Umm Al-Qura University
College of Applied Science
Department of Physics

Time: 10.15 - 12.15

جامعة ام القرى
كلية العلوم التطبيقية
قسم الفيزياء

التاريخ: ١٧ ربيع الثاني ١٤٣٩

Final Exam
Academic Year 1438-1439 (1st Semester)

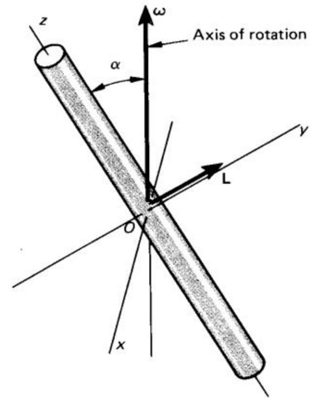
Program: Physics	**Course:** Classical Mechanics (2)	**Course code:** 403321-3

Name: ………………………………………………........	**Group No:** ……………
Academic ID: ……………………………….……….……	**Serial No:** ……………

Question		**Mark**	**Signature**
Question 1	**(10 marks)**		
Question 2	**(10 marks)**		
Question 3	**(10 marks)**		
Question 4	**(10 marks)**		
Question 5	**(5 marks)**		
Question 6	**(5 marks)**		
Question 7	**------**		
Question 8	**------**		
Question 9	**------**		
Question 10	**------**		
Total Mark		**50**	**Exam Committee** Dr. Doaa Abdallah Dr. Fatma El-Sayed

Question 1: (10 marks)

Find the product of inertia and the angular momentum vector $\vec{L}$ for a thin rod of length l and mass m which is constrained to rotate with constant angular velocity $\vec{\omega}$ about an axis passing through the center making an angle α with the rod.



Question 2: (10 marks)

(a) A thin square plate of side a rotates freely under zero torque. If the axis of rotation makes an angle of 45^{o} with the symmetry axis of the plate. Find the period of the precession of the axis of rotation about the symmetry axis, and the period of the precession of the symmetry axis about the invariable line. **(5 marks)**

(b) A rigid body having an axis of symmetry rotates freely about a fixed point under no torques. If α is the angle between the axis of symmetry and the instantaneous axis of rotation, show that the angle between the axis of rotation and the invariable line is

$$tan^{-1}\left[\frac{(I_s - I)tan\alpha}{I_s + Itan^2\alpha}\right]$$

where I_s, the moment of inertia about the symmetry axis, is greater than I, the moment of inertia about an axis normal to the symmetry axis. **(5 marks)**

Question 3: (10 marks)

(a) Find the acceleration of a solid uniform sphere rolling down a perfectly rough fixed inclined plane. **(5 marks)**

(b) Find the acceleration of an Atwood's machine system which consists of two weights of mass m_1 and m_2, connected by a light inextensible of length l which passes over a pulley. **(5 marks)**

Question 4: (10 marks)

Find the Hamiltonian equations of motion for:

1- A one-dimensional harmonic oscillator.

2- A particle in a central field.

Question 5: (5 marks)

Deduce the relation between Impulse $\hat{p}$ and coefficient of restitution ε.

Question 6: (5 marks)

Prove that the position of the center of oscillation of the physical pendulum relative to the center of mass is $\ell' = \dfrac{k_{cm}^2}{\ell}$.

Umm Al-Qura University
College of Applied Science
Department of Physics
Time: 10.15 - 12.15

جامعة ام القرى
كلية العلوم التطبيقية
قسم الفيزياء
التاريخ: ١٧ ربيع الثاني ١٤٣٩

Final Exam
Academic Year 1438-1439 (1st Semester)

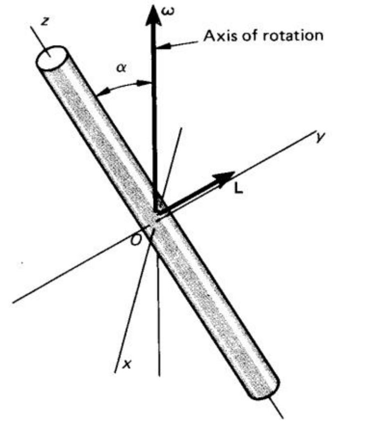
Program: Physics **Course:** Classical Mechanics (2) **Course code:** 403321-3

