose that for each positive number $\varepsilon > 0$ there exists a positive number $\delta > 0$ such that for each point $x \in X$, $S_{d_2}(x;\delta) \subset S_{d_1}(x;\varepsilon)$. If S for τ_{d_1} and $x \in S$ then by Theorem 8.2.2, there exists $\varepsilon > 0$ such that $S_{d_1}(x;\varepsilon) \subset S$. Then by the assumption there exists $\delta > 0$ such that $S_{d_2}(x;\delta) \subset S_{d_1}(x;\varepsilon) \subset S$. Hence, $\tau_{d_1} \subset \tau_{d_2}$.

Theorem 8.2.4. *If* (X, d_1) *and* (X, d_2) *are two metric spaces and the function* $d:(X \times Y) \times (X \times Y) \to R$ *is such that* $d(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2)$ *for each* $x, y \in R^2$ *where* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *then* d *is a distance on* $X \times Y$. *If* τ_d *is the metric topology generated on* $X \times Y$ *by* d *and* τ_{d_1}, τ_{d_2} *are the metric topologies on* X *by* d_1 *and on* Y *by* d_2. *Then* $\tau_d = \tau_{d_1} \times \tau_{d_2}$.

Proof: To prove that d is a distance on $X \times Y$ consider the arbitrary points $x, y, z \in X \times Y$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$ then

$$\begin{aligned} & d_1(x_1, y_1) \geq 0, d_2(x_2, y_2) \geq 0 \Rightarrow \\ & \Rightarrow d(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0 \qquad (i) \end{aligned}$$

and

$$\begin{aligned} & 0 = d(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2) \Leftrightarrow d_1(x_1, y_1) = 0, \\ & d_2(x_2, y_2) = 0 \Leftrightarrow x_1 = y_1, x_2 = y_2 \Leftrightarrow (x_1, x_2) = (y_1, y_2) \\ & \Leftrightarrow x = y \qquad (ii) \end{aligned}$$

which implies that d satisfies D_1.

Secondly

$$\begin{aligned} & d(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2) = d_1(y_1, x_1) \\ & + d_2(y_2, x_2) = d(y,x) \qquad (iii) \end{aligned}$$

this means that d satisfies D_2.

Thirdly

$$\begin{aligned} d(x,z) &= d_1(x_1, z_1) + d_2(x_2, z_2) \leq d_1(x_1, y_1) + d_1(y_1, z_1) \\ & + d_2(x_2, y_2) + d_2(y_2, z_2) = d_1(x_1, y_1) + d_2(x_2, y_2) \\ & + d_1(y_1, z_1) + d_2(y_2, z_2) = d(x,y) + d(y,z) \\ & \Rightarrow d(x,z) \leq d(x,y) + d(y,z) \qquad (vi) \end{aligned}$$

this means that d satisfies D_3.

Now to prove that $\tau_d = \tau_{d_1} \times \tau_{d_2}$ let $W \in \tau_d$ and $(x_{1,} x_2) \in W$. Then there exists $\varepsilon > 0$ such that $S_d((x_1, x_2); \varepsilon) \subset W$ and if $U = S_{d_1}(x_1; \frac{\varepsilon}{2})$ and $V = S_{d_2}(x_2; \frac{\varepsilon}{2})$ then $U \in \tau_{d_1}$, $V \in \tau_{d_2}$ and $(x_1, x_2) \in U \times V$. Hence $(t_1, t_2) \in U \times V \Rightarrow x_1, t_1 \in U, x_2, t_2 \in V$ and we gets

$$d_1(x_1, t_1) < \frac{\varepsilon}{2}, d_2(x_2, t_2) < \frac{\varepsilon}{2} \Rightarrow d((x_1, x_2), (t_1, t_2))$$

$$= d_1(x_1, t_1) + d_2(x_2, t_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow (t_1, t_2) \in S_d((x_1, x_2); \varepsilon) \Rightarrow U \times V \subset S_d((x_1, x_2); \varepsilon)$$

$$\Rightarrow (x_1, x_2) \in U \times V \subset S_d((x_1, x_2); \varepsilon) \subset W$$

