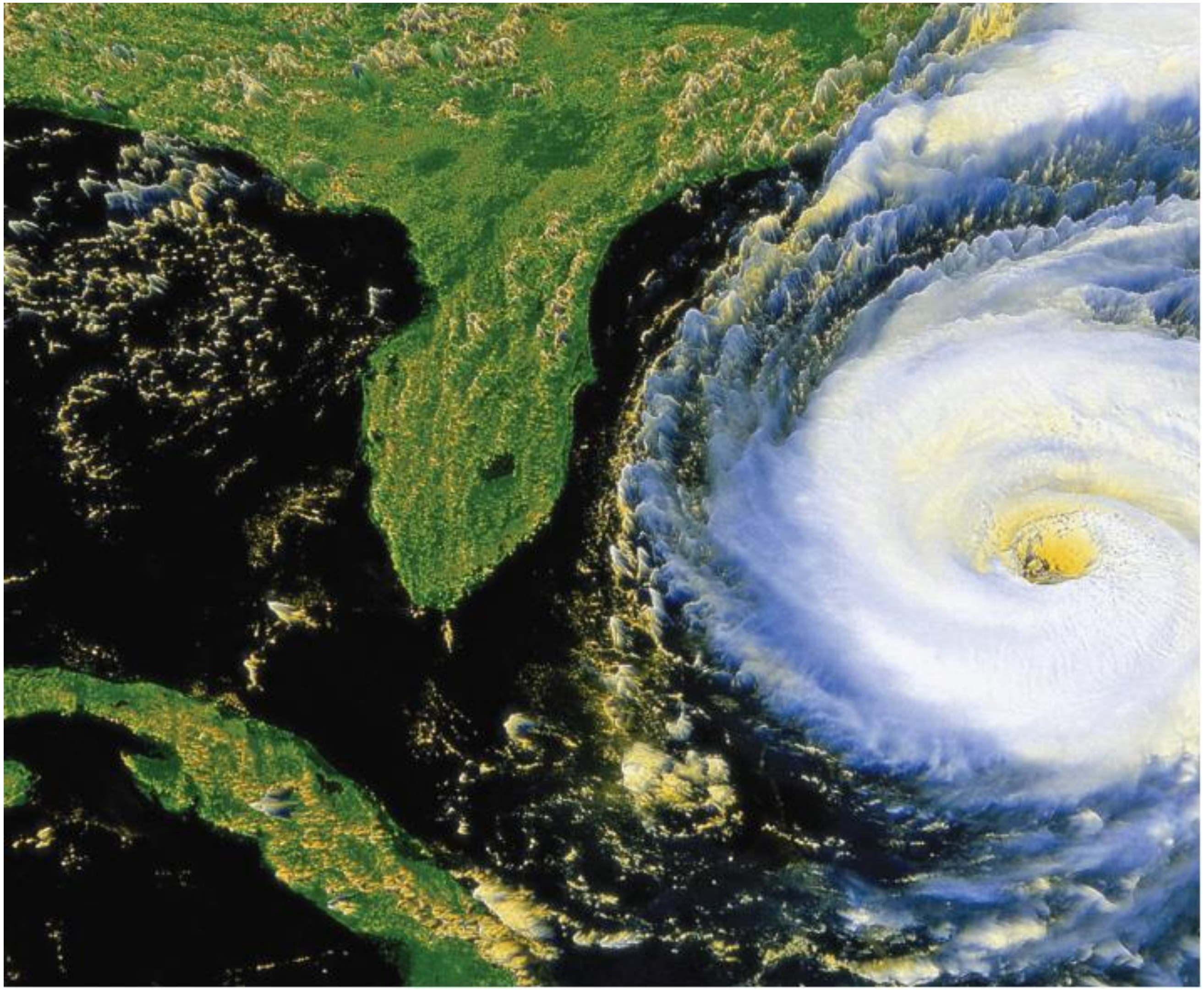
Fluid Mechanics: Fundamentals and Applications, 4th edition

Yunus A. Cengel, John M. Cimbala
Lecture slides by Mehmet Kanoglu

Chapter 3

FLUID KINEMATICS



© StockTrek/Getty Images RF

Satellite image of a hurricane near the Florida coast; water droplets move with the air, enabling us to visualize the counterclockwise swirling motion. However, the major portion of the hurricane is actually *irrotational*, while only the core (the eye of the storm) is *rotational*.

Objectives

- Understand the role of the material derivative in transforming between Lagrangian and Eulerian descriptions
- Distinguish between various types of flow visualizations and methods of plotting the characteristics of a fluid flow
- Appreciate the many ways that fluids move and deform
- Distinguish between rotational and irrotational regions of flow based on the flow property vorticity
- Understand the usefulness of the Reynolds transport theorem

3–1 ■ LAGRANGIAN AND EULERIAN DESCRIPTIONS

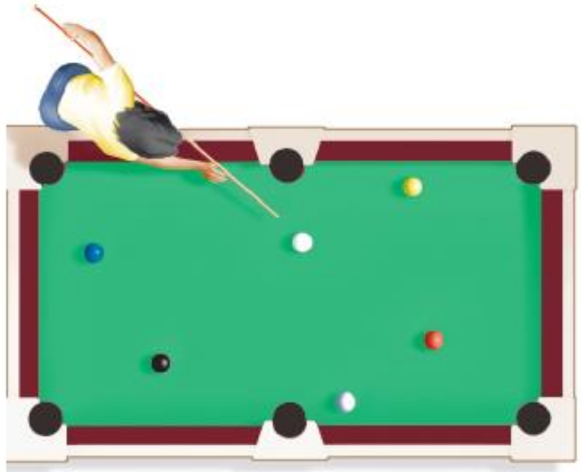
Kinematics: The study of motion.

Fluid kinematics: The study of how fluids flow and how to describe fluid motion.

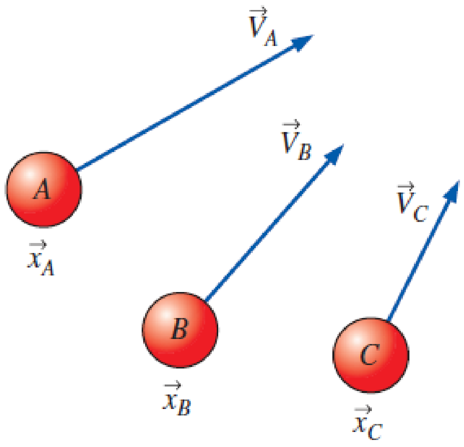
There are two distinct ways to describe motion: Lagrangian and Eulerian

Lagrangian description: To follow the path of individual objects.

This method requires us to track the position and velocity of each individual fluid parcel (fluid particle) and take to be a parcel of fixed identity.



With a small number of objects, such as billiard balls on a pool table, individual objects can be tracked.



In the Lagrangian description, one must keep track of the position and velocity of individual particles.

A more common method is **Eulerian description** of fluid motion.

In the Eulerian description of fluid flow, a finite volume called a **flow domain** or **control volume** is defined, through which fluid flows in and out.

Instead of tracking individual fluid particles, we define **field variables**, functions of space and time, within the control volume.

The field variable at a particular location at a particular time is the value of the variable for whichever fluid particle happens to occupy that location at that time.

For example, the **pressure field** is a **scalar field variable**. We define the **velocity field** as a **vector field variable**.

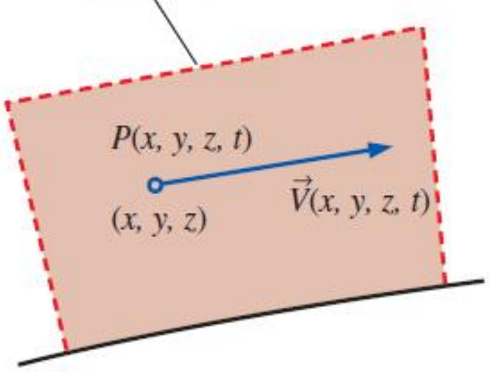
$$\text{Pressure field}: \qquad P = P(x, y, z, t)$$

$$\text{Velocity field}: \qquad \vec{V} = \vec{V}(x, y, z, t)$$

$$\text{Acceleration field}: \qquad \vec{a} = \vec{a}(x, y, z, t)$$

Collectively, these (and other) field variables define the **flow field**. The velocity field can be expanded in Cartesian coordinates as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\,\vec{i} + v(x, y, z, t)\,\vec{j} + w(x, y, z, t)\,\vec{k}$$



(a)


(b)

(Bottom) Photo by John M. Cimbala.

In the Eulerian description we don't really care what happens to individual fluid particles; rather we are concerned with the pressure, velocity, acceleration, etc., of whichever fluid particle happens to be at the location of interest at the time of interest.

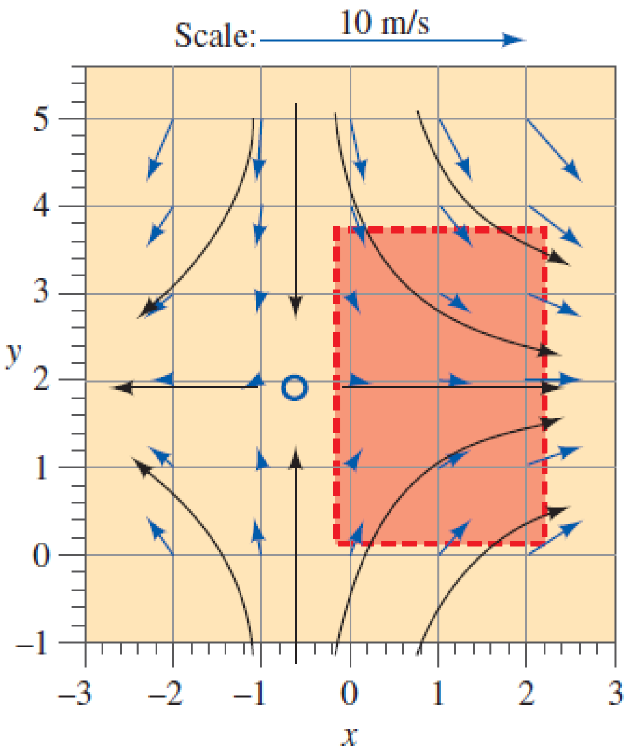
While there are many occasions in which the Lagrangian description is useful, the Eulerian description is often more convenient for fluid mechanics applications.