Answer Model

Question		Mark	Signature
Question 1	(10 marks)		
Question 2	(10 marks)		
Question 3	(10 marks)		
Question 4	(10 marks)		
Question 5	(5 marks)		
Question 6	(5 marks)		
Question 7	------		
Question 8	------		
Question 9	------		
Question 10	------		
Total Mark		50	**Exam Committee** Dr. Doaa Abdallah Dr. Fatma El-Sayed

Question 1: (10 marks)

Find the products of inertia and the angular momentum vector $\vec{L}$ for a thin rod of length l and mass m which is constrained to rotate with constant angular velocity $\vec{\omega}$ about an axis passing through the center making an angle α with the rod.



In yz-plane, the components of $\boldsymbol{\omega}$ are

$$\omega_x = 0 \qquad \omega_y = \omega \, sin\alpha \qquad \omega_z = \omega \, cos\alpha$$

$$I_{xx} = \int (y^2 + z^2) dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} (0 + z^2) \rho \, dz = \rho \left[\frac{z^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{\rho}{3} \frac{2l^3}{8} = \frac{m}{l} \cdot \frac{l^3}{12} = \frac{1}{12} ml^2$$

$$I_{yy} = \int (x^2 + z^2) dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} (0 + z^2) \rho \, dz = \rho \left[\frac{z^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{\rho}{3} \frac{2l^3}{8} = \frac{m}{l} \cdot \frac{l^3}{12} = \frac{1}{12} ml^2$$

$$I_{zz} = \int (x^2 + y^2) dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} (0 + 0) \rho \, dz = 0$$

$$I_{xy} = I_{yx} = -\int xy \, dm = 0 \quad I_{xz} = I_{yz} = -\int xz \, dm = 0 \quad I_{yz} = I_{zy} = -\int yz \, dm = 0$$

The angular momentum vector is

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} \frac{m\,l^2}{12} & 0 & 0 \\ 0 & \frac{m\,l^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega \, sin\alpha \\ \omega \, cos\alpha \end{bmatrix}$$

$$\therefore \boldsymbol{L} = \boldsymbol{j} \frac{m\,l^2}{12} \omega \, sin\alpha$$

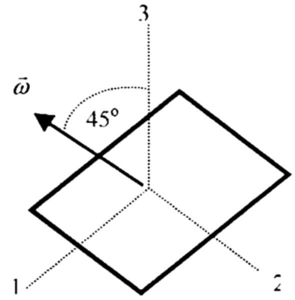
Question 2: (10 marks)

(a) A thin square plate of side a rotates freely under zero torque. If the axis of rotation makes an angle of 45° with the symmetry axis of the plate. Find the period of the precession of the axis of rotation about the symmetry axis, and the period of the precession of the symmetry axis about the invariable line. **(5 marks)**

From symmetry,

$$I_s = I_z = 2I \quad \text{and} \quad I_x = I_y = I$$

$$\Omega = \omega \cos(45^o) = \omega/\sqrt{2}$$



The period of precession of $\vec{\omega}$ about $\hat{k}$ is

$$T_1 = \frac{2\pi}{\Omega} = \frac{\omega\,(1s)}{\omega/\sqrt{2}} = \sqrt{2}\,s = 1.414\,s$$

$$\dot{\varphi} = \omega\sqrt{\left(1 + \left(\left(\frac{I_s}{I}\right)^2 - 1\right)cos^2\alpha\right)} = \sqrt{\frac{5}{2}}\,\omega$$

The period of precession of the symmetry axis $\hat{k}$ about $\vec{L}$ is

$$T_2 = \frac{2\pi}{\dot{\varphi}} = \frac{\omega\,(1s)}{\sqrt{5/2}\,\,\omega} = \sqrt{\frac{2}{5}}\,s = 0.632\,s$$

