Applications of Laplace transform to circuit analysis

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PROPERTIES OF THE LAPLACE TRANSFORM

The properties of the Laplace transform help us to obtain transform pairs without directly using Eq. As we derive each of these properties, we should keep in mind the definition of the Laplace transform in Eq.

Table 1 provides a list of the properties of the Laplace transform. The last property (on convolution) will be proved later. There are other properties, but these are enough for present purposes.

Table 2 summarizes the Laplace transforms of some common functions. We have omitted the factor *u(t)* except where it is necessary.

TABLE | Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$

TABLE | Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_1(t)$	$F_1(s)F_2(s)$

TABLE .2 Laplace transform pairs.

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$

TABLE .2 Laplace transform pairs.

$f(t)$	$F(s)$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

EXAMPLE

Obtain the Laplace transform of $f(t) = \delta(t) + 2u(t) - 3e^{-2t}$, $t \geq 0$.

SOLUTION

By the linearity property,

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}]$$

$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2 + s + 4}{s(s+2)}$$

EXAMPLE

Find the Laplace transform of $f(t) = \cos 2t + e^{-3t}$, $t \geq 0$.

SOLUTION

$$F(s) = \frac{S}{S^2 + 4} + \frac{1}{S+3} = \frac{S(S+3) + (S^2 + 4)}{(S^2 + 4)(S+3)}$$

$$= \frac{2S^2 + 3S + 4}{(S^2 + 4)(S+3)}$$

THE INVERSE LAPLACE TRANSFORM

Steps to Find the Inverse Laplace Transform:

1. Decompose $F(s)$ into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Table .2.

Let us consider the three possible forms $F(s)$ may take and how to apply the two steps to each form.

I Simple Poles

a simple pole is a first-order pole. If $F(s)$ has only simple poles, then $D(s)$ becomes a product of factors, so that

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad (\ .48)$$

where $s = -p_1, -p_2, \ldots, -p_n$ are the simple poles, and $p_i \neq p_j$ for all $i \neq j$ (i.e., the poles are distinct). Assuming that the degree of $N(s)$ is less than the degree of $D(s)$, we use partial fraction expansion to decompose $F(s)$ in Eq. (.48) as

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n} \qquad (\ .49)$$

The expansion coefficients $k_1, k_2, \ldots, k_n$ are known as the *residues* of $F(s)$. There are many ways of finding the expansion coefficients. One way is using the *residue method*. If we multiply both sides of Eq. (.49) by $(s+p_1)$, we obtain

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n} \qquad (\ .50)$$

Since $p_i \neq p_j$, setting $s = -p_1$ in Eq. (.50) leaves only k_1 on the right-hand side of Eq. (.50). Hence,

$$(s+p_1)F(s)\Big|_{s=-p_1} = k_1 \qquad (\ .51)$$

2 Repeated Poles

Suppose $F(s)$ has n repeated poles at $s = -p$. Then we may represent $F(s)$ as

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s) \qquad (\quad .54)$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$. We determine the expansion coefficient k_n as

$$k_n = (s+p)^n F(s) \Big|_{s=-p} \qquad (\quad .55)$$

as we did above. To determine k_{n-1}, we multiply each term in Eq. (.54) by $(s + p)^n$ and differentiate to get rid of k_n, then evaluate the result at $s = -p$ to get rid of the other coefficients except k_{n-1}. Thus, we obtain

$$k_{n-1} = \frac{d}{ds}[(s + p)^n F(s)] \Big|_{s=-p} \qquad (.56)$$

Repeating this gives

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s + p)^n F(s)] \Big|_{s=-p} \qquad (.57)$$

The mth term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s + p)^n F(s)] \Big|_{s=-p} \qquad (.58)$$

where $m = 1, 2, \ldots, n - 1$. One can expect the differentiation to be difficult to handle as m increases. Once we obtain the values of k_1, $k_2, \ldots, k_n$ by partial fraction expansion, we apply the inverse transform

$$\mathcal{L}^{-1}\left[\frac{1}{(s + a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n - 1)!} \qquad (.59)$$

to each term in the right-hand side of Eq. (.54) and obtain

$$f(t) = k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!} t^2 e^{-pt} + \cdots + \frac{k_n}{(n - 1)!} t^{n-1} e^{-pt} + f_1(t) \qquad (.60)$$

