CHAPTER 10
Tensor Analysis

Lecture 12

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Section 1: INTRODUCTION

Concepts of Scalar, Vector, and Tensor

- Scalar α, a physical quantity that can be completely described by a real number.

  The expression of its component is independent of the choice of the coordinate system.

Example: *Temperature; Mass; Density; Potential*….

- Vector **a,** a physical quantity that has both direction and length.

  The expression of its components is dependent of the choice of the coordinate system.

Example: *Displacement; Velocity; Force; Heat flow;* ….

- Tensor **A,** a 2nd order tensor defines an operation that transforms a vector to another vector

In general, Scalar is a 0th order tensor; Vector is A 1st order tensor; 2nd order tensor; 3rd order tensor…

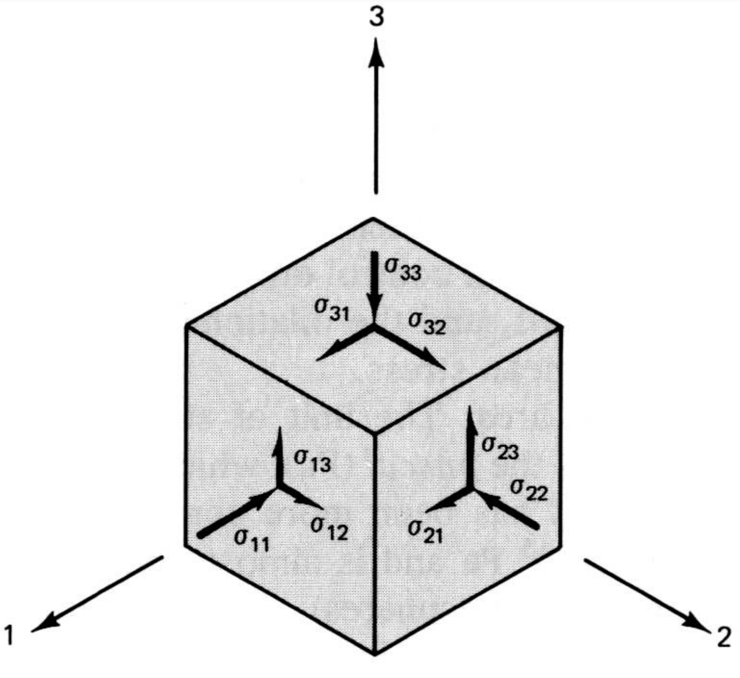
A tensor of rank *n* has 3*n* components

Section 1: INTRODUCTION

- **Example: The stress tensor**

$$P_{ij} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$x = 1, y = 2, z = 3$$

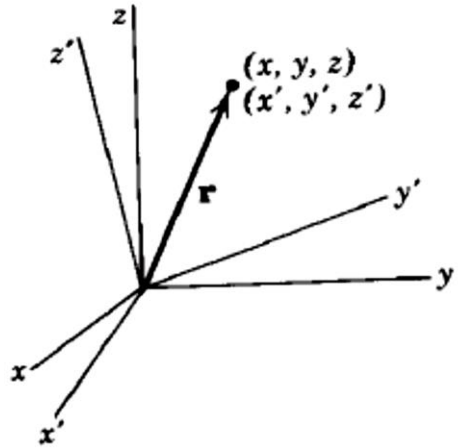


(a)

This is a second-rank tensor known as the stress tensor. The forces (per unit area) P_{xx}, P_{yy}, P_{zz} are pressures or tensions; the others are shear forces (per unit area).
For example P_{zy} is a force per unit area in the *y* direction acting across a plane perpendicular to the *z* direction; this force tends to shear the beam.

Section 2: CARTESIAN TENSORS

- 3D vector rotation:



	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

In the table, l_2 means the cosine of the angle between the x axis and the y' axis, etc. A vector r has components x, y, z or x', y', z' relative to the two coordinate systems; we want to find the relations between the two sets of components.