Then

$$\tau_d \subset \tau_{d_1} \times \tau_{d_2} \quad --(I)$$

Conversely suppose that $W \in \tau_{d_1} \times \tau_{d_2}$ and $(x_{1,} x_2) \in W$ then there are two positive real numbers ε_1 and ε_2 such that $S_{d_1}(x_1; \varepsilon_1) \times S_{d_2}(x_2; \varepsilon_2) \subset W$ and if $\varepsilon = \min.\{\varepsilon_1, \varepsilon_2\}$ so if $(t_1, t_2) \in S_d((x_{1,} x_2); \varepsilon)$ then

$$\begin{aligned} d((x_1, x_2), (t_1, t_2)) &= d_1(x_1, t_1) + d_2(x_2, t_2) \\ &\Rightarrow d_1(x_1, t_1) \le d((x_1, x_2), (t_1, t_2)) < \varepsilon \le \varepsilon_1 \\ &\Rightarrow t_1 \in S_{d_1}(x_1; \varepsilon_1) --(II) \end{aligned}$$

Similarly

$$d_2(x_2,t_2)\le d((x_1,x_2),(t_1,t_2))<\varepsilon\le\varepsilon_2$$
$$\Rightarrow t_2\in S_{d_2}(x_2;\varepsilon_2)--(III)$$

From (II) and (III) we finds

$$S_d((x_1,x_2);\varepsilon)\subset S_{d_1}(x_1;\varepsilon_1)\times S_{d_2}(x_2;\varepsilon_2)\subset W$$

from which $W\in\tau_d$ and so

$$\tau_{d_1}\times\tau_{d_2}\subset\tau_d--(VI)$$

So from (II) and (III), $\tau_d=\tau_{d_1}\times\tau_{d_2}$.

8.3. The metric spaces and separation axioms:

Theorem 8.3.1. *Each metric space (X,d) is a T_2-topological space that is each (X,τ_d) is a T_2-space.*

Proof: Suppose that (X,d) is a metric space and $x,y\in X$ such that $x\neq y$, if we let $d(x,y)=\varepsilon$, $U=S_d(x;\frac{\varepsilon}{2})$ and $V=S_d(y;\frac{\varepsilon}{2})$ then $U,V\in\tau_d$, $x\in U$ and $y\in V$ and we shall prove that $U\cap V=\varnothing$ for we finds

$$t\in U\cap V\Rightarrow t\in S_d(x;\frac{\varepsilon}{2})\cap S_d(y;\frac{\varepsilon}{2})\Rightarrow d(x,t)+d(t,y)$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon=d(x,y)$$

which contradicts the triangle inequality which means that the assumption $U\cap V\neq\varnothing$ is incorrect and hence $U\cap V=\varnothing$. Therefore (X,τ_d) is a T_2-space.

Theorem 8.3.2. *If (X,d) is a metric space then (X,τ_d) is normal that is for each two closed disjoint subsets A and B of X there are two open sets $U,V\in\tau_d$ such that $A\subset U$, $B\subset V$ and $U\cap V=\varnothing$.*

Proof: Suppose that $x\in A$ then $x\notin\overline{B}$ because $A\cap B=\varnothing$ and $B=\overline{B}$ because B is a closed set and so

$$x\notin\overline{B}\Rightarrow\exists\,\varepsilon_x>0:B\cap S_d(x;\varepsilon_x)=\varnothing\ --(I)$$

Similarly

$$y\in B\Rightarrow y\notin\overline{A}\Rightarrow\exists\,\varepsilon_y>0:A\cap S_d(y;\varepsilon_y)=\varnothing\ --(II)$$

If we let $U=\bigcup_{x\in A}S_d(x;\frac{\varepsilon_x}{3})$ and $V=\bigcup_{y\in B}S_d(y;\frac{\varepsilon_y}{3})$ and clearly that $A\subset U$, $B\subset V$ and we are going to prove that $U\cap V=\varnothing$ for if $U\cap V\neq\varnothing$ then there is a point $z\in U\cap V$ according to which there are two points $x\in A$ and $y\in B$ such that $z\in S_d(x;\frac{\varepsilon_x}{3})\cap S_d(y;\frac{\varepsilon_y}{3})$ from which $d(y,z)<\frac{\varepsilon_y}{3}$ and $d(x,z)<\frac{\varepsilon_x}{3}$ then by setting $\varepsilon=\max.\{\varepsilon_x,\varepsilon_y\}$ we finds

$$d(x,y)\le d(x,z)+d(z,y)<\frac{\varepsilon_x}{3}+\frac{\varepsilon_y}{3}=\frac{2}{3}\varepsilon$$

and we have two possibilities $\varepsilon=\varepsilon_x\ --(i)$ or $\varepsilon=\varepsilon_y\ --(ii)$

and we gets

$$\begin{aligned}\varepsilon=\varepsilon_x&\Rightarrow d(x,y)<\frac{2}{3}\varepsilon_x<\varepsilon_x\Rightarrow y\in S_d(x;\varepsilon_x)\\&\Rightarrow B\cap S_d(x;\varepsilon_x)\neq\varnothing\end{aligned}$$

which contradicts (I) or

$$\varepsilon=\varepsilon_y \Rightarrow d(x,y)<\frac{2}{3}\varepsilon_y<\varepsilon_y \Rightarrow x\in S_d(y;\varepsilon_y)$$
$$\Rightarrow A\cap S_d(y;\varepsilon_y)\neq\varnothing$$

which contradicts (II).

