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Department of Electrical Engieerig

Controls (802331)

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Solution Home Work 6

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You do not have to submit this home work. This homework will be part of midterm (March 29 & 30).

**Q1.** Second order system is described as

$$G\left(s\right)=\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}}$$

Find the “step response” and “impulse response” for the following cases

1. $ξ=0$
2. $0<ξ<1$
3. $ξ=1$
4. $1<ξ$

|  |  |
| --- | --- |
| $$1-\cos(ωt)$$ | $$\frac{ω\_{n}^{2}}{s\left(s^{2}+ω\_{n}^{2}\right)}$$ |
| $$1-\frac{e^{-ξω\_{n}t}}{\sqrt{1-ξ^{2}}}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t+cos^{-1}ξ\right))$$$$(0<ξ<1)$$ | $$\frac{ω\_{n}^{2}}{s\left(s^{2}+2ξω\_{n}s+ω\_{n}^{2}\right)}$$ |
| $$\frac{1}{a^{2}}\left(1-e^{-at}-ate^{-at}\right)$$ | $$\frac{1}{s(s+a)^{2}}$$ |
| $$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$$ | $$\frac{1}{s(s+a)(s+b)}$$ |
| $$\sin(ω\_{n}t)$$ | $$\frac{ω\_{n}}{s^{2}+ω\_{n}^{2}}$$ |
| $$\frac{ω\_{n}}{\sqrt{1-ξ^{2}}}e^{-ξω\_{n}t}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t\right))$$$$(0<ξ<1)$$ | $$\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}}$$ |
| $$te^{-at}$$ | $$\frac{1}{(s+a)^{2}}$$ |
| $$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$$ | $$\frac{1}{(s+a)(s+b)}$$ |

**Solution:**

We did the step response in the class. Here we will do the impulse response only.

$$r\left(t\right)=δ\left(t\right) ⇒ R\left(s\right)=L\left\{δ\left(t\right)\right\}=1$$

$$C\left(s\right)=G\left(s\right)R(s)=\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}}$$

1. $ξ=0$

$$C\left(s\right)=\frac{ω\_{n}^{2}}{s^{2}+ω\_{n}^{2}} ⇒ c\left(t\right)=ω\_{n} L^{-1}\left[\frac{ω\_{n}}{s^{2}+ω\_{n}^{2}}\right]=ω\_{n}\sin(ω\_{n}t) u(t)$$

1. $0<ξ<1$

$$C\left(s\right)=\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}} ⇒ c\left(t\right)=L^{-1}\left[\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}}\right]=\frac{ω\_{n}}{\sqrt{1-ξ^{2}}}e^{-ξω\_{n}t}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t\right))u(t)$$

1. $ξ=1$

$$C\left(s\right)=\frac{ω\_{n}^{2}}{s^{2}+2ω\_{n}s+ω\_{n}^{2}}=\frac{ω\_{n}^{2}}{(s+ω\_{n})^{2}} ⇒ c\left(t\right)=ω\_{n}^{2} L^{-1}\left[\frac{1}{(s+ω\_{n})^{2}}\right]=ω\_{n}^{2} te^{-ω\_{n}t} u(t)$$

1. $1<ξ$

$$C\left(s\right)=\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}} $$

$$⇒ c\left(t\right)=ω\_{n}^{2}L^{-1}\left[\frac{1}{\left\{s+\left(ξω\_{n}+ω\_{n}\sqrt{ξ^{2}-1} \right)\right\}\left\{s+\left(ξω\_{n}-ω\_{n}\sqrt{ξ^{2}-1} \right)\right\}}\right]=\frac{ω\_{n}}{2\sqrt{ξ^{2}-1}}\left[e^{-\left(ξ-\sqrt{ξ^{2}-1} \right)ω\_{n}t}-e^{-\left(ξ+\sqrt{ξ^{2}-1} \right)ω\_{n}t}\right]$$

**Q2.** For each of the following second order transfer functions shown below, compute and sketch the step response [Hint: First find the damping ratio $ξ$.].