Section 2: CARTESIAN TENSORS

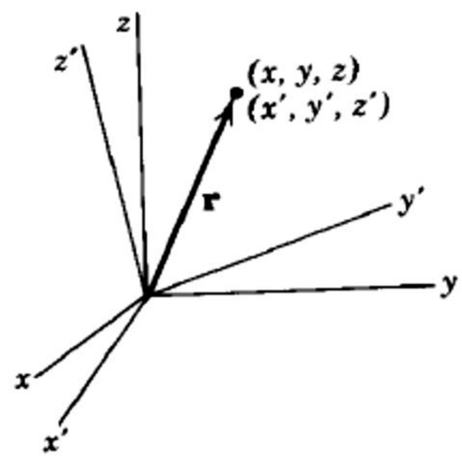
Example 1. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit basis vectors along the (x, y, z) axes and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ be unit basis vectors along the (x', y', z') axes. Then the vector $\mathbf{r}$ can be written in terms of either set of components and basis vectors as follows:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z'.$$

$$\mathbf{r} \cdot \mathbf{i}' = \mathbf{i} \cdot \mathbf{i}'x + \mathbf{j} \cdot \mathbf{i}'y + \mathbf{k} \cdot \mathbf{i}'z = x'$$

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z. \\ y' &= l_2 x + m_2 y + n_2 z, \\ z' &= l_3 x + m_3 y + n_3 z. \end{aligned}\right\}$$

These equations are called the transformation equations from the coordinate system
(x, y, z) to (x', y', z').



Section 2: CARTESIAN TENSORS

Example 2:

In the same way, dotting **r** with **i**, **j**, **k** in turn, we get equations for *x*, *y*, *z* in terms of *x'*, *y'*, *z'*:

$$x = l_1 x' + l_2 y' + l_3 z',$$
$$y = m_1 x' + m_2 y' + m_3 z',$$
$$z = n_1 x' + n_2 y' + n_3 z'.$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 z \\ l_2 & m_2 & n_2 z \\ l_3 & m_3 & n_3 z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \mathrm{r}' = \mathrm{A}\mathrm{r},$$

$$\mathrm{r} = \mathrm{A}^{\mathrm{T}}\mathrm{r}'$$

Section 2: CARTESIAN TENSORS

Definition of Cartesian Vectors A Cartesian vector **V** consists of a set of three numbers (components) in *every* rectangular coordinate system; if V_x, V_y, V_z are the components in one system and V'_x, V'_y, V'_z are the components in a rotated system, these two sets of components are related by an equation similar to

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \mathrm{A} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \quad \text{or} \quad \mathrm{V}' = \mathrm{AV}, \qquad \mathrm{V} = \mathrm{A}^{\mathrm{T}}\mathrm{V}'.$$

Replace x, y, z by x_1, x_2, x_3

Replace x', y', z' by x'_1, x'_2, x'_3

Replace V_x, V_y, V_z by V_1, V_2, V_3

Replace V'_x, V'_y, V'_z by V'_1, V'_2, V'_3

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

In this notation $x'_i = \sum_{j=1}^{3} a_{ij} x_j,$

For any vector

$$V'_i = \sum_{j=1}^{3} a_{ij} V_j, \qquad i = 1, 2, 3.$$

$$V_i = \sum_{j=1}^{3} a_{ji} V'_j.$$

Section 2: CARTESIAN TENSORS

Definition of Cartesian Tensors

A tensor of second rank has nine components in every rectangular coordinate system. If we call the components in one system T_{ij}, the components T_{kl} in a rotated coordinate system are given below where the a's are the direction cosines in the rotation matrix A.

$$T'_{kl} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ki} a_{lj} T_{ij}, \qquad k, l = 1, 2, 3.$$

Example 3. Let **U** and **V** be vectors; we form the following array (in each coordinate system) from the components U_1, U_2, U_3 and V_1, V_2, V_3 of **U** and **V**

$$U'_k = \sum_{i=1}^{3} a_{ki} U_i, \qquad V'_l = \sum_{j=1}^{3} a_{lj} V_j.$$

$$\begin{matrix} U_1V_1 & U_1V_2 & U_1V_3 \\ U_2V_1 & U_2V_2 & U_2V_3 \\ U_3V_1 & U_3V_2 & U_3V_3 \end{matrix}$$

$$U'_k V'_l = \sum_{i=1}^{3} a_{ki} U_i \sum_{j=1}^{3} a_{lj} V_j = \sum_{i,j=1}^{3} a_{ki} a_{lj} U_i V_j,$$

$$T'_{\alpha\beta\gamma\delta} = \sum_{i,j,k,l} a_{\alpha i} a_{\beta j} a_{\gamma k} a_{\delta l} T_{ijkl},$$