The contradiction in both cases (I) and (II) because of incorrect assumption $U\cap V\neq\varnothing$ and the correct one is $U\cap V=\varnothing$.

Theorem 8.3.3. *Any metric space* (X,d) *is a* T_4-*space.*

Proof: From Theorem 8.3.1, any metric space is a T_1-space and from Theorem 8.3.2, any metric space is a normal space. Then any metric space is a T_4-space.

Remark 8.3.1.

$$\textit{Metric space}\rightarrow T_4\rightarrow T_3\rightarrow T_2\rightarrow T_1\rightarrow T_o.$$

Exercise

1 – If $X=\{1,2,3\}$, $Y=\{x,y,z\}$, $\tau=\{X,\varnothing,\{1\},\{1,2\}\}$ and $\tau^*=\{Y,\varnothing,\{x\},\{y,z\}\}$, write $\tau\times\tau^*$.

2 – If (X,τ) and (Y,τ^*) are two topological spaces and $A\subset X$ and $B\subset Y$, prove that $A\times B$ is dense in $X\times Y$ iff A dense in X and B is dense in Y i.e. prove that

$$\overline{A\times B}=X\times Y \Leftrightarrow \overline{A}=X\wedge\overline{B}=Y.$$

3 – Consider the usual topological space $(R^2,\mho^2)$ and suppose that $M=\{(x,y)\in R^2 : x\in(0,1), y\in(2,3)\}$ and find $\overline{M}$, M^o, $ext(M)$ and $b(M)$.

4 – If $f:(X,\tau)\rightarrow(Y,\tau^*)$ is continuous and open and $f\subset X\times Y$ is a closed set prove that (Y,τ^*) is a T_2-space.

5 – Prove that $(X \times Y, \tau \times \tau^*)$ is a separable space iff (X, τ) and (Y, τ^*) are both separable.

6 – Prove that $M = \{(x, y) \in R^2 : x \in Q \vee y \in Q\}$ is a connected set.

7 – If (X, τ_1), (Y, τ_2), (Z, τ_3), (W, τ_4) are topological spaces, $f : Z \to W$, $M \subset Z$ and $f \times g : X \times Z \to Y \times W$ is such that $(f \times g)(x, z) = (f(x), g(z))$; $\forall (x, z) \in X \times Z$.

Prove that $f \times g$ is continuous iff f and g are both continuous.

8 – If d is a distance on X prove that $\rho : X \times X \Rightarrow R$ is a distance on X in each of the following cases:

(i) $\rho(x, y) = \min.\{d(x, y), 1\}$,

(ii) $\rho(x, y) = c\, d(x, y)$ and

(iii) $\rho(x, y) = \begin{cases} c; & x \neq y \\ 0; & x = y \end{cases}$.

9 – Prove that any of ρ_1 and ρ_2 are distances on R^2:

(i) $\rho_1(x, y) = \max.\{|x_1 - y_1|, |x_2 - y_2|\}$ and

(ii) $\rho_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

For each $x, y \in R^2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

10 – If d is a distance on R^2 prove that

$$\rho_1(x, y) \leq d(x, y) \leq \sqrt{2}\, \rho_2(x, y)$$

where ρ_1 and ρ_2 are two distances on R^2 defined in exercise (10). Find the two spheres $S_{\rho_1}(x; 1)$ and $S_{\rho_2}(x; 1)$.

11– If $X \neq \varnothing$ and $d : X \times X \to R$ defined to be such that $\rho(x, y) = \min.\{d(x, y), 1\}$, prove that ρ and d are equivalent.

12 – If (X, d_1) and (X, d_2) are two metric spaces and if the function $d : (X \times Y) \times (X \times Y) \to R$ is such that for each $x, y \in R^2$, $d(x, y) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$ where $x = (x_1, y_1)$ and $y = (x_2, y_2)$, prove that d in this case a distance on $X \times Y$ and if τ_d is the metric topology generated on $X \times Y$ and τ_{d_1} and τ_{d_2} are the metric topologies on X generated by d_1 and on Y generated by d_2, prove that $\tau_d = \tau_{d_1} \times \tau_{d_2}$.

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