3 Complex Poles

A pair of complex poles is simple if it is not repeated; it is a double or multiple pole if repeated. Simple complex poles may be handled the same as simple real poles, but because complex algebra is involved the result is always cumbersome. An easier approach is a method known as *completing the square*. The idea is to express each complex pole pair (or quadratic term) in $D(s)$ as a complete square such as $(s+\alpha)^2+\beta^2$ and then use Table 2 to find the inverse of the term.

Since $N(s)$ and $D(s)$ always have real coefficients and we know that the complex roots of polynomials with real coefficients must occur in conjugate pairs, $F(s)$ may have the general form

$$F(s)=\frac{A_1 s+A_2}{s^2+as+b}+F_1(s) \qquad (\quad .61)$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles. If we complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s + \alpha)^2 + \beta^2 \qquad \text{(1 .62)}$$

and we also let

$$A_1 s + A_2 = A_1(s + \alpha) + B_1 \beta \qquad \text{(.63)}$$

then Eq. (.61) becomes

$$F(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2} + F_1(s) \qquad \text{(.64)}$$

From Table 2, the inverse transform is

$$\boxed{f(t) = A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t + f_1(t)} \qquad \text{(.65)}$$

The sine and cosine terms can be combined using Eq. (12).

EXAMPLE

Determine the inverse Laplace transform of

$$F(s) = 1 + \frac{4}{s+3} - \frac{5s}{s^2+16}$$

SOLUTION

$$\delta(t) + 4e^{-3t} - 5\cos 4t, \; t \geq 0.$$

EXAMPLE

Find $f(t)$ given that

SOLUTION

$$F(s) = \frac{s^2+12}{s(s+2)(s+3)}$$

Unlike in the previous example where the partial fractions have been provided, we first need to determine the partial fractions. Since there are three poles, we let

$$\frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \tag{1}$$

where A, B, and C are the constants to be determined. We can find the constants using two approaches.

METHOD 1 **Residue method:**

$$A = sF(s)\Big|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}\Bigg|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)\Big|_{s=-2} = \frac{s^2+12}{s(s+3)}\Bigg|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)\Big|_{s=-3} = \frac{s^2+12}{s(s+2)}\Bigg|_{s=-3} = \frac{9+12}{(-3)(-1)} = 7$$

Thus $A = 2$, $B = -8$, $C = 7$, and Eq. (15.9.1) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = 2u(t) - 8e^{-2t} + 7e^{-3t}, \qquad t \geq 0.$$

METHOD 2 **Algebraic method:** Multiplying both sides of Eq. (15.9.1) by $s(s+2)(s+3)$ gives

$$s^2 + 12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

or

$$s^2 + 12 = A(s^2 + 5s + 6) + B(s^2 + 3s) + C(s^2 + 2s)$$

Equating the coefficients of like powers of s gives

Constant: $12 = 6A \implies A = 2$

s: $0 = 5A + 3B + 2C \implies 3B + 2C = -10$

s^2: $1 = A + B + C \implies B + C = -1$

Thus $A = 2$, $B = -8$, $C = 7$, and Eq. (15.9.1) becomes

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain

$$f(t) = 2u(t) - 8e^{-2t} + 7e^{-3t}, \qquad t \geq 0.$$

APPLICATION TO CIRCUITS

Having mastered how to obtain the Laplace transform and its inverse, we are now prepared to employ the Laplace transform to analyze circuits. This usually involves three steps.

Steps in applying the Laplace transform :

1. Transform the circuit from the time domain to the *s domain*.

2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.

3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or s domain by Laplace transforming each term in the circuit.

