Umm Al-Qura Universtiy, Makkah

Department of Electrical Engineering

Control (802331)

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Solution Midterm Exam

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Max Marks: 60 Section 2



**Q1.** [5, 5, 5] Consider the system shown here:

1. Find the transfer function $\frac{C\left(s\right)}{R\left(s\right)}$ . What is the ‘order’ of this system?
2. Find the value of “K” such that $ξ=0.7$.
3. What will be value of un-damped natural frequency?

**Solution:**

1. $\frac{C\left(s\right)}{R\left(s\right)}=\frac{\frac{K}{s(s+14)}}{1+\frac{K}{s(s+14)}}=\frac{K}{s\left(s+14\right)+K}=\frac{K}{s^{2}+14s+K}$

It is a 2nd order system.

1. $ω\_{n}=\sqrt{K}$

$$2ξω\_{n}=14 ⇒ ω\_{n}=\frac{14}{2ξ}=\frac{7}{0.7}=10=\sqrt{K} ⇒ K=100$$

1. $ω\_{n}=10$

**Q2.** [8, 7] Consider a PD-controller with $K\_{p}=2$, $K\_{d}=\frac{2}{3}$. The error signal is a ramp function: $e\left(t\right)=3t u(t)$.

1. Find the control signal generated by the PD-controller.
2. Sketch both signals (the error signal and the control signal).

**Solution:**

Transfer function of the PD-controller: $\frac{D\left(s\right)}{E\left(s\right)}=K\_{p}+K\_{d}s=2+\frac{2}{3}s$

$E\left(s\right)=\frac{3}{s^{2}}$ . $D\left(s\right)=\left(2+\frac{2}{3}s\right)\left(\frac{3}{s^{2}}\right)=\frac{6}{s^{2}}+\frac{2}{s}$

Control signal = $d\left(t\right)= \left(6t+2\right)u\left(t\right)$



**Q3.** [15] An LTI system is described by the transfer function

$$G\left(s\right)=\frac{5}{s^{2}+12s+16}$$

Find the output of the system if the input is given by

$$r\left(t\right)=3 u(t)$$

**Solution:**

$$G\left(s\right)=\frac{C\left(s\right)}{R\left(s\right)}=\frac{5}{s^{2}+12s+16}$$

$$C\left(s\right)=\frac{5}{s^{2}+12s+16}R\left(s\right)=\frac{5}{s^{2}+12s+16}\left(\frac{3}{s}\right)=\frac{15}{16} \frac{4^{2}}{s\left[s^{2}+2\left(\frac{3}{2}\right)(4)s+4^{2}\right]}$$

Hence $ξ=1.5>1$

|  |  |
| --- | --- |
| $$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$$ | $$\frac{1}{s(s+a)(s+b)}$$ |

$$C\left(s\right)=\frac{15}{s\left(s^{2}+12s+16\right)}=15\frac{1}{s\left(s+10.4721\right)(s+1.5279)}$$

$$c\left(t\right)=15\frac{1}{\left(10.4721\right)(1.5279)}\left[1+\frac{1}{10.4721-1.5279}\left(1.5279e^{-10.4721t}-10.4721e^{-1.5279t}\right)\right]=\frac{15}{16}\left[1+\frac{1}{8.9442}\left(1.5279e^{-10.4721t}-10.4721e^{-1.5279t}\right)\right]=0.9375+0.1048\left(1.5279e^{-10.4721t}-10.4721e^{-1.5279t}\right)=0.9375+0.1601e^{-10.4721t}-1.0976e^{-1.5279t}$$

**Q4.** [5, 10] For the mechanical system shown here,

a. Show that the differential equations describing the system are

$$m\ddot{x}\_{1}=u-kx\_{1}-k(x\_{1}-x\_{2})$$

$$m\ddot{x}\_{2}=k(x\_{1}-x\_{2})-kx\_{2}$$

b. Find the transfer function matrix.

**Solution:**

1. Considering the forces on the left mass: $m\ddot{x}\_{1}=u-kx\_{1}-k(x\_{1}-x\_{2})$

Considering the forces on the right mass: $m\ddot{x}\_{2}=k(x\_{1}-x\_{2})-kx\_{2}$

1. Taking Laplace transforms:

$ms^{2}X\_{1}\left(s\right)=U\left(s\right)-kX\_{1}\left(s\right)-k\left[X\_{1}\left(s\right)-X\_{2}\left(s\right)\right] ⇒ U\left(s\right)=\left[ms^{2}+2k\right]X\_{1}\left(s\right)-kX\_{2}\left(s\right)$

$$ms^{2}X\_{2}\left(s\right)=k\left[X\_{1}\left(s\right)-X\_{2}\left(s\right)\right]-kX\_{2}\left(s\right) ⇒ 0=-kX\_{1}\left(s\right)+\left[ms^{2}+2k\right]X\_{2}\left(s\right)$$

