Umm Al-Qura University
College of Applied Science
Department of Physics
Time: 10.15 - 12.15 AM

جامعة ام القرى
كلية العلوم التطبيقية
قسم الفيزياء
التاريخ: ١٧ ربيع الثاني ١٤٣٩

Final Exam
Academic Year 1438-1439 (1st Semester)

Program: Physics **Course:** Classical Mechanics (2) **Course code:** 403321-3

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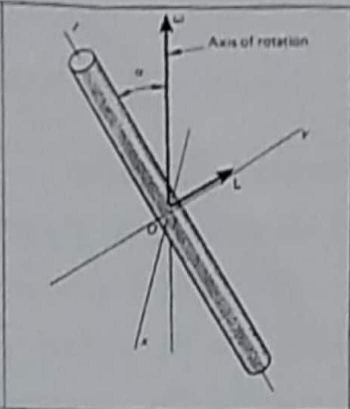
Academic ID: .....436015207........................... **Serial No:** ....23......

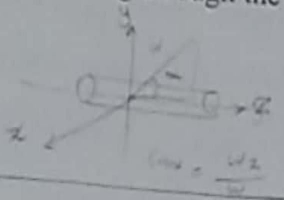
Question		Mark	Signature
Question 1	(10 marks)	10	
Question 2	(10 marks)	10	
Question 3	(10 marks)	10	
Question 4	(10 marks)	10	
Question 5	(5 marks)	5	
Question 6	(5 marks)	5	
Question 7	------		
Question 8	------		
Question 9	------		
Question 10	------		
Total Mark		50 / 50	**Exam Committee** Dr. Doaa Abdallah Dr. Fatma El-Sayed

...estion 1: (10 marks)

$\frac{10}{10}$

...d the product of inertia and the angular momentum vector $\vec{L}$ for ...hin rod of length l and mass m which is constrained to rotate ...th constant angular velocity $\vec{\omega}$ about an axis passing through the ...nter making an angle α with the rod.





$\omega_x = 0 \qquad \omega_y = \omega \sin\alpha \qquad \omega_z = \omega\cos\alpha$

$\because y = 0 \quad x = 0 \qquad \because dm = \lambda\, dl = \lambda\, dz \qquad \because \lambda = \frac{m}{l}$

$$I_{xx} = \int (z^2 + y^2)\, dm = \int_{-l/2}^{l/2} z^2 \lambda\, dz = 2\lambda \left.\frac{z^3}{3}\right|_0^{l/2}$$

$$I_{xx} = \frac{2}{3}\lambda \left(\frac{l}{2}\right)^3 = \frac{2}{3}\lambda \frac{l^3}{8} = \frac{1}{12}\frac{m}{l}\cdot l^3 = \frac{1}{12} m l^2$$

$$I_{yy} = \int (z^2 + x^2)\, dm = \int z^2 \lambda\, dz = \lambda \int_{-l/2}^{l/2} z^2\, dz$$

$$I_{yy} = \frac{2}{3}\lambda\, z^3 \Big|_0^{l/2} = \frac{2}{3}\frac{m}{l}\cdot\frac{l^3}{8} = \frac{1}{12} m l^2$$

$$I_{zz} = \int (x^2 + y^2)\, dm = 0$$

$$I_{xy} = I_{yx} = -\int xy\, dm = 0 \quad , \quad I_{xz} = I_{zx} = -\int xz\, dm = 0$$

$$I_{yz} = I_{zy} = -\int zy\, dm = 0$$

$$\vec{L} = I\vec{\omega}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m l^2 & 0 & 0 \\ 0 & \frac{1}{12} m l^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \omega\sin\alpha \\ \omega\cos\alpha \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m l^2 (0) + \omega\sin\alpha(0) + 0(\omega\cos\alpha) \\ 0(0) + \frac{1}{12} m l^2 (\omega\sin\alpha) + 0(\omega\cos\alpha) \\ 0 + 0(\omega\sin\alpha) + 0(\omega\cos\alpha) \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{12} m\omega l^2 \sin\alpha \\ 0 \end{pmatrix} \Rightarrow \vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$\vec{L} = 0 + \frac{1}{12} m\omega l^2 \sin\alpha\, \hat{j} + 0 = \frac{1}{12} m\omega l^2 \sin\alpha\, \hat{j}$$

$L_x = 0 \qquad L_z = 0$

$$L_y = \frac{1}{12} m\omega l^2 \sin\alpha$$

Question 2: (10 marks)

10/10

(a) A thin square plate of side a rotates freely under zero torque. If the axis of rotation makes an angle of 45° with the symmetry axis of the plate. Find the period of the precession of the axis of rotation about the symmetry axis, and the period of the precession of the symmetry axis about the invariable line. **(5 marks)**