Also make pole-zero diagram for each transfer function.

$$a. G\left(s\right)=\frac{25}{s^{2}+25} b. G\left(s\right)=\frac{25}{s^{2}+2s+25} c. G\left(s\right)=\frac{25}{s^{2}+5s+25} $$

$$d. G\left(s\right)=\frac{25}{s^{2}+10s+25} e. G\left(s\right)=\frac{25}{s^{2}+15s+25} f. G\left(s\right)=\frac{25}{s^{2}+20s+25} $$

**Solution:**

For all of these transfer functions, $ω\_{n}^{2}=25$ and $ω\_{n}=5$.

1. $G\left(s\right)=\frac{25}{s^{2}+25}=\frac{5^{2}}{s^{2}+2\left(0\right)(5)s+5^{2}}$. Hence $ξ=0$. The output will be: $c\left(t\right)=1-\cos(ω\_{n}t)=1-\cos(5t)$
2. $G\left(s\right)=\frac{25}{s^{2}+2s+25}=\frac{5^{2}}{s^{2}+2\left(\frac{1}{5}\right)(5)s+5^{2}}$. Hence $ξ=\frac{1}{5}$. The output will be:

$$c\left(t\right)=1-\frac{e^{-ξω\_{n}t}}{\sqrt{1-ξ^{2}}}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t+tan^{-1}\frac{\sqrt{1-ξ^{2}}}{ξ} \right))$$

$=1-\frac{e^{-\left(\frac{1}{5}\right)(5)t}}{\sqrt{1-\left(\frac{1}{5}\right)^{2}}}\sin(\left(5\sqrt{1-\left(\frac{1}{5}\right)^{2}}t+tan^{-1}\frac{\sqrt{1-\left(\frac{1}{5}\right)^{2}}}{\frac{1}{5}} \right)=1-\frac{5}{2\sqrt{6}}e^{-t}\sin(\left(2\sqrt{6}t+tan^{-1}2\sqrt{6} \right)))$

1. $G\left(s\right)=\frac{25}{s^{2}+5s+25}=\frac{5^{2}}{s^{2}+2\left(\frac{1}{2}\right)(5)s+5^{2}}$. Hence $ξ=\frac{1}{2}$. The output will be:

$$c\left(t\right)=1-\frac{e^{-ξω\_{n}t}}{\sqrt{1-ξ^{2}}}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t+tan^{-1}\frac{\sqrt{1-ξ^{2}}}{ξ} \right))$$

$=1-\frac{e^{-\left(\frac{1}{2}\right)(5)t}}{\sqrt{1-\left(\frac{1}{2}\right)^{2}}}\sin(\left(5\sqrt{1-\left(\frac{1}{2}\right)^{2}}t+tan^{-1}\frac{\sqrt{1-\left(\frac{1}{2}\right)^{2}}}{\frac{1}{2}} \right)=1-\frac{2}{\sqrt{3}}e^{-2.5t}\sin(\left(\frac{5\sqrt{3}}{2}t+\frac{π}{3} \right)))$

1. $G\left(s\right)=\frac{25}{s^{2}+10s+25}=\frac{5^{2}}{s^{2}+2\left(1\right)(5)s+5^{2}}$. Hence $ξ=1$. The output will be:

$$c\left(t\right)=1-e^{-ω\_{n}t}(1+ω\_{n}t)=1-e^{-5t}(1+5t)$$

1. $G\left(s\right)=\frac{25}{s^{2}+15s+25}=\frac{5^{2}}{s^{2}+2\left(1.5\right)(5)s+5^{2}}$. Hence $ξ=1.5$. The output will be:

 $c\left(t\right)=1+\frac{1}{2\sqrt{ξ^{2}-1}}\left[\left[ξ-\sqrt{ξ^{2}-1}\right]e^{-\left[ξ+\sqrt{ξ^{2}-1} \right]ω\_{n}t}-\left[ξ+\sqrt{ξ^{2}-1}\right]e^{-\left[ξ-\sqrt{ξ^{2}-1} \right]ω\_{n}t}\right]=1+\frac{1}{\sqrt{5}}\left[\left[\frac{3}{2}-\frac{\sqrt{5}}{2} \right]e^{-\left[\frac{3}{2}+\frac{\sqrt{5}}{2} \right]5t}-\left[\frac{3}{2}+\frac{\sqrt{5}}{2} \right]e^{-\left[\frac{3}{2}-\frac{\sqrt{5}}{2} \right]5t}\right]$

$$=1+0.171e^{-13.09t}-1.171e^{-1.91t}$$

1. $G\left(s\right)=\frac{25}{s^{2}+20s+25}=\frac{5^{2}}{s^{2}+2\left(2\right)(5)s+5^{2}}$. Hence $ξ=2$. The output will be:

 $c\left(t\right)=1+\frac{1}{2\sqrt{ξ^{2}-1}}\left[\left[ξ-\sqrt{ξ^{2}-1}\right]e^{-\left[ξ+\sqrt{ξ^{2}-1} \right]ω\_{n}t}-\left[ξ+\sqrt{ξ^{2}-1}\right]e^{-\left[ξ-\sqrt{ξ^{2}-1} \right]ω\_{n}t}\right]=1+\frac{1}{2\sqrt{3}}\left[\left[2-\sqrt{3} \right]e^{-\left[2+\sqrt{3} \right]5t}-\left[2+\sqrt{3} \right]e^{-\left[2-\sqrt{3} \right]5t}\right]$

$$=1+0.077e^{-18.66t}-1.077e^{-1.34t}$$





**Q3.** Find the transfer function for the block diagram shown in Figure 5-5(b) [page 175]. Identify the un-damped natural frequency and the damping co-efficient.

**Solution:**

$$\frac{C\left(s\right)}{R\left(s\right)}=\frac{\frac{K}{s\left(Js+B\right)}}{1+\frac{K}{s\left(Js+B\right)}}=\frac{K}{s\left(Js+B\right)+K}=\frac{K}{Js^{2}+Bs+K}=\frac{\frac{K}{J}}{s^{2}+\frac{B}{J}s+\frac{K}{J}}$$

Hence $ω\_{n}^{2}=\frac{K}{J} ⇒ ω\_{n}=\sqrt{\frac{K}{J}} $

$$2ξω\_{n}=\frac{B}{J} ⇒ ξ=\frac{B}{2Jω\_{n}}=\frac{B}{2J}\sqrt{\frac{J}{K}}=\frac{B}{2\sqrt{KJ}}$$

**Q4.** Find the transfer function for the block diagram shown in Figure 5-13(a) [page 186]. Identify the un-damped natural frequency and the damping co-efficient.

**Solution:**

Transfer function for the inner loop: $\frac{\frac{K}{Js+B}}{1+\frac{K}{Js+B}K\_{h}}=\frac{K}{Js+B+KK\_{h}}$

Transfer function for the outer loop:

$$\frac{C\left(s\right)}{R\left(s\right)}=\frac{\frac{K}{Js^{2}+\left(B+KK\_{h}\right)s}}{1+\frac{K}{Js^{2}+\left(B+KK\_{h}\right)s}}=\frac{K}{Js^{2}+\left(B+KK\_{h}\right)s+K}=\frac{K/J}{s^{2}+\left(\frac{B+KK\_{h}}{J}\right)s+\frac{K}{J}}$$

Hence $ω\_{n}^{2}=\frac{K}{J} ⇒ ω\_{n}=\sqrt{\frac{K}{J}} $

$$2ξω\_{n}=\frac{B+KK\_{h}}{J} ⇒ ξ=\frac{B+KK\_{h}}{2Jω\_{n}}=\frac{B+KK\_{h}}{2J\sqrt{K/J}}=\frac{B+KK\_{h}}{2\sqrt{KJ}}$$