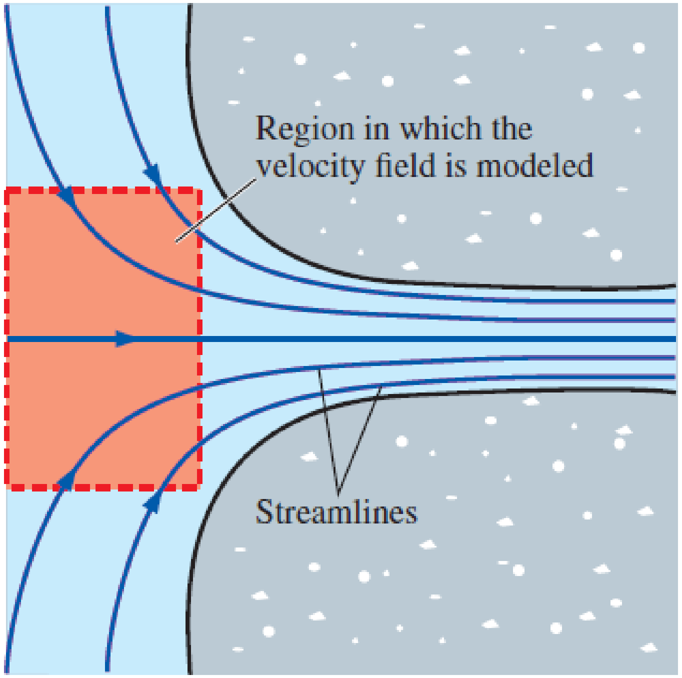
Experimental measurements are generally more suited to the Eulerian description.

(a) In the Eulerian description, we define field variables, such as the pressure field and the velocity field, at any location and instant in time. (*b) For example, the air speed probe* mounted under the wing of an airplane measures the air speed at that location.

A Steady Two-Dimensional Velocity Field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$





Flow field near the bell mouth inlet of a hydroelectric dam; a portion of the velocity field of Example 4-1 may be used as a first-order approximation of this physical flow field.

Velocity vectors for the velocity field of Example 4–1. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the blue circle. The shaded region represents a portion of the flow field that can approximate flow into an inlet.

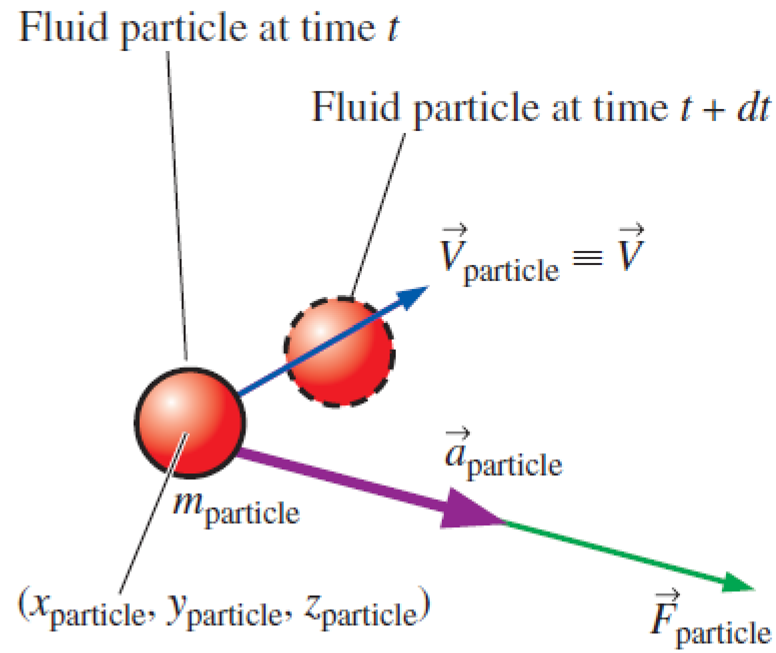
Acceleration Field

The equations of motion for fluid flow (such as Newton's second law) are written for a fluid particle, which we also call a **material particle**.

If we were to follow a particular fluid particle as it moves around in the flow, we would be employing the Lagrangian description, and the equations of motion would be directly applicable.

For example, we would define the particle's location in space in terms of a **material position vector**

$x_{\text{particle}}(t), y_{\text{particle}}(t), z_{\text{particle}}(t)$



Newton's second law applied to a fluid particle; the acceleration vector (purple arrow) is in the same direction as the force vector (green arrow), but the velocity vector (blue arrow) may act in a different direction.

$$\textit{Newton's second law:} \qquad \vec{F}_{\text{particle}} = m_{\text{particle}}\,\vec{a}_{\text{particle}}$$

$$\textit{Acceleration of a fluid particle:} \qquad \vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt}$$

$$\vec{V}_{\text{particle}}(t) \equiv \vec{V}\left(x_{\text{particle}}(t),\ y_{\text{particle}}(t),\ z_{\text{particle}}(t),\ t\right)$$

$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}\left(x_{\text{particle}},\ y_{\text{particle}},\ z_{\text{particle}},\ t\right)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt}$$

$$\vec{a}_{\text{particle}}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

Acceleration of a fluid particle expressed as a field variable:

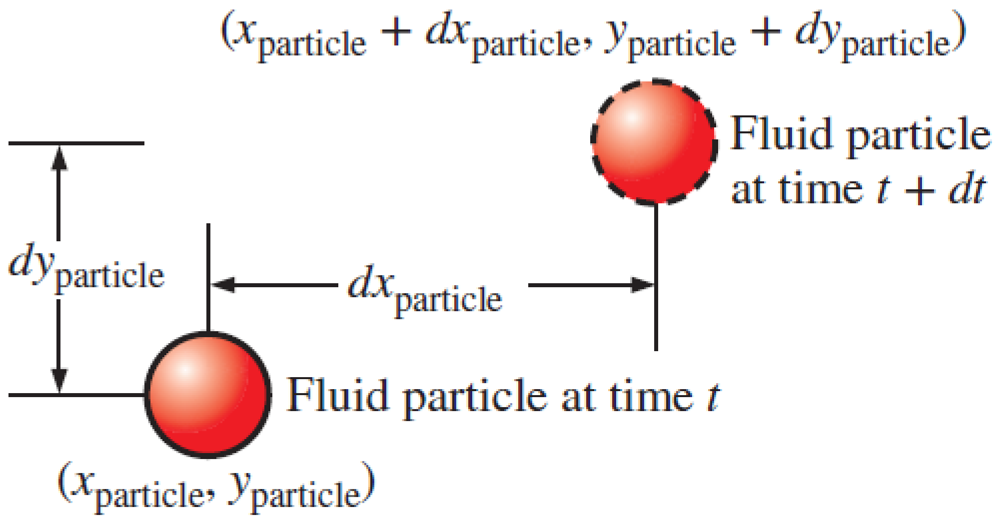
$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

$\dfrac{\partial \vec{V}}{\partial t}$ Local acceleration

$\left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$ Advective (convective) acceleration

Gradient or del operation:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$



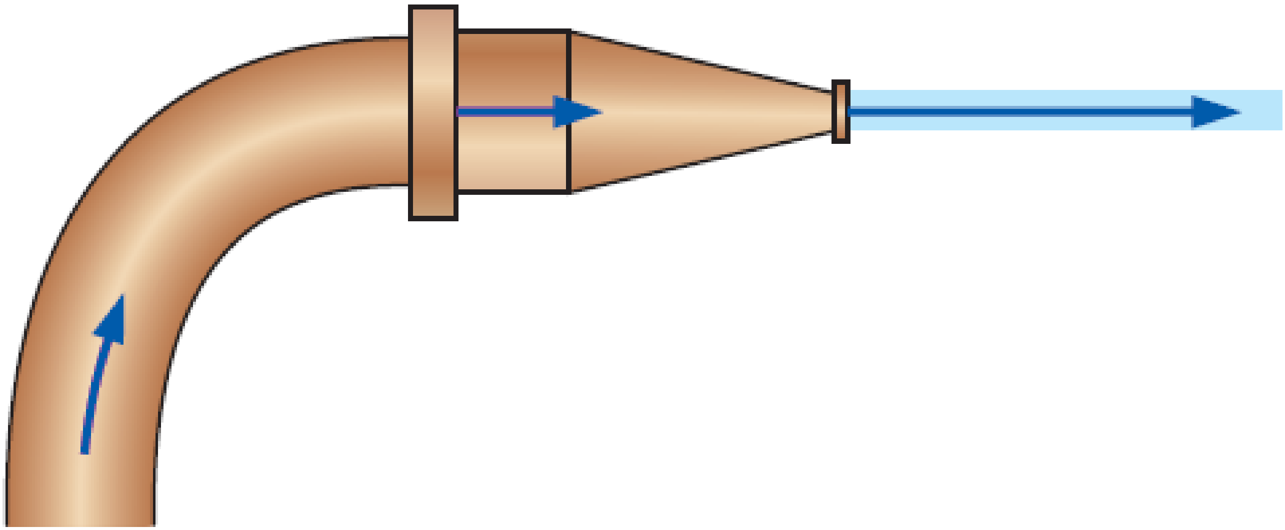
When following a fluid particle, the *x*-component of velocity, *u*, is defined as $dx_{particle}/dt$. Similarly, $v=dy_{particle}/dt$ and $w=dz_{particle}/dt$. Movement is shown here only in two dimensions for simplicity.