$\frac{2\pi}{\Omega}$ $\frac{2\pi}{\dot\varphi}$

$$\Omega = \omega \cos\alpha \left(\frac{I_s - I}{I}\right) = \omega \cos 45^\circ \left(\frac{2I - I}{I}\right)$$

∵ $I_s = 2I$ "Perpendicular axis"

$$\Omega = \frac{\omega}{\sqrt{2}}\left(\frac{I}{I}\right) = \frac{\omega}{\sqrt{2}}$$

suppose $T = \frac{2\pi}{\omega} = 1s$

$$\frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\cdot\sqrt{2} = \frac{2\pi}{2\pi/s}\cdot\sqrt{2} = 1.414\ s$$

∴ $\omega = 2\pi/s$

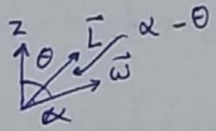
$$\dot\varphi = \omega\left(1 + \left(\left(\frac{I_s}{I}\right)^2 - 1\right)\cos^2\alpha\right)^{1/2} = \omega\left(1 + \left(\left(\frac{2I}{I}\right)^2 - 1\right)(\cos 45)^2\right)^{1/2}$$

$$\dot\varphi = \omega\left(1 + (4-1)\frac{1}{2}\right)^{1/2} = \omega\left(1 + \frac{3}{2}\right)^{1/2} = 1.58\ \omega$$

$$\frac{2\pi}{\dot\varphi} = \frac{2\pi}{1.58\omega} = \frac{2\pi}{1.58(2\pi/s)} = 0.632\ s$$

(b) A rigid body having an axis of symmetry rotates freely about a fixed point under no torques. If α is the angle between the axis of symmetry and the instantaneous axis of rotation, show that the angle between the axis of rotation and the invariable line is

$$tan^{-1}\left[\frac{(I_s - I)tan\alpha}{I_s + I tan^2\alpha}\right]$$



where I_s, the moment of inertia about the symmetry axis, is greater than I, the moment of inertia about an axis normal to the symmetry axis. **(5 marks)**

∵ $I_s > I$ $\quad \alpha > \theta \quad \tan\theta = \frac{I}{I_s}\tan\alpha$

$$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta} = \frac{\tan\alpha - I/I_s \tan\alpha}{1 + \tan\alpha\left(\frac{I}{I_s}\tan\alpha\right)}$$

$$\tan(\alpha - \theta) = \frac{(1 - I/I_s)\tan\alpha}{1 + \frac{I}{I_s}\tan^2\alpha}\cdot\frac{I_s}{I_s}$$

$$\tan(\alpha - \theta) = \frac{(I_s - I)\tan\alpha}{I_s + I\tan^2\alpha}$$

$$\alpha - \theta = \tan^{-1}\left(\frac{(I_s - I)\tan\alpha}{I_s + I\tan^2\alpha}\right)$$

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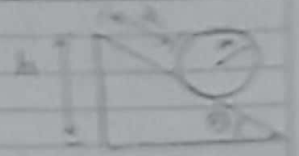
...estion 5: (10 marks)

(a) Find the acceleration of a solid uniform sphere rolling down a perfectly rough fixed inclined plane. (5 marks)



Choose x as displacement coordinate

$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2}\left(m \dot{x}^2 + I\left(\frac{\dot{x}}{a}\right)^2\right)$

$V = -mg\,x\sin\theta$

$L = T - V = \frac{1}{2}\left(m\dot{x}^2 + \frac{I}{a^2}\dot{x}^2\right) + mgx\sin\theta$

$I_{sphere} = \frac{2}{5} m a^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$

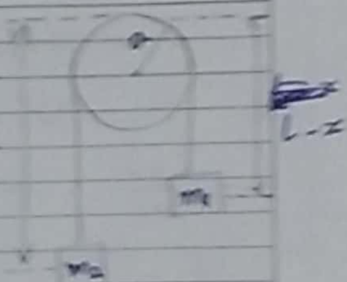
$\ddot{x} = \frac{g\sin\theta}{1+2/5} = \frac{g\sin\theta}{7/5}$

$\ddot{x} = \frac{5}{7} g\sin\theta$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{I}{a^2}\dot{x}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + \frac{I}{a^2}\ddot{x} = \ddot{x}\left(m + \frac{I}{a^2}\right)$

$\frac{\partial L}{\partial x} = mg\sin\theta$, $\ddot{x}\left(m + \frac{I}{a^2}\right) = mg\sin\theta$, $\ddot{x} = \frac{mg\sin\theta}{m + I/a^2} = \frac{mg\sin\theta}{m + \frac{2}{5}\frac{ma^2}{a^2}} = \frac{mg\sin\theta}{m + 2/5\,m}$