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t)$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)}$$

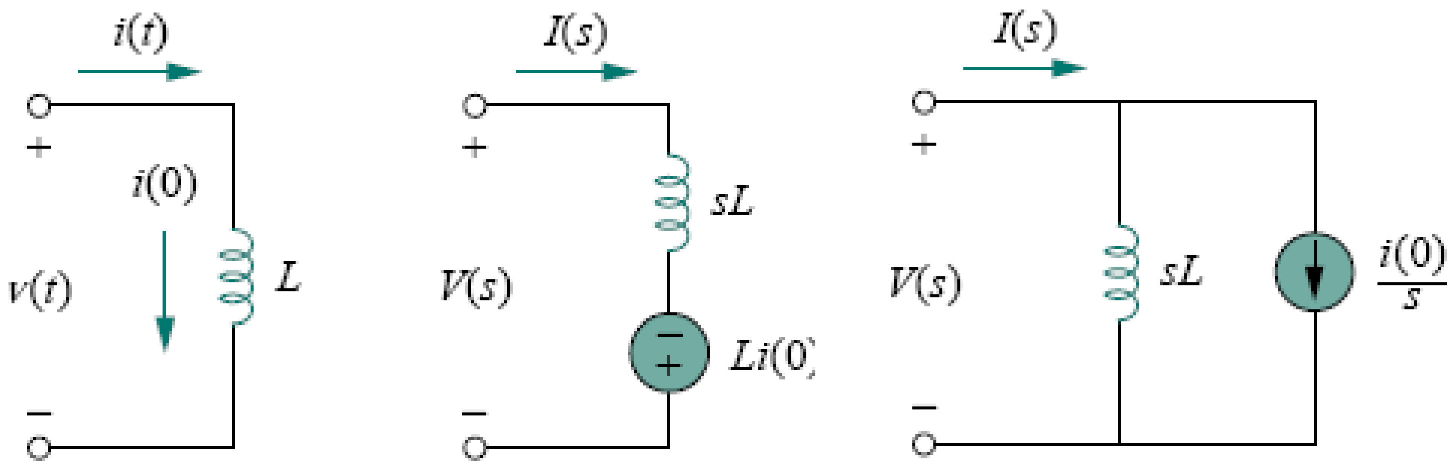
For an inductor, the voltage-current relationship in the time domain is

$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform of both sides gives

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

The S-domain equivalents are shown in Fig, where the initial condition is modeled as a voltage or current source.



(a) Time-domain (b,c) S-domain equivalents

For a capacitor, the voltage-current relationship in the time domain is

$$i(t) = C\frac{dv(t)}{dt}$$

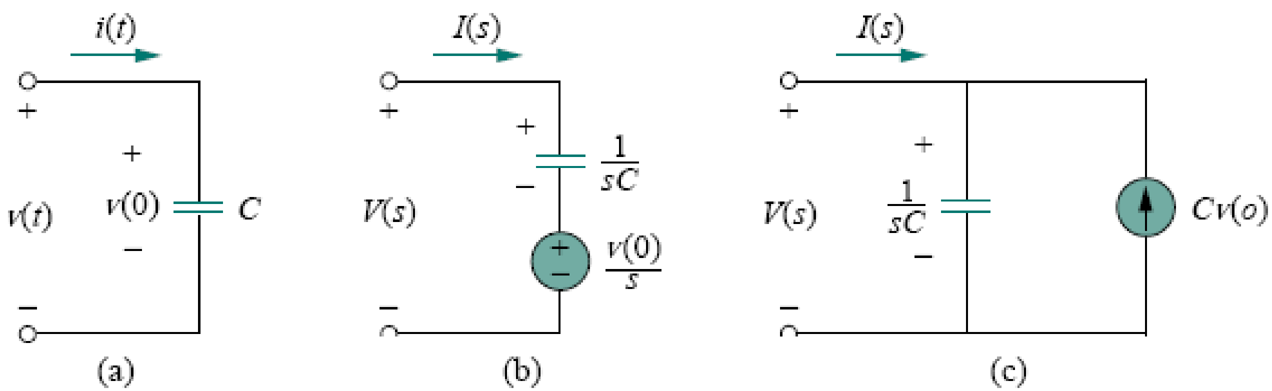
which transforms into the s domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

or

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

The S-domain equivalents are shown in Fig, where the initial condition is modeled as a voltage or current source.