The components of the acceleration vector in cartesian coordinates:

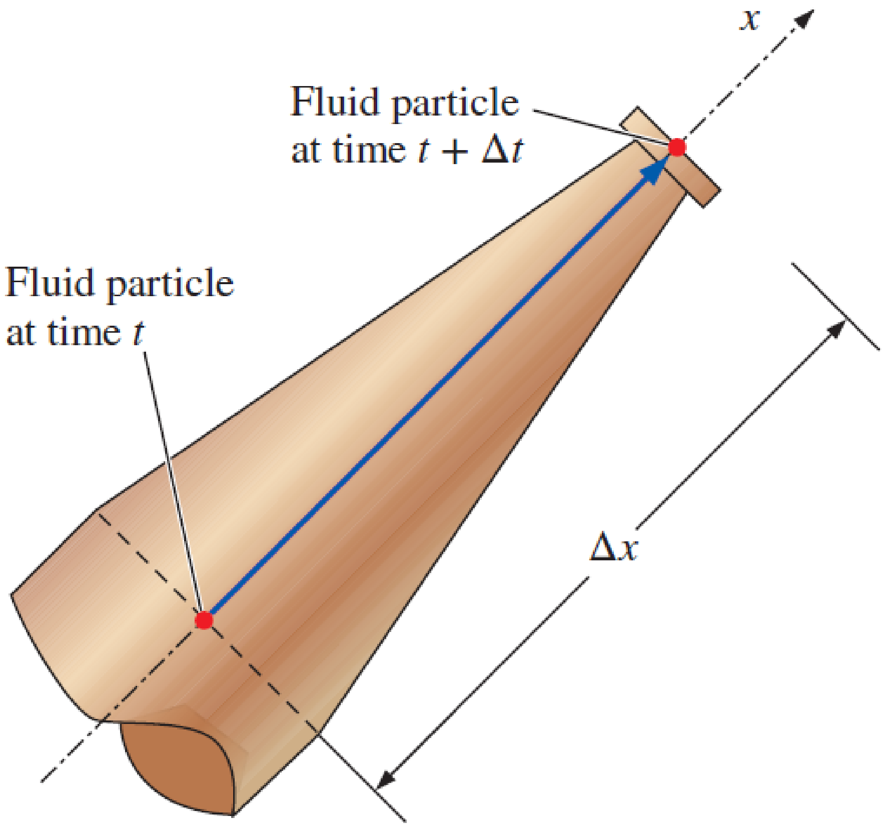
$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

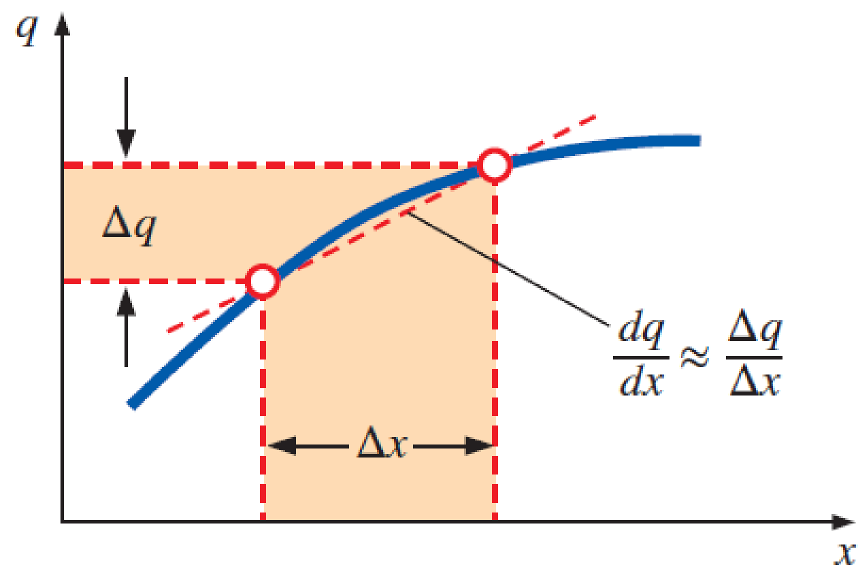
$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$



Flow of water through the nozzle of a garden hose illustrates that fluid particles may accelerate, even in a steady flow. In this example, the exit speed of the water is much higher than the water speed in the hose, implying that fluid particles have accelerated even though the flow is steady.



Residence time Δt is defined as the time it takes for a fluid particle to travel through the nozzle from inlet to outlet (distance Δx).



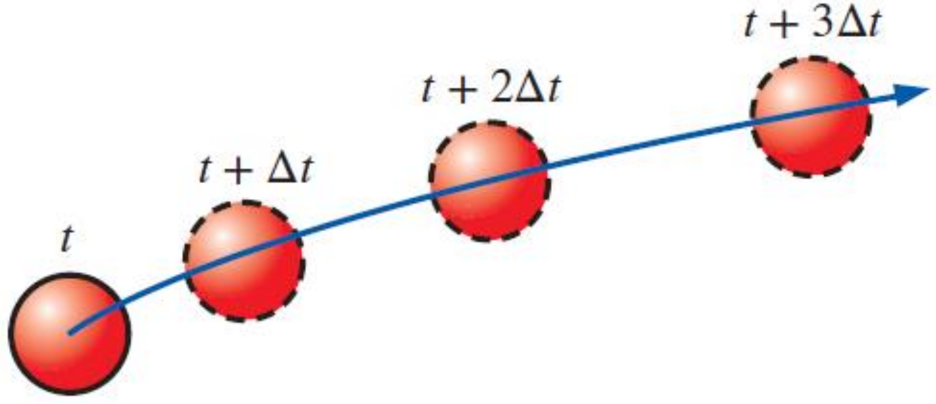
A first-order finite difference approximation for derivative dq/dx is simply the change in dependent variable (q) divided by the change in independent variable (x).

Material Derivative

$$\vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

The total derivative operator *d/dt* in this equation is given a special name, the **material derivative**; it is assigned a special notation, *D/Dt*, in order to emphasize that it is formed by *following a fluid particle as it moves through the flow field*.

Other names for the material derivative include **total, particle, Lagrangian, Eulerian**, and **substantial derivative**.

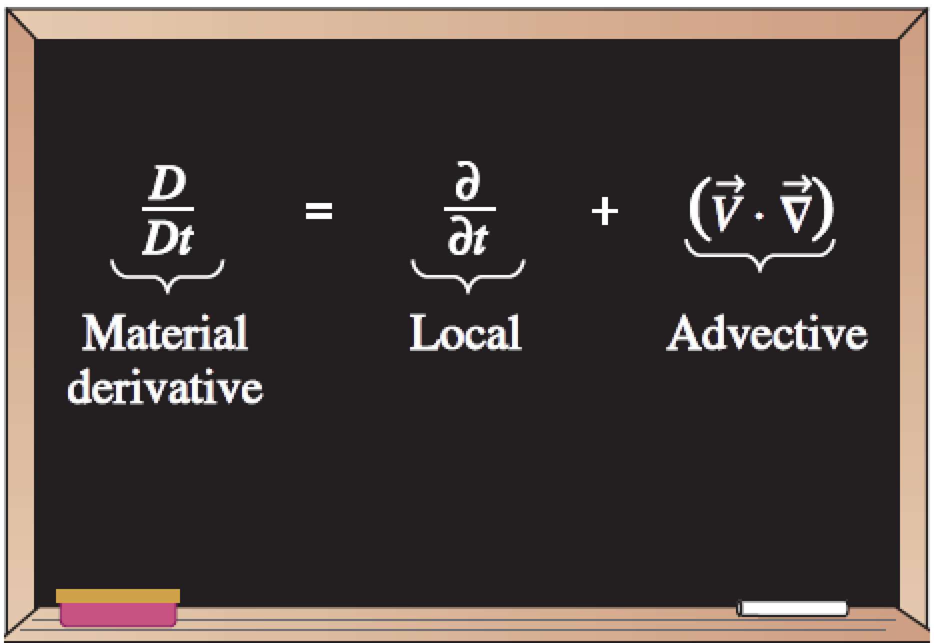


The material derivative *D/Dt* is defined by following a fluid particle as it moves throughout the flow field. In this illustration, the fluid particle is accelerating to the right as it moves up and to the right.