Section 3: TENSOR NOTATION AND OPERATIONS

- **Summation Convention:** we omit the summation signs in equations and simply sum over any index which appears exactly twice in one term.

Examples.

$$a_{ii} \text{ or } a_{jj} \text{ or } a_{\beta\beta}, \text{ etc.} \quad \text{means} \quad a_{11} + a_{22} + a_{33};$$

$$x_i x_i \text{ or } x_\alpha x_\alpha, \text{ etc.} \quad \text{means} \quad x_1^2 + x_2^2 + x_3^2;$$

$$a_{ij} b_{jk} \quad \text{means} \quad a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k};$$

$$(3.1) \qquad \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \quad \text{means} \quad \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x_i'} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x_i'} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial x_i'};$$

$$T_{ijkl} S_{ij} V_k U_l \quad \text{means} \quad \sum_i \sum_j \sum_k \sum_l T_{ijkl} S_{ij} V_k U_l;$$

- **Contraction:** Repeating an index implies a summation over that index.
  → result is a tensor of rank = original rank - 2

- **Tensors and Matrices:** The components T_{ij} of a 2nd-rank tensor can be written as the elements of a square matrix. Then note that in the tensor equation, $U_i = T_{ij} V_j$, the contraction (sum on j) corresponds exactly to row times column multiplication for matrices.

Section 3: TENSOR NOTATION AND OPERATIONS

- **Symmetric and Antisymmetric Tensors:**

  - A 2nd-rank tensor T_{ij} is called
    - ***symmetric*** → **if $T_{ij} = T_{ji}$,**
    - ***antisymmetric*** → **if $T_{ij} = -T_{ji}$.**

  - Any 2nd-rank tensor can be written as a sum of a symmetric tensor and an antisymmetric tensor

$$T_{ij} = (T_{ij} + T_{ji}) + (T_{ij} - T_{ji}).$$

  - For tensors of higher rank, similar terminology is used. If an exchange of two indices leaves the tensor component unchanged, we say that the tensor is symmetric with respect to those two indices. If an exchange of two indices changes the tensor component to its negative, we say that the tensor is antisymmetric with respect to those two indices.

Section 3: TENSOR NOTATION AND OPERATIONS

- **Combining tensors:**

  - ***Addition and Subtract of Two Tensors***

$$A_{i_1 i_2 \ldots i_s} \pm B_{i_1 i_2 \ldots i_s} = C_{i_1 i_2 \ldots i_s}$$

- The sum or difference of two tensors of rank n is a tensor of rank n
- Addition is not defined for tensors of different ranks.

Quotient Rule: Let us suppose we know that, for every vector V_j, the quantities $U_i = T_{ij} V_j$ are the components of a non-zero vector and that this holds true in all rotated coordinate systems. Then we can prove that the quantities T_{ij} are the components of a 2^{nd}-rank tensor.