=V(0)/s/1/sC

Figure 11 Representation of a capacitor: (a) time-domain, (b,c) s-domain equivalents.

If we assume zero initial conditions for the inductor and capacitor, the above equations reduced to:

$$\text{Resistor:} \quad V(s) = R\,I(s)$$

$$\text{Inductor:} \quad V(s) = sL\,I(s)$$

$$\text{Capacitor:} \quad V(s) = \frac{1}{sC}I(s)$$

The S-domain equivalents are shown in Figure

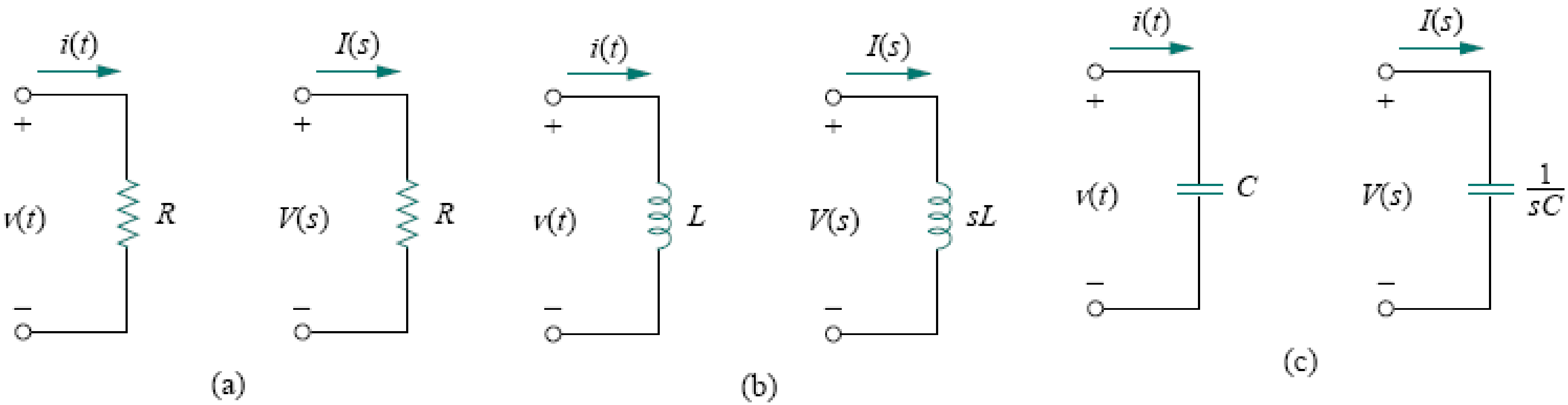


Figure 12 Time-domain and s-domain representations of passive elements under zero initial conditions.

We define the impedance in the s-domain as the ratio of the voltage transform to the current transform under zero initial conditions, this is

$$Z(s) = \frac{V(s)}{I(s)}$$

Thus the impedances of the three elements are

Resistor: $Z(s) = R$

Inductor: $Z(s) = sL$

Capacitor: $Z(s) = \frac{1}{sC}$

Table 3 summarizes these. The admittance in the s domain in the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal.

TABLE .3 Impedance of an element in the s domain.

Element	$Z(s) = V(s)/I(s)$
Resistor	R
Inductor	sL
Capacitor	$1/sC$

EXAMPLE

Find $v_o(t)$ in the circuit in Fig. 13, assuming zero initial conditions.