Material derivative: $\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)$

Material acceleration: $\vec{a}(x, y, z, t) = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$

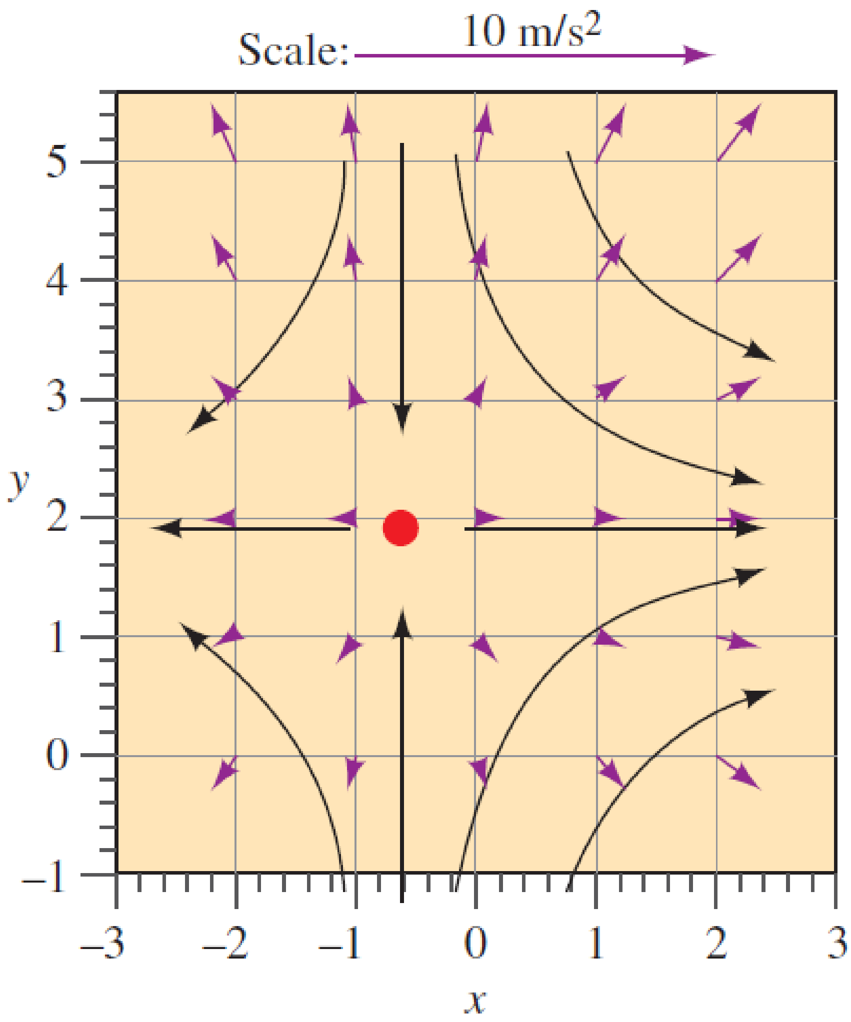
Material derivative of pressure: $\frac{DP}{Dt} = \frac{dP}{dt} = \frac{\partial P}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)P$



The material derivative *D*/*Dt* is composed of a *local* or *unsteady* part and a *convective* or *advective* part.

Material Acceleration of a Steady Velocity Field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$



Acceleration vectors for the velocity field of Examples 4–1 and 4–3. The scale is shown by the top arrow, and the solid black curves represent the approximate shapes of some streamlines, based on the calculated velocity vectors. The stagnation point is indicated by the red circle.

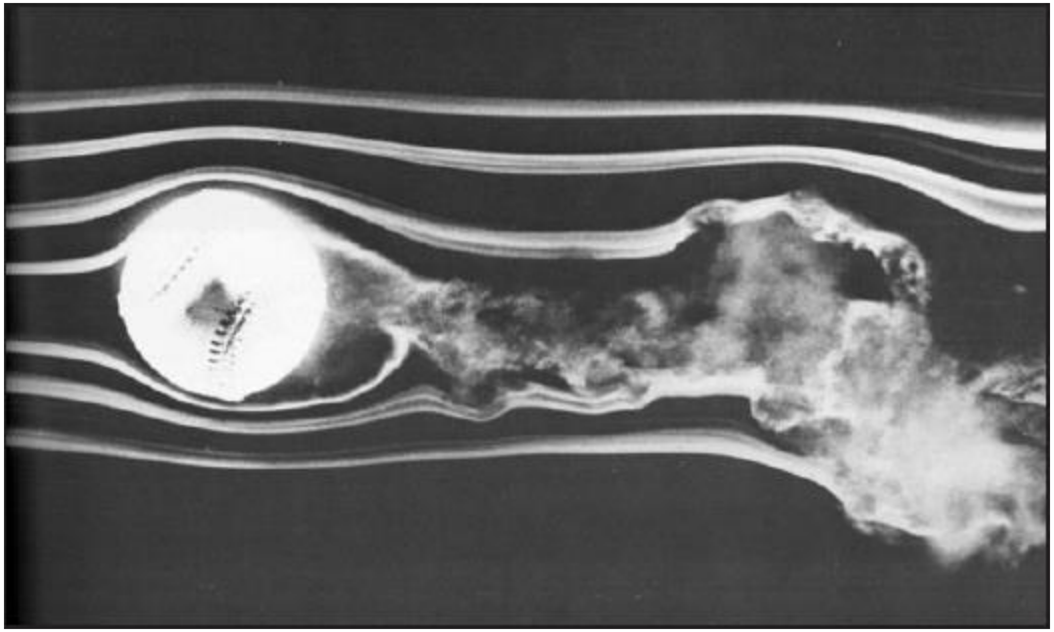
3–2 ■ FLOW PATTERNS AND FLOW VISUALIZATION

Flow visualization: The visual examination of flow field features.

While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from flow visualization.

Flow visualization is useful not only in physical experiments but in *numerical* solutions as well [**computational fluid dynamics (CFD)**].

In fact, the very first thing an engineer using CFD does after obtaining a numerical solution is simulate some form of flow visualization.



Courtesy of Professor Thomas J. Mueller from the Collection of Professor F.N.M. Brown.

Spinning baseball. The late F. N. M. Brown devoted many years to developing and using smoke visualization in wind tunnels at the University of Notre Dame. Here the flow speed is about 23 m/s and the ball is rotated at 630 rpm.

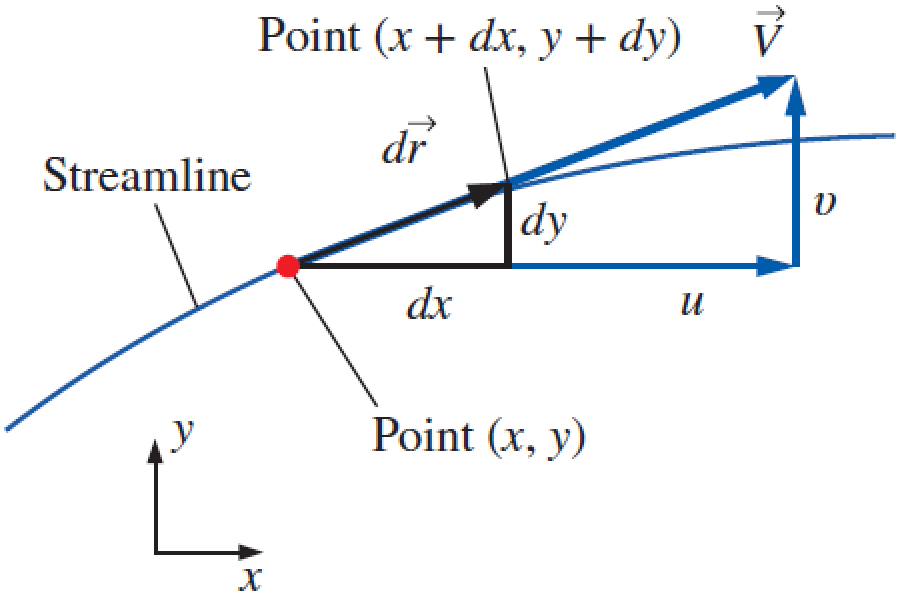
Streamlines and Streamtubes

Streamline: A curve that is everywhere tangent to the instantaneous local velocity vector.

Streamlines are useful as indicators of the instantaneous direction of fluid motion throughout the flow field.

For example, regions of recirculating flow and separation of a fluid off of a solid wall are easily identified by the streamline pattern.

Streamlines cannot be directly observed experimentally except in steady flow fields.



For two-dimensional flow in the *xy*-plane, arc length $d\vec{r} = (dx, dy)$ along a *streamline* is everywhere tangent to the local instantaneous velocity vector $\vec{V} = (u, v)$.

Consider an infinitesimal arc length $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ along a streamline; $d\vec{r}$ must be parallel to the local velocity vector $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$ by definition of the streamline. By simple geometric arguments using similar triangles, we know that the components of $d\vec{r}$ must be proportional to those of $\vec{V}$ (Fig. 4–16). Hence,

$$\text{Equation for a streamline:} \quad \frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \textbf{(4-15)}$$