From 2nd equation: $X\_{1}\left(s\right)=\left[\frac{ms^{2}+2k}{k}\right]X\_{2}\left(s\right)$

Substituting in the 1st equation:

$$U\left(s\right)=\left[ms^{2}+2k\right]X\_{1}\left(s\right)-kX\_{2}\left(s\right)=\left[ms^{2}+2k\right]\left[\frac{ms^{2}+2k}{k}\right]X\_{2}\left(s\right)-kX\_{2}\left(s\right)=\left[\frac{\left(ms^{2}+2k\right)\left(ms^{2}+2k\right)}{k}-k\right]X\_{2}\left(s\right)=\left[\frac{m^{2}s^{4}+4mks^{2}+4k^{2}-k^{2}}{k}\right]X\_{2}\left(s\right)=\left[\frac{m^{2}s^{4}+4mks^{2}+3k^{2}}{k}\right]X\_{2}\left(s\right)$$

$$X\_{2}\left(s\right)=\frac{k}{m^{2}s^{4}+4mks^{2}+3k^{2}}U\left(s\right)$$

Remember $X\_{2}\left(s\right)=\left[\frac{k}{ms^{2}+2k}\right]X\_{1}\left(s\right)$. Substituting in above

$$\frac{1}{U\left(s\right)}\left[\frac{k}{ms^{2}+2k}\right]X\_{1}\left(s\right)=\frac{k}{m^{2}s^{4}+4mks^{2}+3k^{2}}$$

$$X\_{1}\left(s\right)=\frac{ms^{2}+2k}{m^{2}s^{4}+4mks^{2}+3k^{2}}U\left(s\right)$$

Writing in matrix form

$$\left[\begin{matrix}X\_{1}\left(s\right)\\X\_{2}\left(s\right)\end{matrix}\right]=\left[\begin{matrix}\frac{ms^{2}+2k}{m^{2}s^{4}+4mks^{2}+3k^{2}}\\\frac{k}{m^{2}s^{4}+4mks^{2}+3k^{2}}\end{matrix}\right] \left[U\left(s\right)\right]$$

Laplace transform

|  |  |
| --- | --- |
| Unit impulse: $δ\left(t\right)$ | 1 |
| Unit step: 1 | $$\frac{1}{s}$$ |
| Unit ramp: $t$ | $$\frac{1}{s^{2}}$$ |
| $$\frac{t^{n-1}}{\left(n-1\right)!}$$$$(n=1, 2, 3, \cdots )$$ | $$\frac{1}{s^{n}}$$ |
| $$t^{n} (n=1, 2, 3, \cdots )$$ | $$\frac{n!}{s^{n+1}}$$ |
| $$e^{-at}$$ | $$\frac{1}{s+a}$$ |
| $$\frac{1}{a}\left(1-e^{-at}\right)$$ | $$\frac{1}{s\left(s+a\right)}$$ |
| $$te^{-at}$$ | $$\frac{1}{(s+a)^{2}}$$ |
| $$\frac{1}{a^{2}}\left(1-e^{-at}-ate^{-at}\right)$$ | $$\frac{1}{s(s+a)^{2}}$$ |
| $$\sin(ωt)$$ | $$\frac{ω}{s^{2}+ω^{2}}$$ |
| $$\cos(ωt)$$ | $$\frac{s}{s^{2}+ω^{2}}$$ |
| $$1-\cos(ωt)$$ | $$\frac{ω\_{n}^{2}}{s\left(s^{2}+ω\_{n}^{2}\right)}$$ |
| $$1-\frac{e^{-ξω\_{n}t}}{\sqrt{1-ξ^{2}}}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t+cos^{-1}ξ\right))$$$$(0<ξ<1)$$ | $$\frac{ω\_{n}^{2}}{s\left(s^{2}+2ξω\_{n}s+ω\_{n}^{2}\right)}$$ |
| $$\frac{1}{a^{2}}\left(1-e^{-at}-ate^{-at}\right)$$ | $$\frac{1}{s(s+a)^{2}}$$ |
| $$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$$ | $$\frac{1}{s(s+a)(s+b)}$$ |
| $$\sin(ω\_{n}t)$$ | $$\frac{ω\_{n}}{s^{2}+ω\_{n}^{2}}$$ |
| $$\frac{ω\_{n}}{\sqrt{1-ξ^{2}}}e^{-ξω\_{n}t}\sin(\left(ω\_{n}\sqrt{1-ξ^{2}}t\right))$$$$(0<ξ<1)$$ | $$\frac{ω\_{n}^{2}}{s^{2}+2ξω\_{n}s+ω\_{n}^{2}}$$ |
| $$te^{-at}$$ | $$\frac{1}{(s+a)^{2}}$$ |
| $$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$$ | $$\frac{1}{(s+a)(s+b)}$$ |