SOLUTION

We first transform the circuit from the time to
The s domain

$$u(t) \implies \frac{1}{s}$$

$$1\text{ H} \implies sL = s$$

$$\frac{1}{3}\text{ F} \implies \frac{1}{sC} = \frac{3}{s}$$

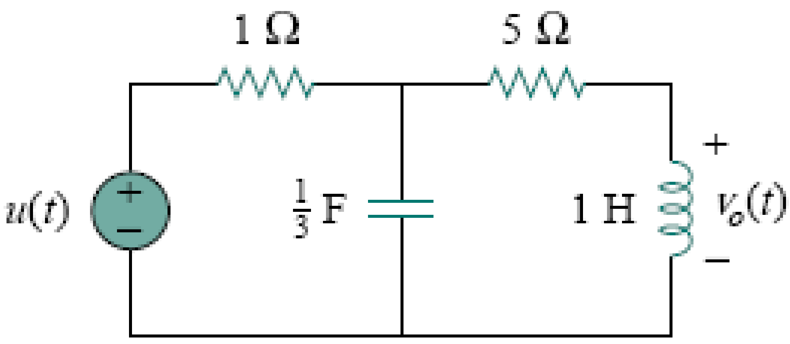


Figure .13

The resulting s-domain circuit is in Fig. 14. we now apply mesh analysis.
analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 \qquad (.1)$$

For mesh 2,

$$0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2$$

or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3) I_2 \qquad (.2)$$

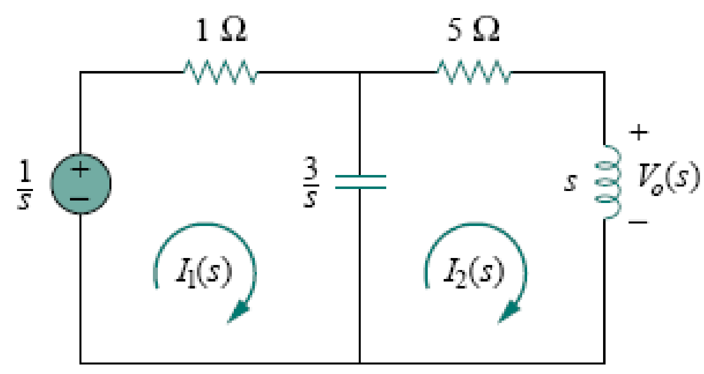


Figure 14 Mesh analysis of the frequency-domain equivalent of the same circuit.

Substituting this into Eq. (.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \qquad \Longrightarrow \qquad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t \text{ V}, \qquad t \geq 0$$

EXAMPLE

Find $v_o(t)$ in the circuit of Fig. 16. Assume $v_o(0) = 5$ V.

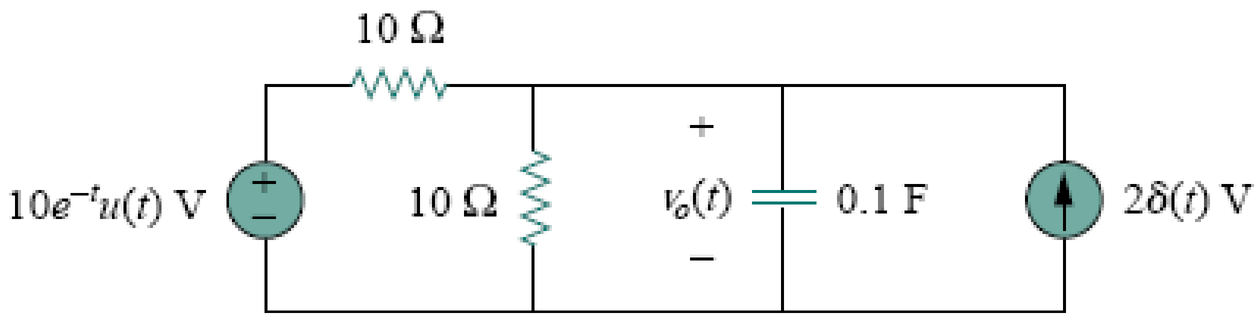


Figure 16

We transform the circuit to the s domain as shown in Fig. 17. **the initial condition is included in the from of the current source $CV_o(0)= 0.1(5)=0.5A$.**