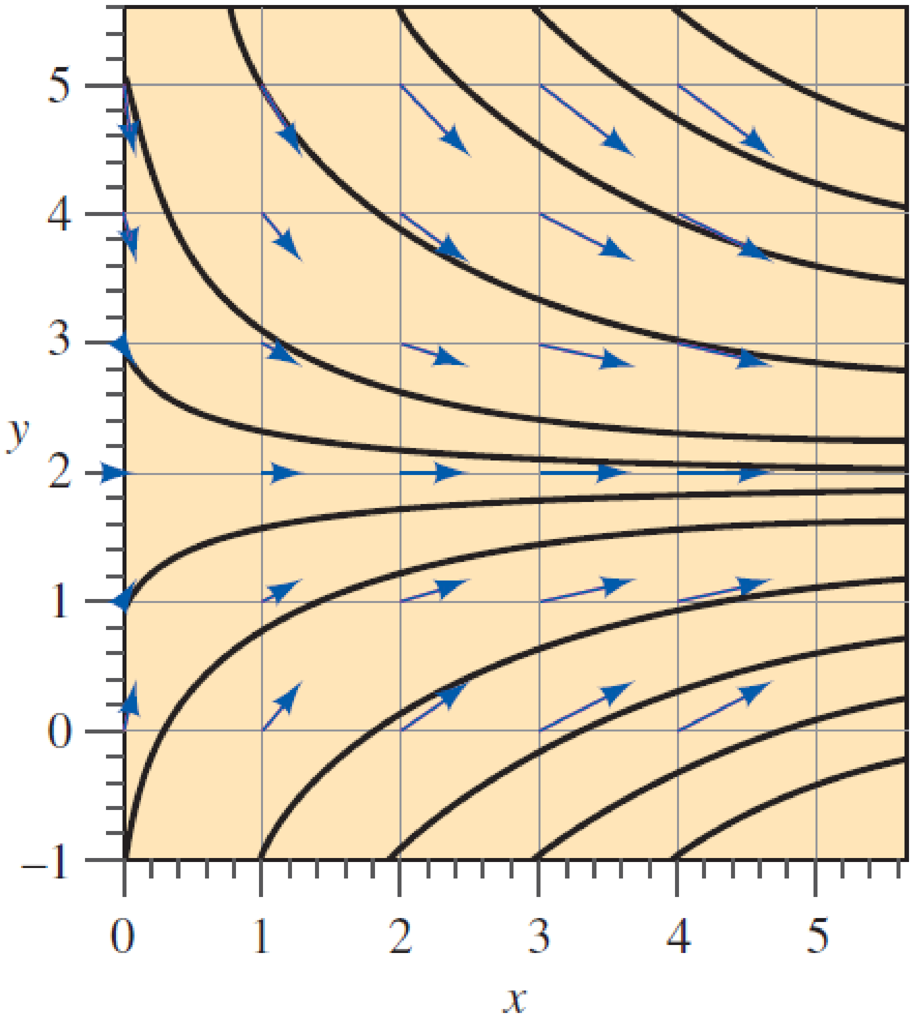
where *dr* is the magnitude of $d\vec{r}$ and *V* is the speed, the magnitude of velocity vector $\vec{V}$. Equation 4–15 is illustrated in two dimensions for simplicity in Fig. 4–16. For a known velocity field, we integrate Eq. 4–15 to obtain equations for the streamlines. In two dimensions, (*x*, *y*), (*u*, v), the following differential equation is obtained:

$$\text{Streamline in the xy-plane:} \quad \left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \qquad \textbf{(4-16)}$$

In some simple cases, Eq. 4–16 may be solvable analytically; in the general case, it must be solved numerically. In either case, an arbitrary constant of integration appears. Each chosen value of the constant represents a different streamline. The *family* of curves that satisfy Eq. 4–16 therefore represents streamlines of the flow field.

Streamlines for a steady, incompressible, two-dimensional velocity field

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$



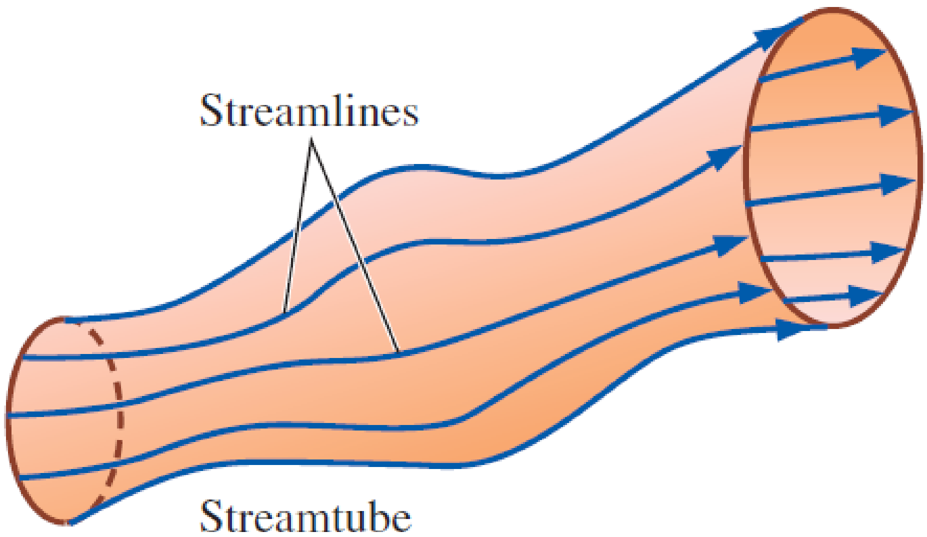
Streamlines (solid black curves) for the velocity field of Example 4–4; velocity vectors (blue arrows) are superimposed for comparison.

The agreement is excellent in the sense that the velocity vectors point everywhere tangent to the streamlines. Note that speed cannot be determined directly from the streamlines alone.

A **streamtube** consists of a bundle of streamlines much like a communications cable consists of a bundle of fiber-optic cables.

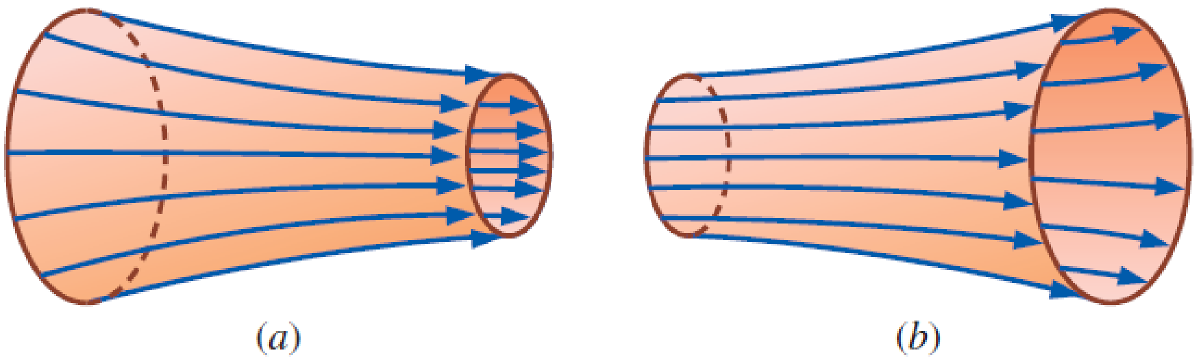
Since streamlines are everywhere parallel to the local velocity, fluid cannot cross a streamline by definition.

***Fluid within a streamtube must remain there and cannot cross the boundary of the streamtube*.**



A *streamtube* consists of a bundle of individual streamlines.

Both streamlines and streamtubes are instantaneous quantities, defined at a particular instant in time according to the velocity field at that instant.



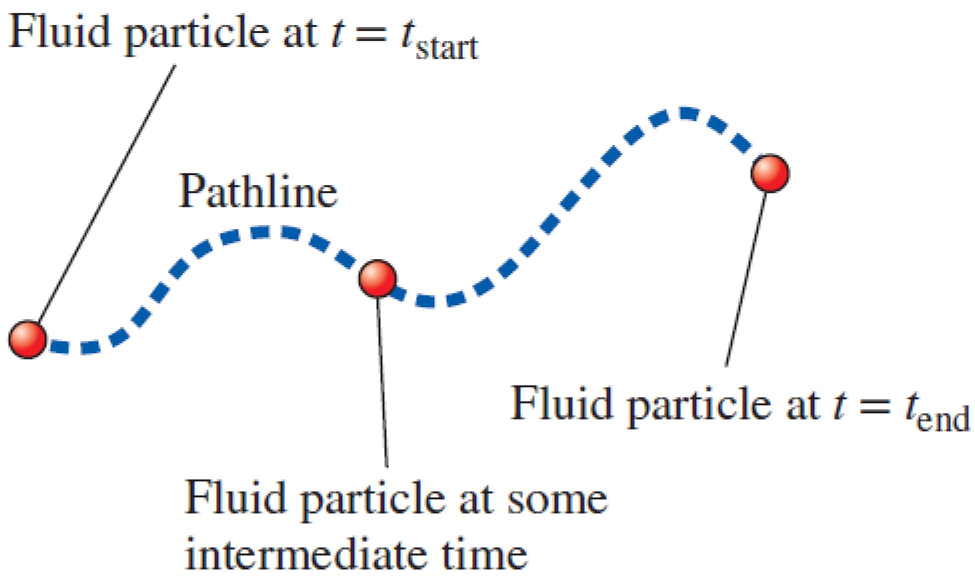
In an incompressible flow field, a streamtube (*a*) decreases in diameter as the flow accelerates or converges and (*b*) increases in diameter as the flow decelerates or diverges.

Pathlines

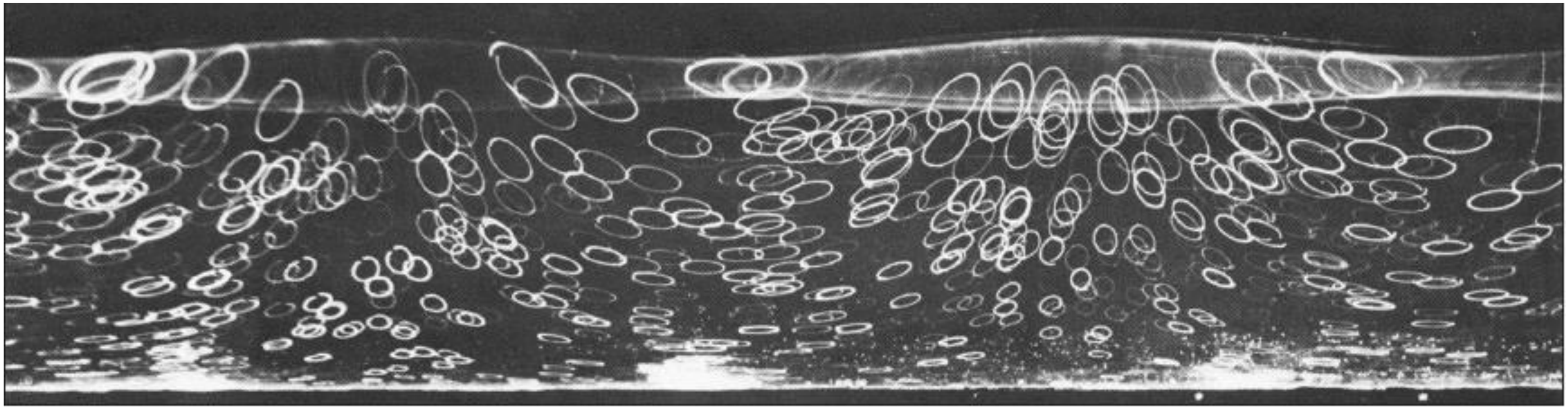
Pathline: The actual path traveled by an individual fluid particle over some time period.