(b) Find the acceleration of an Atwood's machine system which consists of two weights of mass m_1 and m_2, connected by a light inextensible of length l which passes over a pulley. (5 marks)



Choose x as displacement coordinate

$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \frac{1}{2} I\omega^2$

$T = \frac{1}{2}\left(m_1\dot{x}^2 + m_2\dot{x}^2 + I\frac{\dot{x}^2}{a^2}\right) = \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{a^2}\right)$

$V = -m_2 g x - m_1 g(L - x)$

$L = T - V = \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{a^2}\right) + m_2 g x + m_1 g(L-x)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$

$\frac{\partial L}{\partial \dot{x}} = \dot{x}\left(m_1 + m_2 + \frac{I}{a^2}\right)$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \ddot{x}\left(m_1 + m_2 + \frac{I}{a^2}\right)$

$\frac{\partial L}{\partial x} = m_2 g - m_1 g = g(m_2 - m_1)$

$\ddot{x}\left(m_1 + m_2 + \frac{I}{a^2}\right) = g(m_2 - m_1)$

$\ddot{x} = \frac{g(m_2 - m_1)}{m_1 + m_2 + I/a^2}$

Q4 ~~ب~~ $\frac{\partial H}{\partial r} = -\dot{P}_r \Rightarrow \frac{\partial V}{\partial r} - \frac{P_\theta^2}{mr^3} = -\dot{P}_r$

$-\frac{\partial V}{\partial r} + \frac{P_\theta^2}{mr^3} = \dot{P}_r$ $\quad \dot{P}_r = \dot{P}_r - \frac{P_\theta^2}{mr^3} = \dot{P}_r - \frac{h^2}{mr^3}$

$\frac{\partial H}{\partial P_r} = \dot{r} \Rightarrow \frac{P_r}{m} = \dot{r} \quad P_r = m\dot{r}$

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Question 4: (10 marks)

Find the Hamiltonian equations of motion for:

1- A one-dimensional harmonic oscillator.

2- A particle in a central field.

[1] $\frac{\partial H}{\partial q_k} = -\dot{P}_k \qquad \frac{\partial H}{\partial P_k} = \dot{q}_k \qquad H(P_k, q_k) = V + T$

1- Choos x as displacment Coordinates

2- $T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2 = \frac{P_x^2}{2m}$

3- ~~$V = \frac{1}{2} K x^2$~~ $V = \frac{1}{2} K x^2$

4- $H = V + T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} Kx^2 = \frac{P_x^2}{2m} + \frac{1}{2} Kx^2$

5- $\frac{\partial H}{\partial x} = Kx = -\dot{P}_x \quad , \quad \frac{\partial H}{\partial P_x} = \frac{P_x}{m} = \dot{x}$

$-\dot{P}_x = Kx \qquad \boxed{P_x = m\dot{x}}$

$\boxed{m\ddot{x} = -Kx} \qquad \dot{P}_x = m\ddot{x}$

[2]

[1] Choose The Polar Coordinate $q_1 = r$, $q_2 = \theta$

[2] $T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$

[3] $V = V(r)$

[4] $H = T + V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

[5] $H = \frac{1}{2} m . \frac{P_r^2}{m^2} + \frac{1}{2} m r^2 \frac{P_\theta^2}{(mr^2)^2} + V(r)$

$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$

$\frac{1}{2} m r^2 \dot{\theta}^2$

$P_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}$

$\dot{r} = \frac{P_r}{m}$

$P_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$\dot{\theta} = \frac{P_\theta}{mr^2}$

~~$\frac{\partial H}{\partial r} = \dot{P}_r \quad \frac{\partial V}{\partial r} \frac{P_\theta^2}{mr^3} = \dot{P}_r \quad P_r = \frac{P_\theta}{m}$~~

~~$\frac{\partial H}{\partial P_r} = \dot{r} \Rightarrow \frac{P_r}{m}$~~

$$\frac{\partial H}{\partial \theta} = -\dot{P}_\theta \Rightarrow \quad 0 = -\dot{P}_\theta \qquad P_\theta = \text{Constant} = h$$

$$\frac{\partial H}{\partial P_\theta} = \dot{\theta} \qquad \frac{P_\theta}{mr^2} = \dot{\theta} \qquad P_\theta = mr^2\dot{\theta}$$

$$h = mr^2\dot{\theta}$$

Question 5: (5 marks)

Deduce the relation between Impulse $\hat{p}$ and coefficient of restitution ε.