Section 4: INERTIA TENSOR

- If a rigid body is rotating about a fixed axis, then

$$\boldsymbol{\tau} = d\boldsymbol{L}/dt$$

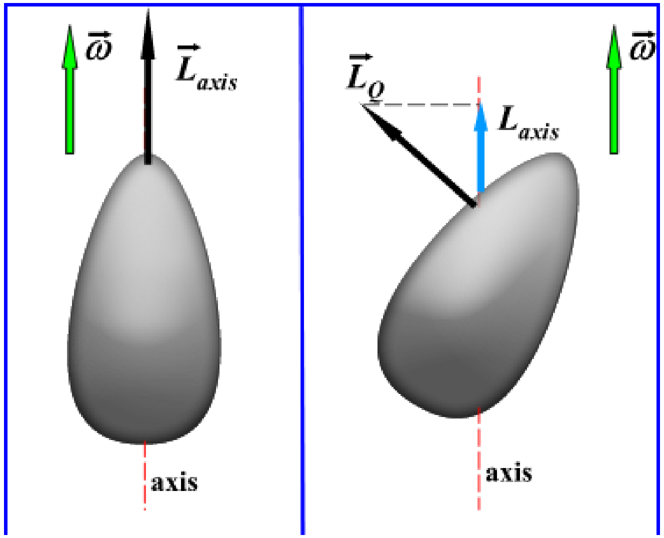
where $\boldsymbol{\tau}$ is the torque and $\boldsymbol{L}$ is the angular momentum about the rotation axis.

- The angular velocity $\boldsymbol{\omega}$ and the angular momentum $\boldsymbol{L}$ are related by the equation

$$\boldsymbol{L} = I\boldsymbol{\omega}$$

where I is the moment of inertia of the body about the rotation axis.

- For rotation about a fixed axis, $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are parallel vectors, and I is a scalar.
- For rotation axis is not fixed, the angular velocity and the angular momentum may not be parallel.



Since $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are vectors, we see by the quotient rule that (when $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are not parallel) the scalar I must be replaced by a 2nd-rank tensor with components I_{jk}. Then in component form we have

$$L_j = I_{jk}\omega_k$$

Angular Momentum for an Arbitrary Angular Velocity

- We will write an arbitrary angular velocity vector as $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$.
- The angular momentum is then

$$L = \sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha = \sum m_\alpha \mathbf{r}_\alpha \times \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right).$$

- For any position $\mathbf{r} = (x, y, z)$, the terms $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ can be explicitly written in the rather ugly form

$$\begin{aligned} \mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right) = [&(y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z, \\ &-yx\omega_x + (z^2 + x^2)\omega_y - yz\omega_z, \\ &-zx\omega_x - zy\omega_y + (x^2 + y^2)\omega_z]. \end{aligned}$$

- A double cross-product like this can be written down with the aid of the BAC-CAB rule ($\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$). Try it.
- Then the general expression for the angular momentum has components

$$\left.\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}\right\}.$$

where

$$\begin{aligned} I_{xx} &= \sum m_\alpha (y_\alpha^2 + z_\alpha^2) \\ I_{xy} &= -\sum m_\alpha x_\alpha y_\alpha \quad . \end{aligned}$$

etc.

Simpler Forms

- We can write this equation for L in simpler forms. Instead of writing $x, y, z,$ we can use subscripts $1, 2, 3$ to get

$$L_i = \sum_{j=1}^{3} I_{ij}\omega_j. \quad \text{or just} \quad L_i = I_{ij}\omega_j \quad \text{in Einstein summation notation.}$$

- Or, we can write it in matrix form $\mathbf{L} = \mathbf{I\omega}$,

where $\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$, $\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$, and $\mathbf{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}$.

- In this form, $\mathbf{I}$ is known as the **inertia tensor**. To distinguish this from the identity tensor you may be familiar with, the text uses

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Note that the term tensor refers to a higher-order vector. A vector is written as a column, as in $\mathbf{L}$ and $\boldsymbol{\omega}$ above, while a tensor is written as a matrix.

Properties of the Inertia Tensor

- You can see from the elements of the moment of inertia tensor

$$I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2)$$
$$I_{xy} = -\sum m_\alpha x_\alpha y_\alpha \quad .$$
$$\text{etc.}$$

  that it has the property that $I_{ij} = I_{ji}$. The elements I_{ii} are called the diagonal elements, so we can say that the inertia tensor is unchanged by swapping off-diagonal elements mirrored about the diagonal.
- Such a swap (replacing I_{ij} with I_{ji} and vice versa) is an operation called taking the transpose of the matrix, so we can say

$$\mathbf{I} = \mathbf{I}^{\mathrm{T}} .$$

- A matrix that is its own transpose is said to be symmetric, and this symmetric property plays a key role in the mathematical theory of the moment of inertia tensor.

