Umm Al-Qura University
Faculty of Applied Science
Department of Physics
Classical Mechanics (2)
Code: 403220-3
Date: 6 / 11 /2017

جامعة ام القرى
كلية العلوم التطبيقية
قسم الفيزياء
مقرر: ميكانيكا كلاسيكية (٢)
كود المقرر: ٤٠٣٢٢٠-٣
الاثنين ١٧ صفر ١٤٣٩

العام الجامعي ١٤٣٨-١٤٣٩
الفصل الدراسي الأول

Periodic Exam
[Total marks: 30]

الاسم: ....

الرقم الجامعي: ....

الشعبة: ....

الرقم التسلسلي: ....

30

أستاذة المقرر: د/ فاطمة السيد محروس

Answer the following questions:

Question (1) (9 marks)

A system consists of three particles, each of unit mass, with positions and velocities are

$$\vec{\mathbf{r}}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}} \qquad \vec{\mathbf{r}}_2 = \hat{\mathbf{j}} + \hat{\mathbf{k}} \qquad \vec{\mathbf{r}}_3 = \hat{\mathbf{k}}$$

$$\vec{\mathbf{v}}_1 = 2\hat{\mathbf{i}} \qquad \vec{\mathbf{v}}_2 = \hat{\mathbf{j}} \qquad \vec{\mathbf{v}}_3 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Find:

(1) The position of the center of mass.

(2) The velocity of the center of mass.

(3) The linear momentum of the system.

(4) The angular momentum of the system.

(5) The kinetic energy of the system.

(6) The value of $mv_{cm}^2/2$.

Question (2) (6 marks)

A proton of mass m_p with initial velocity $\mathbf{v}_o$ collides with a helium atom, mass $4m_p$, that is initially at rest. If the proton leaves the point of impact at an angle 45^o with its original line of motion, find the final velocities of each particle. Assume that the collision is inelastic and that Q is equal to $\frac{1}{4}$ of the initial energy of the proton.

Question (3) (6 marks)

Prove that "the moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes".

Question (4) (9 marks)

(a) Find the center of mass of a solid homogeneous hemisphere of radius a.

(b) Show that the moment of inertia for a thin uniform rod of length $2a$ and mass m about an axis perpendicular to the rod at one end is $\frac{4}{3}ma^2$.

(c) Find the radius of gyration of a thin uniform rod of length a and mass m about an axis passing through one end.

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$\vec{\mathbf{v}}_1 = 2\hat{\mathbf{i}}$ $\qquad$ $\vec{\mathbf{v}}_2 = \hat{\mathbf{j}}$ $\qquad$ $\vec{\mathbf{v}}_3 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Find:

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(3) The linear momentum of the system.

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(5) The kinetic energy of the system.

(6) The value of $mv_{cm}^2/2$.

$$\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$$

$$\vec{r}_{cm} = \frac{1}{3}\Big((\hat{\mathbf{i}} + \hat{\mathbf{j}}) + (\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (\hat{\mathbf{k}})\Big) = \frac{1}{3}(\hat{\mathbf{i}} + \mathbf{2}\hat{\mathbf{j}} + \mathbf{2}\hat{\mathbf{k}})$$

$$\vec{v}_{cm} = \frac{1}{m}\sum_i m_i \vec{v}_i$$

$$\vec{v}_{cm} = \frac{1}{3}\Big((2\hat{\mathbf{i}}) + (\hat{\mathbf{j}}) + (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\Big) = \frac{1}{3}(3\hat{\mathbf{i}} + \mathbf{2}\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$\vec{p} = \sum_i m_i \vec{v}_i$$

$$\vec{p} = \Big((2\hat{\mathbf{i}}) + (\hat{\mathbf{j}}) + (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\Big) = (3\hat{\mathbf{i}} + \mathbf{2}\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$\vec{\mathrm{L}} = \sum_i \vec{r}_i \times m\vec{v}_i$$

$$\vec{\mathrm{L}} = [(\hat{\mathbf{i}} + \hat{\mathbf{j}})\times(2\hat{\mathbf{i}}) + (\hat{\mathbf{j}} + \hat{\mathbf{k}})\times(\hat{\mathbf{j}}) + (\hat{\mathbf{k}})\times(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})] = -2\hat{\mathbf{k}} - \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{i}} = -\mathbf{2}\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$\mathrm{T} = \frac{1}{2}[(2)^2 + (1)^2 + ((1)^2 + (1)^2 + (1)^2)] = \frac{8}{2} = 4$$

$$\frac{1}{2}mv_{cm}^2 = \frac{1}{2}(3)\left(\frac{1}{9}((3)^2 + (2)^2 + (1)^2)\right) = \frac{1}{2}(3)\left(\frac{1}{9}(9 + 4 + 1)\right) = \frac{7}{3}$$