(b) A rigid body having an axis of symmetry rotates freely about a fixed point under no torques. If α is the angle between the axis of symmetry and the instantaneous axis of rotation, show that the angle between the axis of rotation and the invariable line is

$$tan^{-1}\left[\frac{(I_s - I)tan\alpha}{I_s + I tan^2\alpha}\right]$$

where where I_s, the moment of inertia about the symmetry axis, is greater than I, the moment of inertia about an axis normal to the symmetry axis. **(5 marks)**

$$tan\theta = \frac{I}{I_s}tan\alpha$$

Since $I_s > I$, $\theta < \alpha$, and the angle between the axis of rotation $\vec{\omega}$ and the invariable line $\vec{L}$ is $\alpha - \theta$. From trigonometric identities

$$\tan(\alpha - \theta) = \frac{tan\alpha - tan\theta}{1 + tan\alpha\, tan\theta}$$

$$\tan(\alpha - \theta) = \frac{tan\alpha - \frac{I}{I_s}tan\alpha}{1 + tan\alpha\left(\frac{I}{I_s}tan\alpha\right)} = \frac{\left(1 - \frac{I}{I_s}\right)tan\alpha}{1 + \frac{I}{I_s}tan^2\alpha} = \frac{(I_s - I)tan\alpha}{I_s + I\,tan^2\alpha}$$

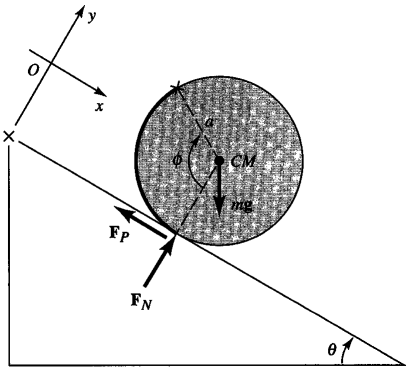
$$\alpha - \theta = tan^{-1}\left[\frac{(I_s - I)tan\alpha}{I_s + I\,tan^2\alpha}\right]$$

Question 3: (10 marks)

(a) Find the acceleration of a solid uniform sphere rolling down a perfectly rough fixed inclined plane. **(5 marks)**

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}ma^2\right)\left(\frac{\dot{x}}{a}\right)^2 = \frac{7}{10}ma^2\dot{x}^2$$

For $V = 0$ at the initial position of the sphere.



$$V = -mgx\,sin\theta$$

$$L = T - V = \frac{7}{10}ma^2\dot{x}^2 + mgx\,sin\theta$$

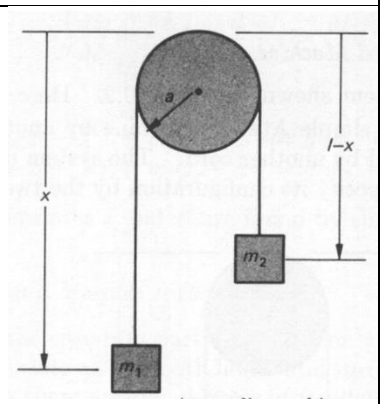
$$\frac{\partial L}{\partial \dot{x}} = \frac{7}{5}m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{7}{5}m\ddot{x} \qquad \frac{\partial L}{\partial x} = mg\,sin\theta$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\therefore mg\,sin\theta = \frac{7}{5}m\ddot{x} \quad \rightarrow \quad \ddot{x} = \frac{5}{7}g\,sin\theta$$

(b) Find the acceleration of an Atwood's machine system which consists of two weights of mass m_1 and m_2, connected by a light inextensible of length l which passes over a pulley. **(5 marks)**

The angular speed of the pulley is clearly $\frac{\dot{x}}{a}$, where a is the radius.