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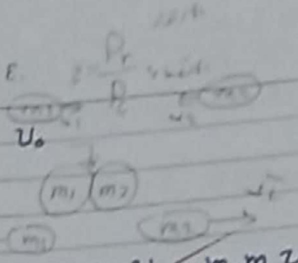


$\hat{P}_r$: restitution implulse , $\hat{P}_c$: Compressing impluse v_0

For m_1

$$\hat{P}_c = \Delta P = -m_1 v_1 + m_1 v_0 \Rightarrow m_2 \hat{P}_c = m_1 m_2 v_0 - m_1 m_2 v_1 \Rightarrow \boxed{1}$$

$$\hat{P}_r = m_1 v_1' - m_1 v_0 \Rightarrow m_2 \hat{P}_r = m_1 m_2 v_1' - m_1 m_2 v_0 \Rightarrow \boxed{2}$$

For m_2

$$\hat{P}_c = m_2 v_0 - m_2 v_2 \Rightarrow m_1 \hat{P}_c = m_1 m_2 v_0 - m_1 m_2 v_2 \Rightarrow \boxed{3}$$

$$\hat{P}_r = m_2 v_2' - m_2 v_0 \Rightarrow m_1 \hat{P}_r = m_1 m_2 v_2' - m_1 m_2 v_0 \Rightarrow \boxed{4}$$

subtract [1] From [3]

$$\hat{P}_c (m_1 - m_2) = -m_1 m_2 v_2 + m_1 m_2 v_1 = m_1 m_2 (v_1 - v_2) \Rightarrow \boxed{5}$$

subtract [2] From [4]

$$\hat{P}_r (m_1 - m_2) = m_1 m_2 v_2' - m_1 m_2 v_1' = m_1 m_2 (v_2' - v_1') \Rightarrow \boxed{6}$$

~~Divide $\hat{P}_r$ By $\hat{P}_c$~~ Divide eq [6] by eq [5]

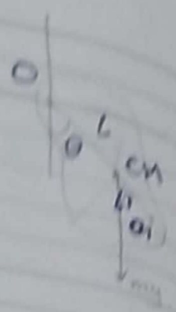
$$\frac{\hat{P}_r (m_1 - m_2)}{\hat{P}_c (m_1 - m_2)} = \frac{m_1 m_2 (v_2' - v_1')}{m_1 m_2 (v_1 - v_2)}$$

$$\left| \hat{P}_r / \hat{P}_c \right| = \frac{|v_2' - v_1|}{|v_1 - v_2|} = \frac{|v_2' - v_1|}{|v_2 - v_1|} \quad \because \varepsilon = \frac{|v_2' - v_1'|}{|v_2 - v_1|}$$

$$\boxed{\frac{\hat{P}_r}{\hat{P}_c} = \varepsilon}$$

Question 6: (5 marks)

Prove that the position of the center of oscillation of the physical pendulum relative to the center of mass is $\ell' = \frac{k^2_{cm}}{\ell}$



$$N = I \frac{d^2\theta}{dt^2}$$

$$N = -mgl\sin\theta = -mg\theta \quad \because \theta \text{ small} \quad \sin\theta \simeq \theta$$

$$I\ddot{\theta} = -mg\theta l \Rightarrow \ddot{\theta} + \frac{mgl}{I}\theta = 0$$

equation of harmonic motion

$$\theta = \theta_0 \cos(\omega t) \qquad \because \omega = \sqrt{\frac{mgl}{I}}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{mgl/I}}$$

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{k^2 m}{mgl}} = 2\pi\sqrt{\frac{k^2}{lg}} \qquad \because I = k^2 m$$

By Parallel axis Theory

$$I = I_{cm} + ml^2 \Rightarrow k^2 m = k_{cm}^2 m + l^2 \Rightarrow k^2 = k_{cm}^2 + l^2$$

$$T = 2\pi\sqrt{\frac{k_{cm}^2 + l^2}{gl}} \qquad \because \text{The Periodic O suspending Point}$$

$$T' = 2\pi\sqrt{\frac{k_{cm}^2 + l'^2}{gl'}} \qquad \because \text{" " " O' " "}$$

if O is a center of oscillation of O'

$T = T'$

$$2\pi\sqrt{\frac{K_{cm}^2 + l^2}{gl}} = 2\pi\sqrt{\frac{K_{cm}^2 + l'^2}{gl'}}$$

$$\frac{K_{cm}^2 + l^2}{gl} = \frac{K_{cm}^2 + l'^2}{gl'} \Rightarrow \frac{K_{cm}^2 + l^2}{l} = \frac{K_{cm}^2 + l'^2}{l'}$$

$$\frac{ll'(K_{cm}^2 + l^2)}{l} = \frac{(K_{cm}^2 + l'^2)ll'}{l'}$$

$$l'K_{cm}^2 + l'l^2 = K_{cm}^2 l + ll'^2$$

$$l'K_{cm}^2 - lK_{cm}^2 = ll'^2 - l'l^2$$

$$K_{cm}^2(l'-l) = ll'(l'-l)$$

$$K_{cm}^2 = ll'$$

$$l' = \frac{K_{cm}^2}{l}$$