Question (2) (6 marks)

A proton of mass m_p with initial velocity $\mathbf{v}_o$ collides with a helium atom, mass $4m_p$, that is initially at rest. If the proton leaves the point of impact at an angle 45^o with its original line of motion, find the final velocities of each particle. Assume that the collision is inelastic and that Q is equal to $\frac{1}{4}$ of the initial energy of the proton.

Conservation of momentum:

$$v_o = \acute{v}_p cos45 + 4\acute{v}_\alpha cos\phi \qquad \acute{v}_p sin45 - 4\acute{v}_\alpha sin\phi = 0$$

$$4\acute{v}_\alpha cos\phi = v_o - \frac{\acute{v}_p}{\sqrt{2}} \qquad 4\acute{v}_\alpha sin\phi = \frac{\acute{v}_p}{\sqrt{2}}$$

$$16v_\alpha'^2 = v_o^2 - \sqrt{2}v_o\acute{v}_p + v_p'^2$$

Conservation of energy:

$$\frac{1}{2}m_p v_o^2 = \frac{1}{2}m_p v_p'^2 + \frac{1}{2}4m_p v_\alpha'^2 + \frac{1}{4}\left(\frac{1}{2}m_p v_o^2\right)$$

$$16v_\alpha'^2 = 3v_o^2 - 4v_p'^2$$

subtracting:

$$v_o^2 - \sqrt{2}v_o\acute{v}_p + v_p'^2 - \left(3v_o^2 - 4v_p'^2\right) = 0$$

$$-2v_o^2 - \sqrt{2}v_o\acute{v}_p + 5v_p'^2 = 0$$

$$\acute{v}_p = \frac{\sqrt{2}v_o \pm \sqrt{2v_o^2 + 40v_o^2}}{10} = \frac{v_o}{10}\left(\sqrt{2} \pm \sqrt{42}\right)$$

$$\acute{v}_p = 0.7895v_o\,, \ \ \acute{v}_{px} = \acute{v}_{px} = \frac{\acute{v}_p}{\sqrt{2}} = 0.558\, v_o$$

$$\acute{v}_\alpha = \sqrt{\left(\frac{3}{16}v_o^2 - \frac{1}{4}v_p'^2\right)} = \frac{v_o}{2}\sqrt{(0.75 - 0.7895^2)} = 0.1780\, v_o$$

$$tan\phi = \frac{\frac{\acute{v}_p}{\sqrt{2}}}{v_o - \frac{\acute{v}_p}{\sqrt{2}}} = \frac{\acute{v}_p}{\sqrt{2}v_o - \acute{v}_p} = \frac{0.7895}{\sqrt{2} - 0.7895} = 1.2638$$

$$\phi = tan^{-1}(1.2638) = 51.65^o$$

$$\acute{v}_{\alpha x} = \acute{v}_\alpha cos\phi = 0.110v_o\,, \quad \acute{v}_{\alpha y} = -\acute{v}_\alpha sin\phi = -0.140\, v_o$$

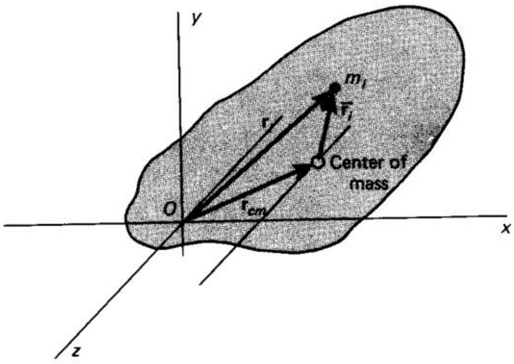
Question (3) (6 marks)

Prove that "the moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes".