The kinetic energy and potential energy of the system are

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{a^2}$$

$$V = -m_1gx - m_2g(l-x)$$

$$\therefore L = T - V = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2 + g(m_1 - m_2)x + m_2gl$$

and Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k} \qquad \left(m_1 + m_2 + \frac{I}{a^2}\right)\ddot{x} = g(m_1 - m_2)$$

$$\therefore \ddot{x} = g\frac{(m_1 - m_2)}{\left(m_1 + m_2 + \frac{I}{a^2}\right)}$$

Question 4: (10 marks)

Find the Hamiltonian equations of motion for:

1- A one-dimensional harmonic oscillator.

2- A particle in a central field.

(1)

$$T = \frac{1}{2}m\dot{x}^2 \qquad V = \frac{1}{2}kx^2 \qquad p = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \qquad \dot{x} = \frac{p}{m}$$

$$H = T + V = \frac{p^2}{2m} + \frac{kx^2}{2}$$

The equations of motion

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \qquad \frac{\partial H}{\partial x} = kx = -\dot{p}$$

$$kx = -\frac{d}{dt}(m\dot{x}) \qquad \Longrightarrow \qquad m\ddot{x} + kx = 0$$

(2)

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right), \qquad V = V(r)$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\therefore H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

The Hamiltonian equations

$$\frac{\partial H}{\partial p_r} = \dot{r} \qquad \frac{\partial H}{\partial r} = -\dot{p}_r \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \frac{\partial H}{\partial \theta} = -\dot{p}_\theta$$

then

$$\frac{p_r}{m} = \dot{r} \qquad \frac{\partial V(r)}{\partial r} - \frac{p_\theta^2}{mr^3} = -\dot{p}_r \qquad \frac{p_\theta}{mr^2} = \dot{\theta} \qquad 0 = -\dot{p}_\theta$$

The last two equations yield the constancy of angular momentum:

$$p_\theta = constant = mr^2\dot{\theta} = h$$

From which the first two give

$$\dot{p}_r = m\ddot{r} = \frac{h^2}{mr^3} - \frac{\partial V(r)}{\partial r} = \frac{h^2}{mr^3} + F_r$$

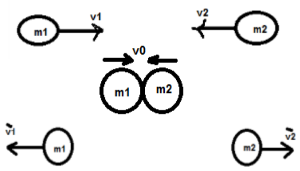
Question 5: (5 marks)

Deduce the relation between Impulse $\hat{p}$ and coefficient of restitution ε.

Before collision

At the moment of collision

After collision



We shall divide the impulse into two parts, namely

(1) The impulse of compression $\hat{p}_c$ (2)The impulse of restitution $\hat{p}_r$

For the compression, we can write

$$m_1\vec{v}_0 - m_1\vec{v}_1 = \hat{p}_c \quad \ldots..(1)$$
$$m_2\vec{v}_0 - m_2\vec{v}_2 = -\hat{p}_c \quad \ldots\ldots..(2)$$

where $\vec{v}_0$ is the common velocity of both particles at the instant their relative speed is zero

$$v_2 - v_1 = 0$$

Similarly, for the restitution we have

$$m_1\overrightarrow{v_1'} - m_1\overrightarrow{v_0} = \hat{p}_r \ \ldots\ldots(3)$$
$$m_2\overrightarrow{v_2'} - m_2\overrightarrow{v_0} = -\hat{p}_r \ldots..(4)$$

Eliminating $\overrightarrow{v_0}$ from equations (1) and (2) by multiply eq.(1) by m_2 and eq.(2) by m_1 then subtract eq.(2) from eq.(1)

$$m_1m_2\overrightarrow{v_0} - m_1m_2\overrightarrow{v_1} = m_2\hat{p}_c$$
$$m_1m_2\overrightarrow{v_0} - m_1m_2\overrightarrow{v_2} = -m_1\hat{p}_c$$

We get

$$m_1m_2\overrightarrow{v_2} - m_1m_2\overrightarrow{v_1} = (m_1+m_2)\hat{p}_c$$
$$m_1m_2(\overrightarrow{v_2} - \overrightarrow{v_1}) = (m_1+m_2)\hat{p}_c \ \ldots.(5)$$