A pathline is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.

Thus, a pathline is the same as the fluid particle's material position vector ($x_{\text{particle}}(t)$, $y_{\text{particle}}(t)$, $z_{\text{particle}}(t)$) traced out over some finite time interval.



A *pathline* is formed by following the actual path of a fluid particle.



Wallet, A & Ruellan, F. 1950, La Houille Blanche 5: 483–489. Used by permission.

Pathlines produced by white tracer particles suspended in water and captured by time-exposure photography; as waves pass horizontally, each particle moves in an elliptical path during one wave period.

Particle image velocimetry (PIV): A modern experimental technique that utilizes short segments of particle pathlines to measure the velocity field over an entire plane in a flow.

Recent advances also extend the technique to three dimensions.

In PIV, tiny tracer particles are suspended in the fluid. However, the flow is illuminated by two flashes of light (usually a light sheet from a laser) to produce two bright spots (recorded by a camera) for each moving particle.

Then, both the magnitude and direction of the velocity vector at each particle location can be inferred, assuming that the tracer particles are small enough that they move with the fluid.

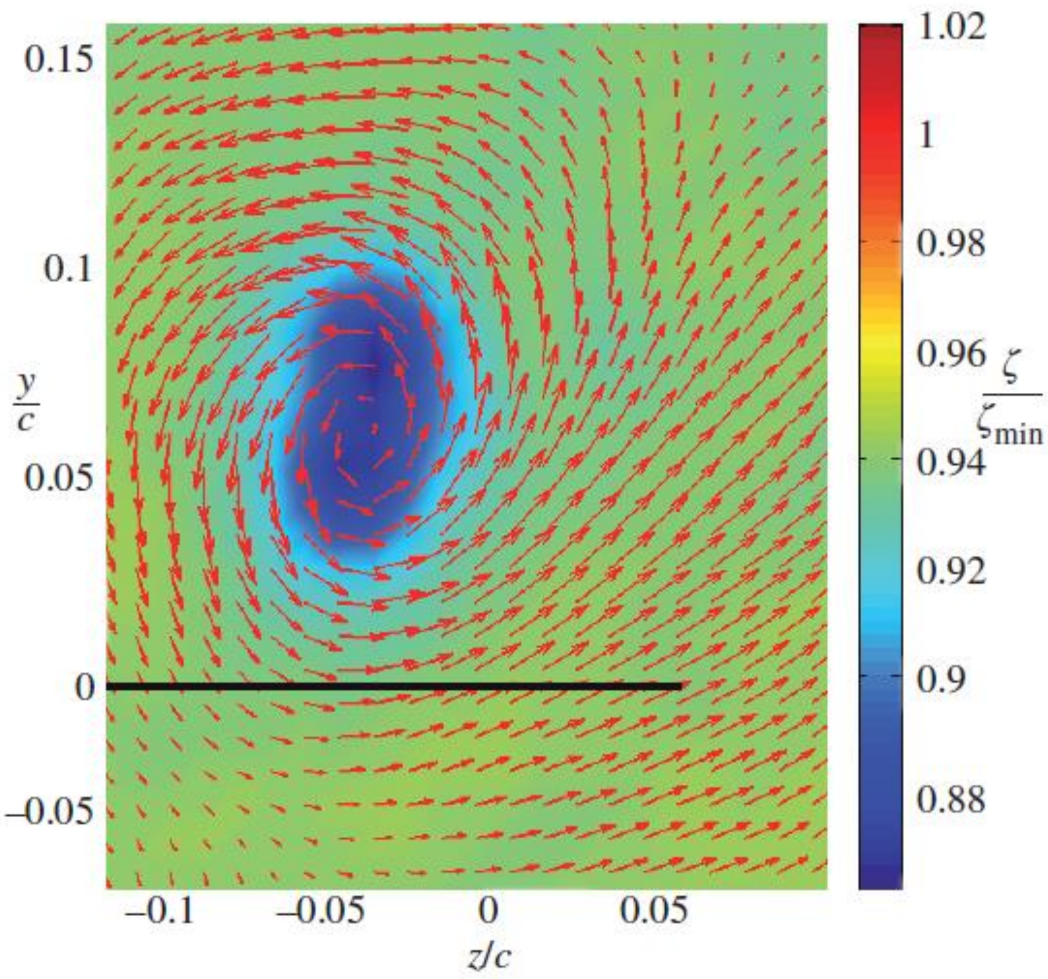


Photo by Michael H. Krane, ARL-Penn State.

Stereo PIV measurements of the wing tip vortex in the wake of a NACA-66 airfoil at angle of attack. Color contours denote the local vorticity, normalized by the minimum value, as indicated in the color map. Vectors denote fluid motion in the plane of measurement. The black line denotes the location of the upstream wing trailling edge. Coordinates are normalized by the airfoil chord, and the origin is the wing root.

3–2 ■ FLOW PATTERNS AND FLOW VISUALIZATION(9)

Pathlines can also be calculated numerically for a known velocity field. Specifically, the location of the tracer particle is integrated over time from some starting location $\vec{x}_{\text{start}}$ and starting time t_{start} to some later time *t*.

$$\textit{Tracer particle location at time } t: \quad \vec{x} = \vec{x}_{\text{start}} + \int_{t_{\text{start}}}^{t} \vec{V}\, dt \qquad \textbf{(4-17)}$$

When Eq. 4–17 is calculated for *t* between t_{start} and t_{end}, a plot of $\vec{x}(t)$ is the pathline of the fluid particle during that time interval, as illustrated in Fig. 4–20. For some simple flow fields, Eq. 4–17 can be integrated analytically. For more complex flows, we must perform a numerical integration.

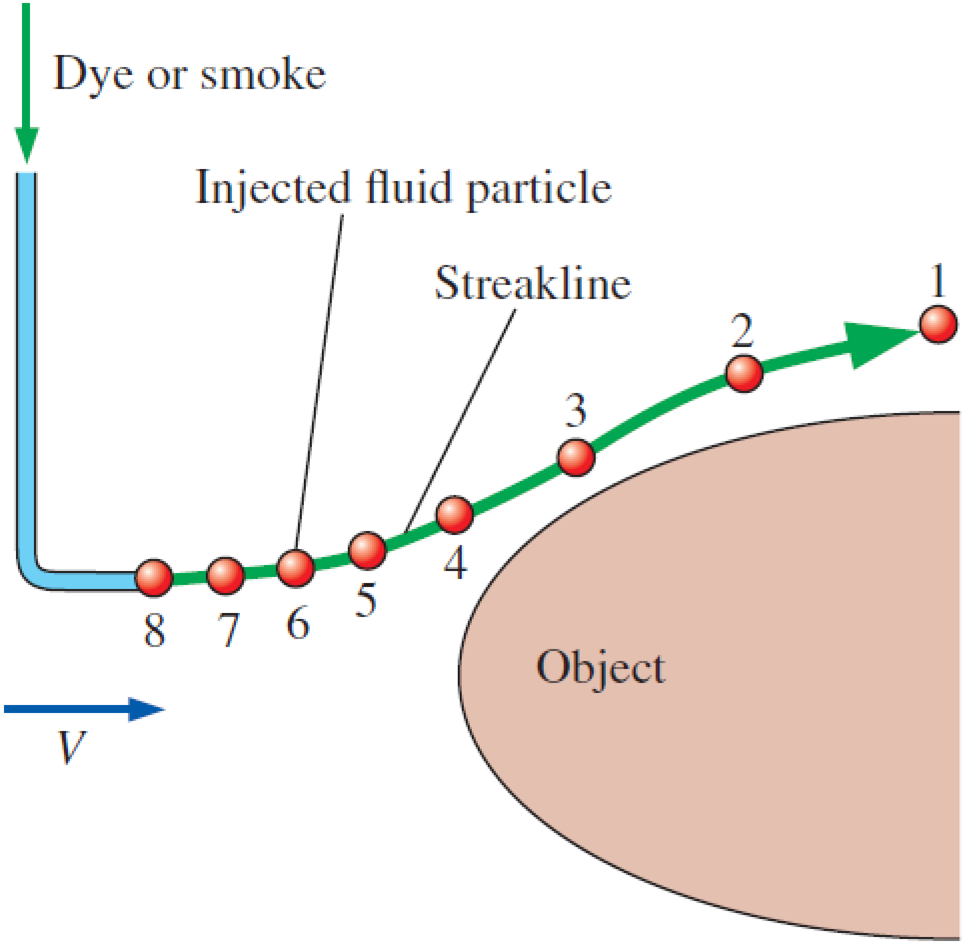
If the velocity field is steady, individual fluid particles follow streamlines. Thus, *for steady flow, pathlines are identical to streamlines*.