Consider the equation of the moment of inertia about some axis, say the *z*-axis,

$$I = \sum_i m_i (x_i^2 + y_i^2)$$

Now we can express x_i and y_i in terms of the coordinates of the center of mass (x_{cm}, y_{cm}, z_{cm}) and the coordinates relative to the center of mass $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ as follows:



$$x_i = x_{cm} + \bar{x}_i \ , \qquad y_i = y_{cm} + \bar{y}_i$$

$$I = \sum_i m_i \left(x_i^2 + y_i^2\right) = \sum_i m_i [(x_{cm} + \bar{x}_i)^2 + (y_{cm} + \bar{y}_i)^2]$$

$$I = \sum_i m_i \left(\bar{x}_i^2 + \bar{y}_i^2\right) + \sum_i m_i (x_{cm}^2 + y_{cm}^2) + 2x_{cm} \sum_i m_i \, \bar{x}_i + 2y_{cm} \sum_i m_i \, \bar{y}_i$$

The first sum on the right is just the moment of inertia about an axis parallel to the *z*-axis and passing through the center of mass. We call it I_{cm}. The second sum is equal to the mass of the body multiplied by the square of the distance between the center of mass and the *z*-axis. we call this distance l. That is $l^2 = \left(x_{cm}^2 + y_{cm}^2\right)$. From the definition of the center of mass,

$$\sum_i m_i \, \bar{x}_i = \sum_i m_i \, \bar{y}_i = 0 \qquad \Rightarrow \qquad \therefore I = I_{cm} + m \, l^2$$

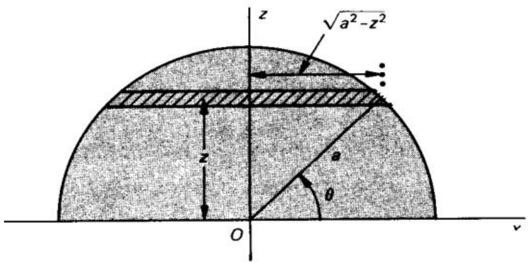
Question (4) (9 marks)

(a) Find the center of mass of a solid homogeneous hemisphere of radius a.

(b) Show that the moment of inertia for a thin uniform rod of length $2a$ and mass m about an axis perpendicular to the rod at one end is $\frac{4}{3}ma^2$.

(c) Find the radius of gyration of a thin uniform rod of length a and mass m about an axis passing through one end.

To find the center of mass of a solid homogeneous hemisphere of radius a, we know from symmetry that the center of mass lies on the radius that is normal to the plane face.



$$dV = \pi\, r^2 dz = \pi(a^2 - z^2)dz$$

$$z_{cm} = \frac{\int_0^a \rho\pi z(a^2 - z^2)dz}{\int_0^a \rho\pi(a^2 - z^2)dz} = \frac{\left.\frac{a^2z^2}{2} - \frac{z^4}{4}\right|_0^a}{\left.a^2z - \frac{z^3}{3}\right|_0^a} = \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - \frac{a^3}{3}} = \frac{\frac{a^4}{4}}{\frac{2a^3}{3}} = \frac{3}{8}a$$

For a thin uniform rod of length $2a$ and mass m, for an axis perpendicular to the rod at one end is

Figure 8.3.2 Coordinates for finding the moment of inertia of a disc.

Circular Disc or Cylinder

$$I = \int_0^{2a} x^2\rho\, dx = \rho\left.\frac{a^3}{3}\right|_0^{2a} = \frac{1}{3}\left(\frac{m}{2a}\right)(8a^3) = \frac{4}{3}ma^2$$

$$I = \int_0^{a} x^2\rho\, dx = \rho\left.\frac{a^3}{3}\right|_0^{a} = \frac{1}{3}\left(\frac{m}{a}\right)(a^3) = \frac{1}{3}ma^2$$

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\left(\frac{1}{3}ma^2\right)}{m}} = \frac{a}{\sqrt{3}}$$

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أستاذة المقرر: د/ فاطمة السيد محروس

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$$\vec{\mathbf{v}}_1 = \hat{\mathbf{i}} \qquad \vec{\mathbf{v}}_2 = 2\hat{\mathbf{j}} \qquad \vec{\mathbf{v}}_3 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

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$$\vec{\mathbf{v}}_1 = \hat{\mathbf{i}} \qquad \vec{\mathbf{v}}_2 = 2\hat{\mathbf{j}} \qquad \vec{\mathbf{v}}_3 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