Eliminating $\overrightarrow{v_0}$ from equations (3) and (4) by multiply eq.(3) by m_2 and eq.(4) by m_1 then subtract eq.(4) from eq.(3)

$$m_1m_2\overrightarrow{v_1'} - m_1m_2\overrightarrow{v_0} = m_2\hat{p}_r$$
$$m_1m_2\overrightarrow{v_2'} - m_1m_2\overrightarrow{v_0} = -m_1\hat{p}_r$$

We get

$$m_1m_2\overrightarrow{v_1'} - m_1m_2\overrightarrow{v_2'} = (m_1+m_2)\hat{p}_r$$
$$m_1m_2\left(\overrightarrow{v_1'} - \overrightarrow{v_2'}\right) = (m_1+m_2)\hat{p}_r \ \ldots.(6)$$

Divide eq.(6) by eq.(5)

$$\frac{m_1m_2\left(\overrightarrow{v_1'} - \overrightarrow{v_2'}\right)}{m_1m_2(\overrightarrow{v_2} - \overrightarrow{v_1})} = \frac{(m_1+m_2)\hat{p}_r}{(m_1+m_2)\hat{p}_c}$$
$$\frac{\left(\overrightarrow{v_1'} - \overrightarrow{v_2'}\right)}{(\overrightarrow{v_2} - \overrightarrow{v_1})} = \frac{\hat{p}_r}{\hat{p}_c}$$

The left-hand side is just the definition of the coefficient of restitution ε

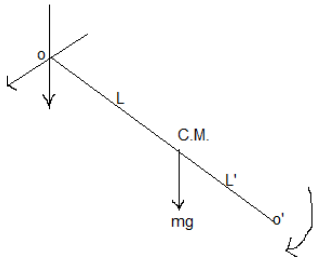
$$\left|\frac{\overrightarrow{v_2'} - \overrightarrow{v_1'}}{\overrightarrow{v_2} - \overrightarrow{v_1}}\right| = \varepsilon = \frac{\hat{p}_r}{\hat{p}_c}$$

The coefficient of restitution id thus equal to the ratio of the impulse of restitution to the impulse of compression.

Question 6: (5 marks)

Prove that the position of the center of oscillation of the physical pendulum relative to the center of mass is $\ell' = \dfrac{k_{cm}^{2}}{\ell}$.

The center of oscillation is a point along the line linking center of mass with the point of suspension o but in the other side, so that if the body suspend at o' it oscillates by same periodic time as at o.



$$T_o = T_{o'}$$

By use of the parallel axis theorem, we can express the radius of gyration k in terms of the radius of gyration about the center of mass k_{cm} as follows:

$$I = I_{cm} + m\,l^2$$

$$mk^2 = mk_{cm}^2 + m\,l^2$$

$$k^2 = k_{cm}^2 + l^2$$

The period of oscillation becomes

$$T = 2\pi\sqrt{\frac{k_{cm}^2 + l^2}{gl}}$$

Suppose that the axis of rotation of a physical pendulum is shifted to a different position O' at a distance l' from the center of mass. The period of oscillation T' about this new axis is given by

$$T' = 2\pi\sqrt{\frac{k_{cm}^2 + l'^2}{gl'}}$$

It follows that the periods of oscillation about O and O' will be equal,

$$\frac{k_{cm}^2 + l^2}{l} = \frac{k_{cm}^2 + l'^2}{l'}$$

$$l(k_{cm}^2 + l'^2) = l'(k_{cm}^2 + l^2) \quad \Longrightarrow \quad k_{cm}^2\left(l - l'\right) = l' l^2 - l l'^2$$

$$k_{cm}^2\left(l - l'\right) = l' l^2 - l l'^2 = l l'\left(l - l'\right)$$

$$k_{cm}^2 = l l'$$

$$\therefore\ l' = k_{cm}^2 / l$$