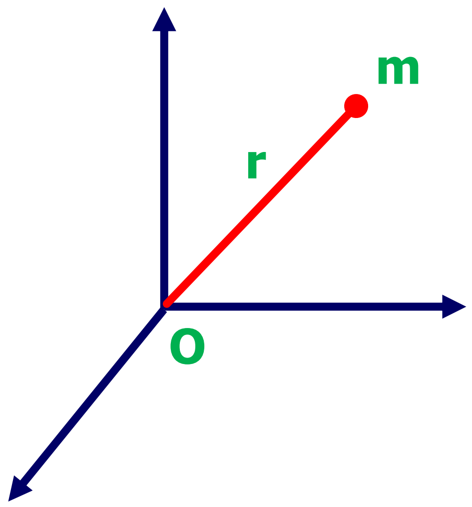
Example 1: Inertia Tensor for Cube

find the components of the inertia tensor. For simplicity, first consider a point mass *m* at the tip of a vector **r** with tail at the origin O

$$\boldsymbol{L} = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{L} = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = m[r^2\boldsymbol{\omega} - (\boldsymbol{r} \cdot \boldsymbol{\omega})\boldsymbol{r}]$$

$$= m[r^2\boldsymbol{\omega} - (x\omega_x + y\omega_y + z\omega_z)\boldsymbol{r}].$$



$$L_x = m[r^2\omega_x - (x\omega_x + y\omega_y + z\omega_z)x]$$

$$= m[(r^2 - x^2)\omega_x - xy\omega_y - xz\omega_z]$$

$$I_{xx} = m(r^2 - x^2) = m(y^2 + z^2),$$

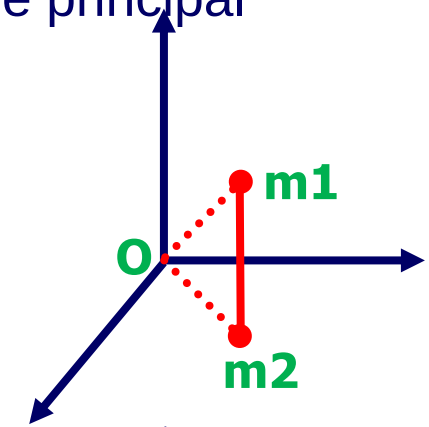
$$I_{xy} = -mxy,$$

$$I_{xz} = -mxz.$$

The other 6 components can be found similarly by taking the *y* and *z* components

Example 1: Inertia Tensor for Cube

- Find the inertia tensor about the origin for the mass distribution consisting of a mass 1 at (0, 1, 1) and a mass 2 at (1, −1, 0). Find the principal moments of inertia and the principal axes.



$$I_{xx} = \sum_i m_i(y_i^2 + z_i^2) \quad \text{or} \quad \int (y^2 + z^2)\, dm,$$

$$I_{xy} = -\sum_i m_i x_i y_i \quad \text{or} \quad -\int xy\, dm, \text{ etc.}$$

Substituting $(x_1, y_1, z_1) = (0, 1, 1), m_1 = 1$, and $(x_2, y_2, z_2) = (1, -1, 0), m_2 = 2$ we find

$$I_{xx} = (1^2 + 1^2) + 2(-1)^2 = 4, I_{xy} = I_{yx} = -0 - 2(-1) = 2.$$

Continuing in the same way, we can find the rest

of the components and write them as an inertia matrix

$$I = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 5 \end{bmatrix}$$

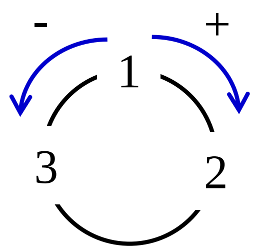
Section 5:KRONECKER DELTA AND LEVI-CIVITA SYMBOL

- The Kronecker δ

$$\delta ij = \begin{cases} 1 & if \;\; i = j \\ 0 & otherwise \end{cases}$$

- Levi-Civita symbol (or permutation symbol)

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any two of } ijk \text{ are equal.} \\ 1, & \text{if } ijk \text{ is an even permutation of 123.} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123.} \end{cases}$$



Look at your watch.