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$$\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$$

$$\vec{r}_{cm} = \frac{1}{3}\left((2\hat{\mathbf{i}} + \hat{\mathbf{j}}) + (\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (2\hat{\mathbf{k}})\right) = \frac{1}{3}(2\hat{\mathbf{i}} + \mathbf{2}\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$\vec{v}_{cm} = \frac{1}{m}\sum_i m_i \vec{v}_i$$

$$\vec{v}_{cm} = \frac{1}{3}\left((\hat{\mathbf{i}}) + (2\hat{\mathbf{j}}) + (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\right) = \frac{1}{3}(2\hat{\mathbf{i}} + \mathbf{3}\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$\vec{p} = \sum_i m_i \vec{v}_i$$

$$\vec{p} = \left((\hat{\mathbf{i}}) + (2\hat{\mathbf{j}}) + (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\right) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$\vec{\mathrm{L}} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$$\vec{\mathrm{L}} = [(2\hat{\mathbf{i}} + \hat{\mathbf{j}})\times(\hat{\mathbf{i}}) + (\hat{\mathbf{j}} + \hat{\mathbf{k}})\times(2\hat{\mathbf{j}}) + (2\hat{\mathbf{k}})\times(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})] = -\hat{\mathbf{k}} - 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{i}}$$

$$= -4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

$$\mathrm{T} = \frac{1}{2}[(1)^2 + (2)^2 + ((1)^2 + (1)^2 + (1)^2)] = \frac{8}{2} = 4$$

$$\frac{1}{2}mv_{cm}^2 = \frac{1}{2}(3)\left(\frac{1}{9}((2)^2 + (3)^2 + (1)^2)\right) = \frac{1}{2}(3)\left(\frac{1}{9}(4 + 9 + 1)\right) = \frac{7}{3}$$

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Conservation of momentum:

$$v_o = \acute{v}_p cos45 + 4\acute{v}_\alpha cos\phi \qquad \acute{v}_p sin45 - 4\acute{v}_\alpha sin\phi = 0$$

$$4\acute{v}_\alpha cos\phi = v_o - \frac{\acute{v}_p}{\sqrt{2}} \qquad 4\acute{v}_\alpha sin\phi = \frac{\acute{v}_p}{\sqrt{2}}$$

$$16v_\alpha'^2 = v_o^2 - \sqrt{2}v_o\acute{v}_p + v_p'^2$$

Conservation of energy:

$$\frac{1}{2}m_p v_o^2 = \frac{1}{2}m_p v_p'^2 + \frac{1}{2}4m_p v_\alpha'^2$$

$$16v_\alpha'^2 = 4v_o^2 - 4v_p'^2$$

subtracting:

$$v_o^2 - \sqrt{2}v_o\acute{v}_p + v_p'^2 - \left(4v_o^2 - 4v_p'^2\right) = 0$$

$$-3v_o^2 - \sqrt{2}v_o\acute{v}_p + 5v_p'^2 = 0$$

$$\acute{v}_p = \frac{\sqrt{2}v_o \pm \sqrt{2v_o^2 + 60v_o^2}}{10} = \frac{v_o}{10}\left(\sqrt{2} \pm \sqrt{62}\right)$$

$$\acute{v}_p = 0.9288v_o\,,\ \ \acute{v}_{px} = \acute{v}_{px} = \frac{\acute{v}_p}{\sqrt{2}} = 0.6568\,v_o$$

$$\acute{v}_\alpha = \frac{1}{2}\sqrt{\left(v_o^2 - v_p'^2\right)} = \frac{v_o}{2}\sqrt{(1 - 0.9288^2)} = 0.1853\,v_o$$

$$tan\phi = \frac{\frac{\acute{v}_p}{\sqrt{2}}}{v_o - \frac{\acute{v}_p}{\sqrt{2}}} = \frac{\acute{v}_p}{\sqrt{2}v_o - \acute{v}_p} = \frac{0.9288}{\sqrt{2} - 0.9288} = 1.9134$$

$$\phi = tan^{-1}(1.9134) = 62.41^o$$

$$\acute{v}_{\alpha x} = \acute{v}_\alpha cos\phi = 0.086v_o\,, \quad \acute{v}_{\alpha y} = -\acute{v}_\alpha sin\phi = -0.164\,v_o$$

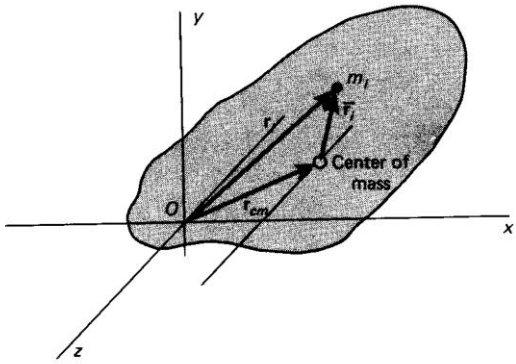
Question (3) (6 marks)

Prove that "the moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes".