Streaklines

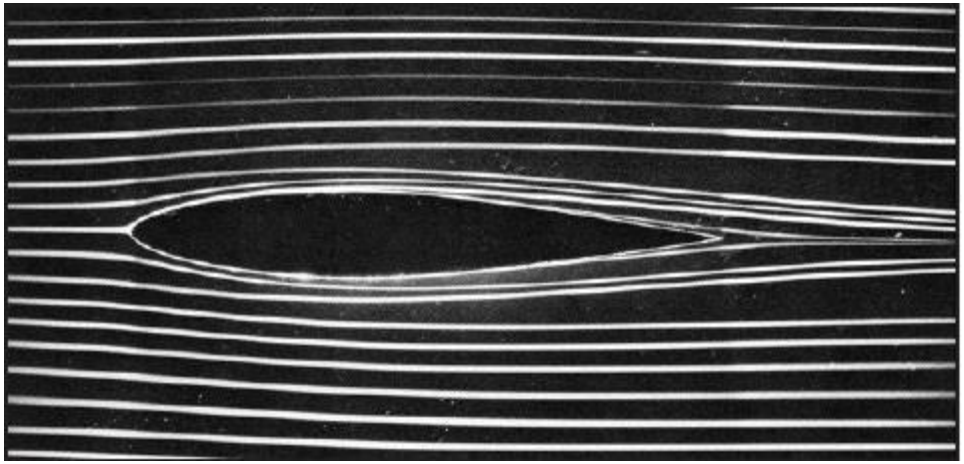
Streakline: The locus of fluid particles that have passed sequentially through a prescribed point in the flow.

Streaklines are the most common flow pattern generated in a physical experiment.

If you insert a small tube into a flow and introduce a continuous stream of tracer fluid (dye in a water flow or smoke in an air flow), the observed pattern is a streakline.



A *streakline* is formed by continuous introduction of dye or smoke from a point in the flow. Labeled tracer particles (1 through 8) were introduced sequentially.



Courtesy of ONERA. Photo by Werlé.

Streaklines produced by colored fluid introduced upstream; since the flow is steady, these streaklines are the same as streamlines and pathlines.

Streaklines, streamlines, and pathlines are identical in steady flow but they can be quite different in unsteady flow.

The main difference is that a streamline represents an *instantaneous* flow pattern at a given instant in time, while a streakline and a pathline are flow patterns that have some *age* and thus a *time history* associated with them.

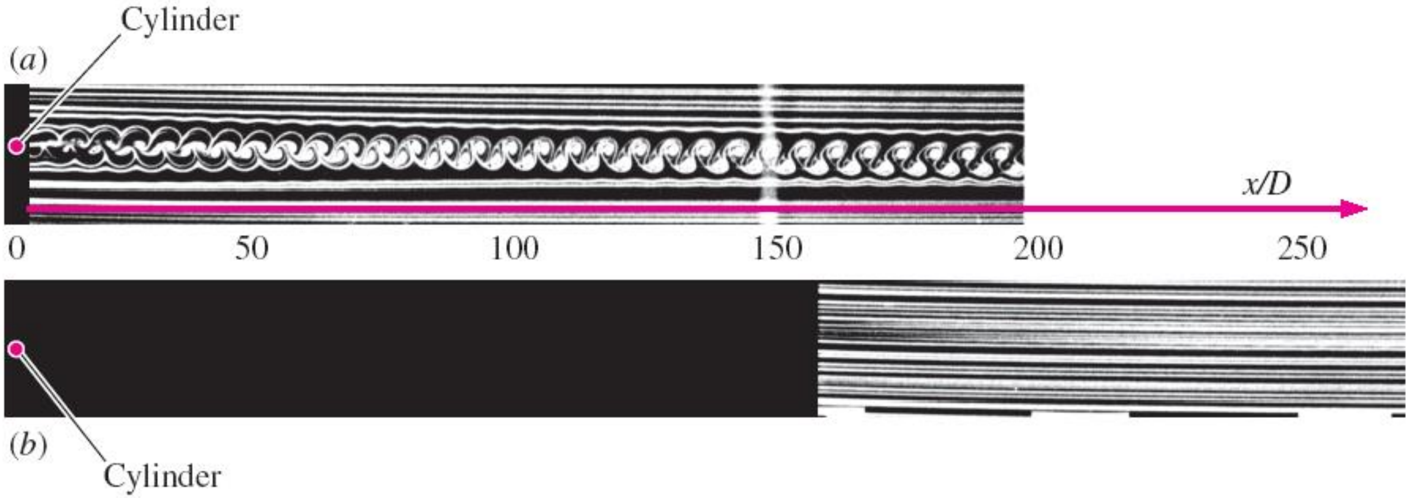
A streakline is an instantaneous snapshot of a *time-integrated* flow pattern.

A pathline, on the other hand, is the *time-exposed* flow path of an individual particle over some time period.

In the figure, streaklines are introduced from a smoke wire located just downstream of a circular cylinder of diameter *D* aligned normal to the plane of view.

When multiple streaklines are introduced along a line, as in the figure, we refer to this as a **rake** of streaklines.

The Reynolds number of the flow is Re = 93.

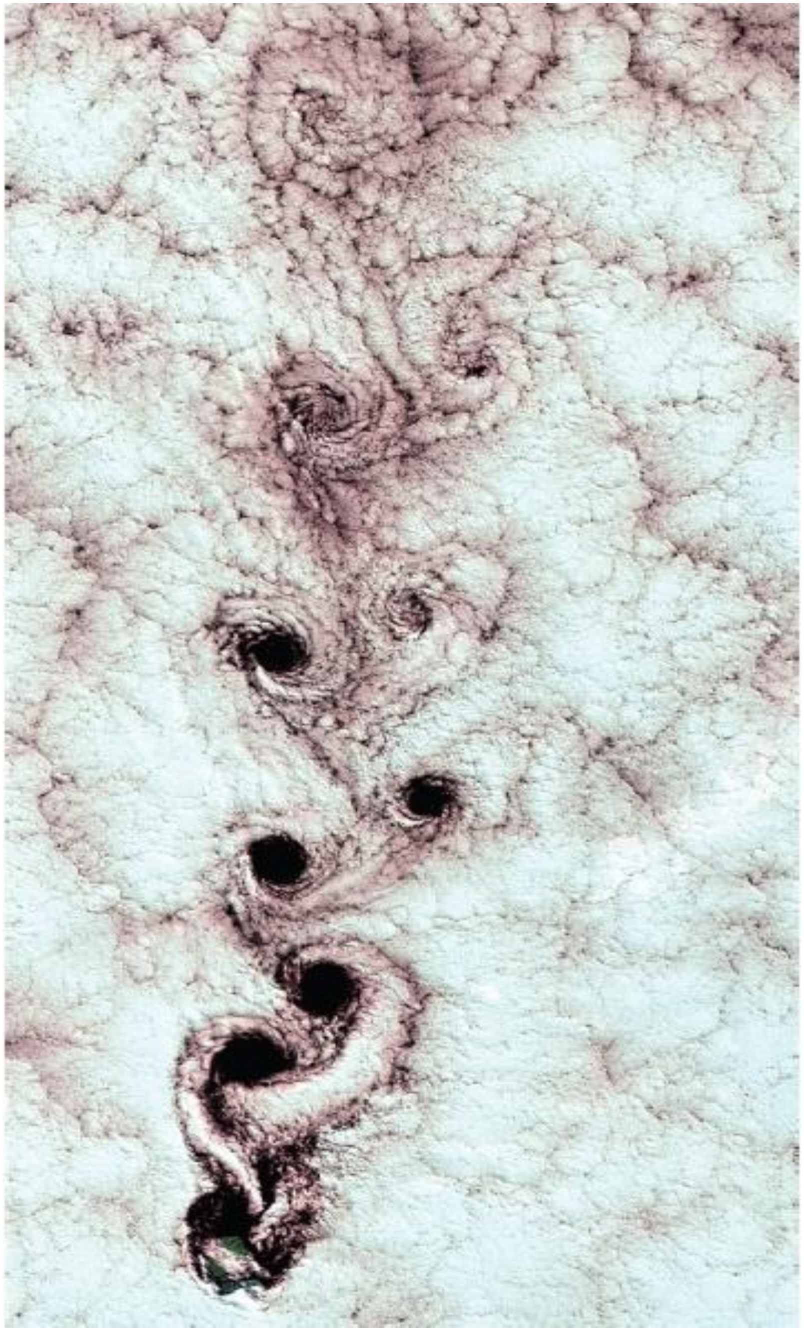


Photos by John M. Cimbala.

Smoke streaklines introduced by a smoke wire at two different locations in the wake of a circular cylinder: (*a*) smoke wire just downstream of the cylinder and (*b*) smoke wire located at $x/D = 150$. The time-integrative nature of streaklines is clearly seen by comparing the two photographs.

Because of unsteady **vortices** shed in an alternating pattern from the cylinder, the smoke collects into a clearly defined periodic pattern called a **Kármán vortex street**.

A similar pattern can be seen at much larger scale in the air flow in the wake of an island.



Kármán vortices visible in the clouds in the wake of Alexander Selkirk Island in the southern Pacific Ocean.

Photo from Landsat 7 WRS Path 6 Row 83, center: -33.18, -79.99, 9/15/1999, earthobservatory.nasa.gov. Courtesy of USGS EROS Data Center Satellite System Branch/NASA.