We apply nodal analysis at the top node,

$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

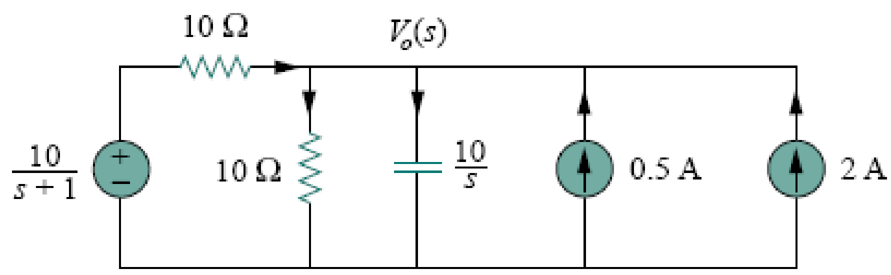


Figure 17 Nodal analysis of the equivalent of the circuit in Fig. 16.

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = (s+1)V_o(s)\ \Big|_{s=-1} = \frac{25s+35}{(s+2)}\Bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\ \Big|_{s=-2} = \frac{25s+35}{(s+1)}\Bigg|_{s=-2} = \frac{-15}{-1} = 15$$

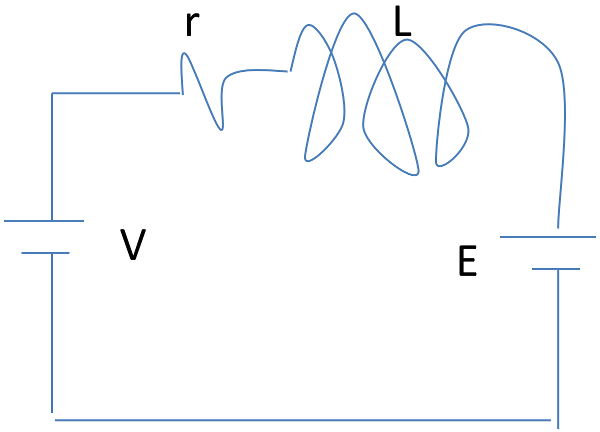
Thus,

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t)$$

The R-L-E load and denoting the initial current at t=0 as I_o



$$V = ri + L\frac{di}{dt} + E$$

Taking Laplace transform

$$\frac{V}{s} = rI(s) + L(sI(s) - I_o) + \frac{E}{S}$$

$$\frac{V}{s} = rI(s) + LsI(s) - LI_o + \frac{E}{S}$$

$$\frac{V-E}{s} = I(s)[r + Ls] - LI_o$$

$$I(s) = \frac{V-E}{Ls(s + {}^{r}/_{L})} + \frac{LI_o}{L(s + {}^{r}/_{L})}$$

$$I(s) = \frac{V - {E}/{s}}{r + Ls} + \frac{LI_o}{r + Ls}$$

The first term of the equation can be resolved as:

$$\frac{V-E}{Ls\left(s+{}^{r}/_{L}\right)} = \frac{A}{s} + \frac{B}{\left(s+{}^{r}/_{L}\right)}$$

Where

$$A = s\frac{V-E}{Ls\left(s+{}^{r}/_{L}\right)}\Bigg|_{s=0} = \frac{V-E}{r}$$

$$B = (s+\frac{r}{L})\frac{V-E}{Ls\left(s+{}^{r}/_{L}\right)}\Bigg|_{s=\frac{-r}{L}} = -\frac{V-E}{r}$$

$$I(s) = \frac{V-E}{r}\frac{1}{s} + \frac{V-E}{s+{}^{r}/_{L}} + \frac{I_o}{\left(s+{}^{r}/_{L}\right)}$$

Taking the inverse Laplace transform:

$$i(t) = \frac{V-E}{r} - \frac{V-E}{r}e^{-\frac{rt}{L}} + I_o e^{-\frac{rt}{L}}$$

$$i(t) = \frac{V-E}{r}(1-e^{-\frac{rt}{L}}) + I_o e^{-\frac{rt}{L}}$$