Consider the equation of the moment of inertia about some axis, say the *z*-axis,

$$I = \sum_i m_i (x_i^2 + y_i^2)$$

Now we can express x_i and y_i in terms of the coordinates of the center of mass (x_{cm}, y_{cm}, z_{cm}) and the coordinates relative to the center of mass $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ as follows:



$$x_i = x_{cm} + \bar{x}_i \ , \qquad y_i = y_{cm} + \bar{y}_i$$

$$I = \sum_i m_i (x_i^2 + y_i^2) = \sum_i m_i [(x_{cm} + \bar{x}_i)^2 + (y_{cm} + \bar{y}_i)^2]$$

$$I = \sum_i m_i (\bar{x}_i^2 + \bar{y}_i^2) + \sum_i m_i (x_{cm}^2 + y_{cm}^2) + 2x_{cm} \sum_i m_i \bar{x}_i + 2y_{cm} \sum_i m_i \bar{y}_i$$

The first sum on the right is just the moment of inertia about an axis parallel to the *z*-axis and passing through the center of mass. We call it I_{cm}. The second sum is equal to the mass of the body multiplied by the square of the distance between the center of mass and the *z*-axis. we call this distance l. That is $l^2 = (x_{cm}^2 + y_{cm}^2)$. From the definition of the center of mass,

$$\sum_i m_i \bar{x}_i = \sum_i m_i \bar{y}_i = 0 \qquad \Rightarrow \qquad \therefore I = I_{cm} + m\, l^2$$

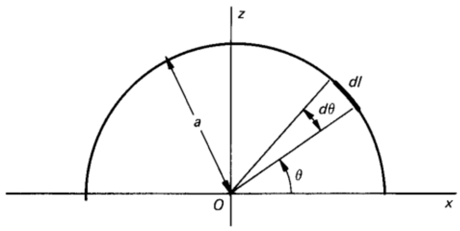
Question (4) (9 marks)

(a) Find the center of mass of a thin wire bent into the form of a semicircle of radius a.

(b) Find the radius of gyration of a thin uniform rod of length a and mass m about an axis passing through the center.

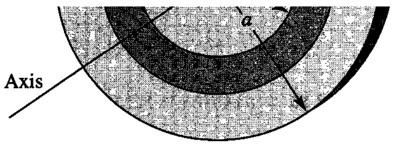
(c) Show that the moment of inertia for a thin uniform rod of length $2a$ and mass m about an axis perpendicular to the rod at one end is $\frac{4}{3}ma^2$.

To find the center of mass of a thin wire bent into the form of a semicircle of a radius a, we use axes as shown in the figure



$$dl = a\,d\theta \qquad \text{and} \qquad z = a\,sin\theta$$

$$z_{cm} = \frac{\int_0^a \rho(a\,sin\theta)\,a\,d\theta}{\int_0^a \rho\,a\,d\theta} = \frac{\int_0^{\pi/2}(a\,sin\theta)\,d\theta}{\int_0^{\pi/2} d\theta} = \frac{a\,[-cos\theta]_0^{\pi/2}}{[\theta]_0^{\pi/2}} = \frac{a}{\frac{\pi}{2}} = \frac{2}{\pi}a$$



For a thin uniform rod of length a and mass m, for an axis perpendicular to the rod at the center is

$$I = \int_{-a/2}^{a/2} x^2\rho\,dx = \frac{1}{12}\rho\,a^3 = \frac{1}{12}\left(\frac{m}{a}\right)a^3 = \frac{1}{12}ma^2$$

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{\left(\frac{1}{12}ma^2\right)}{m}} = \frac{a}{\sqrt{12}}$$

$$I = \int_0^{2a} x^2\rho\,dx = \rho\left.\frac{a^3}{3}\right|_0^{2a} = \frac{1}{3}\left(\frac{m}{2a}\right)(8a^3) = \frac{4}{3}ma^2$$