For a known velocity field, a streakline can be generated numerically. We need to follow the paths of a continuous stream of tracer particles from the time of their injection into the flow until the present time, using Eq. 4–17. Mathematically, the location of a tracer particle is integrated over time from the time of its injection t_{inject} to the present time t_{present}. Equation 4–17 becomes

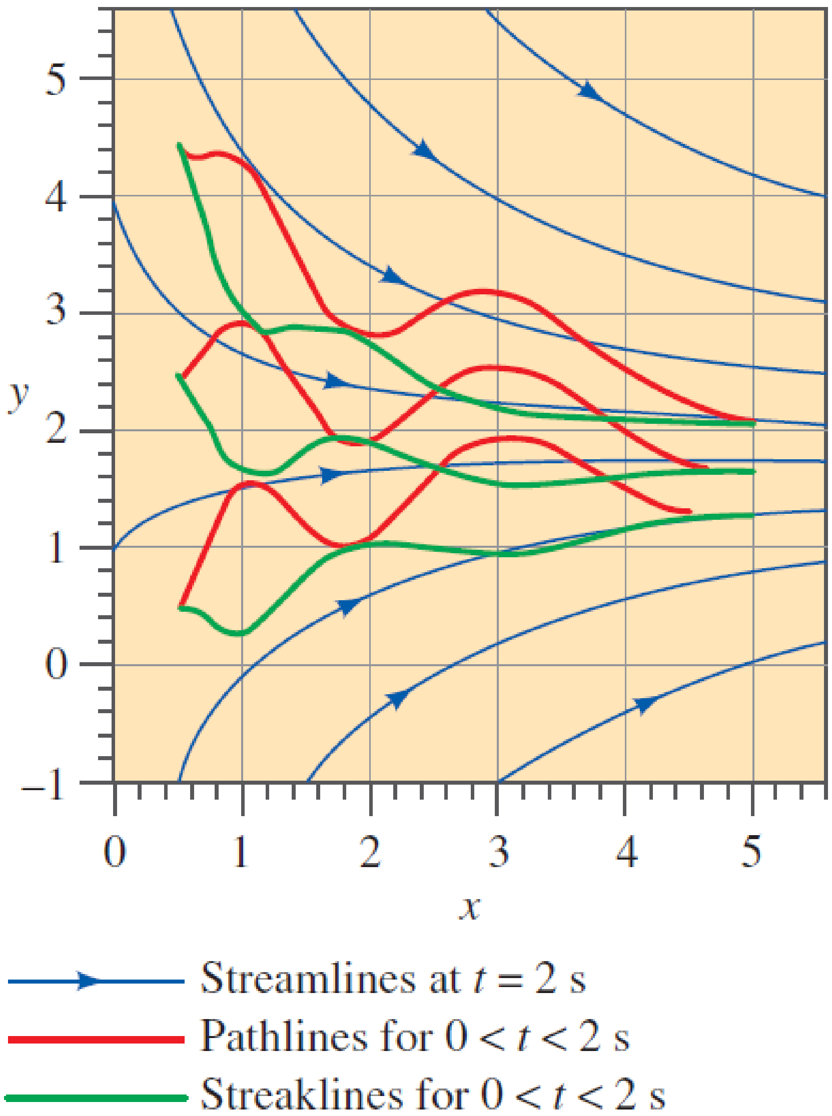
$$\textit{Integrated tracer particle location:} \quad \vec{x} = \vec{x}_{\text{injection}} + \int_{t_{\text{inject}}}^{t_{\text{present}}} \vec{V}\, dt \qquad \textbf{(4-18)}$$

In a complex unsteady flow, the time integration must be performed numerically as the velocity field changes with time. When the locus of tracer particle locations at $t = t_{\text{present}}$ is connected by a smooth curve, the result is the desired streakline.

$$\textit{Tracer particle location at time } t\textit{:} \quad \vec{x} = \vec{x}_{\text{start}} + \int_{t_{\text{start}}}^{t} \vec{V}\, dt \qquad \textbf{(4-17)}$$

Comparison of Flow Patterns in an Unsteady Flow

$$\vec{V}=(u,v)=(0.5+0.8x)\,\vec{i}+\left(1.5+2.5\sin(\omega t)-0.8y\right)\vec{j}$$



An *unsteady*, incompressible, two-dimensional velocity field

Streamlines, pathlines, and streaklines for the oscillating velocity field of Example 4–5. The streaklines and pathlines are wavy because of their integrated time history, but the streamlines are not wavy since they represent an instantaneous snapshot of the velocity field.

3–3 ■ PLOTS OF FLUID FLOW DATA

Regardless of how the results are obtained (analytically, experimentally, or computationally), it is usually necessary to *plot* flow data in ways that enable the reader to get a feel for how the flow properties vary in time and/or space.

You are already familiar with *time plots*, which are especially useful in turbulent flows (e.g., a velocity component plotted as a function of time), and *xy*-plots (e.g., pressure as a function of radius).

In this section, we discuss three additional types of plots that are useful in fluid mechanics:

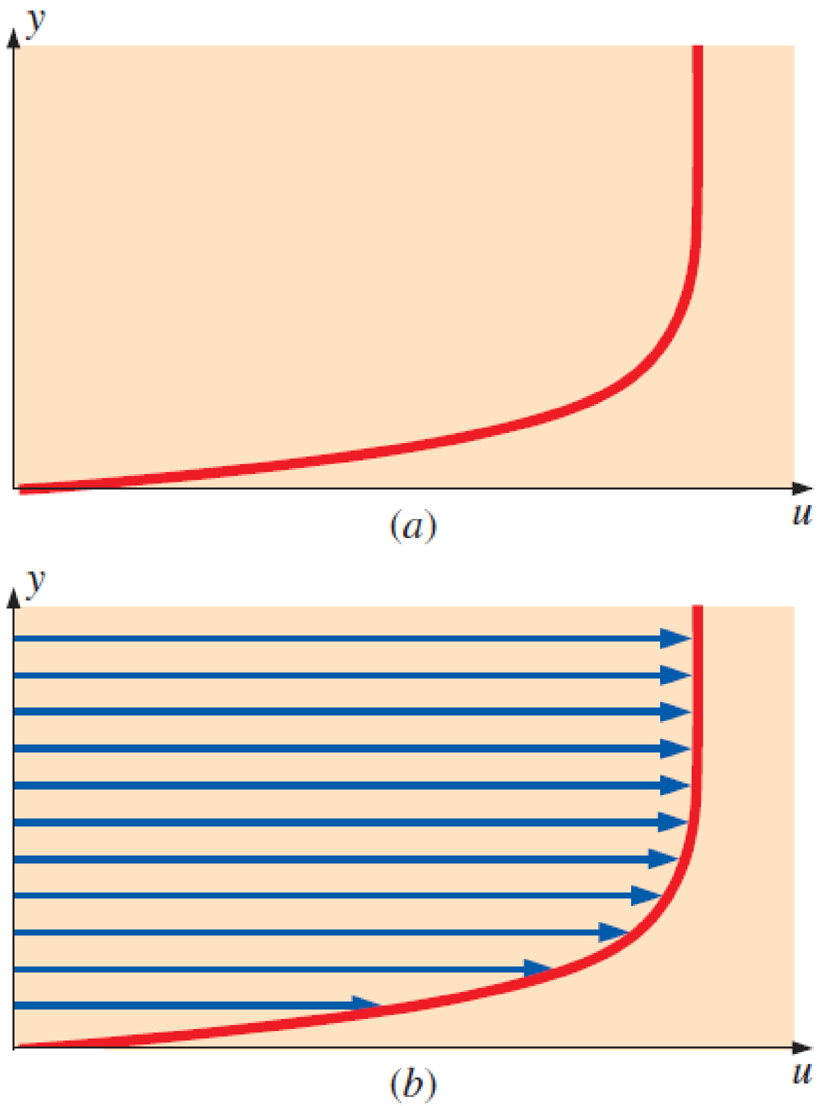
- profile plots
- vector plots
- contour plots

Profile Plots

A **profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.

In fluid mechanics, profile plots of *any* scalar variable (pressure, temperature, density, etc.) can be created, but the most common one used in this book is the ***velocity profile plot***.

Since velocity is a vector quantity, we usually plot either the magnitude of velocity or one of the components of the velocity vector as a function of distance in some desired direction.



Profile plots of the horizontal component of velocity as a function of vertical distance; flow in the boundary layer growing along a horizontal flat plate: (*a*) standard profile plot and (*b*) profile plot with arrows.

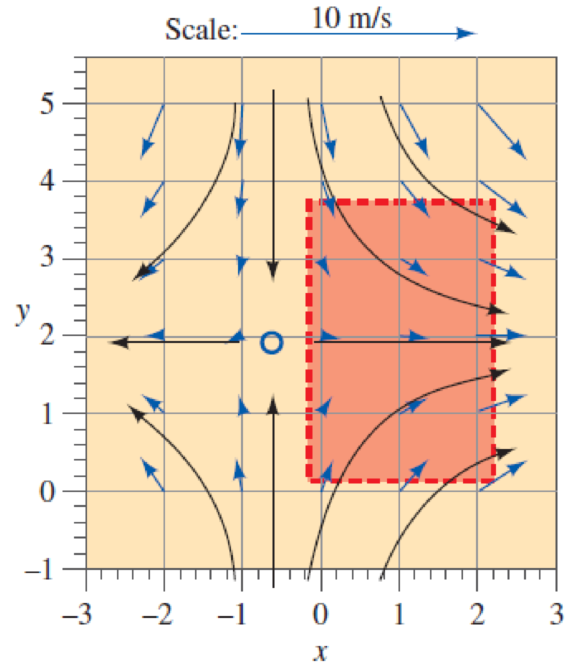
Vector Plots

A vector plot is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.

Streamlines indicate the *direction* of the instantaneous velocity field, they do not directly indicate the *magnitude* of the velocity (i.e., the speed).

A useful flow pattern for both experimental and computational fluid flows is thus the vector plot, which consists of an array of arrows that indicate both magnitude *and* direction of an instantaneous vector property.

Vector plots can also be generated from experimentally obtained data (e.g., from PIV measurements) or numerically from CFD calculations.



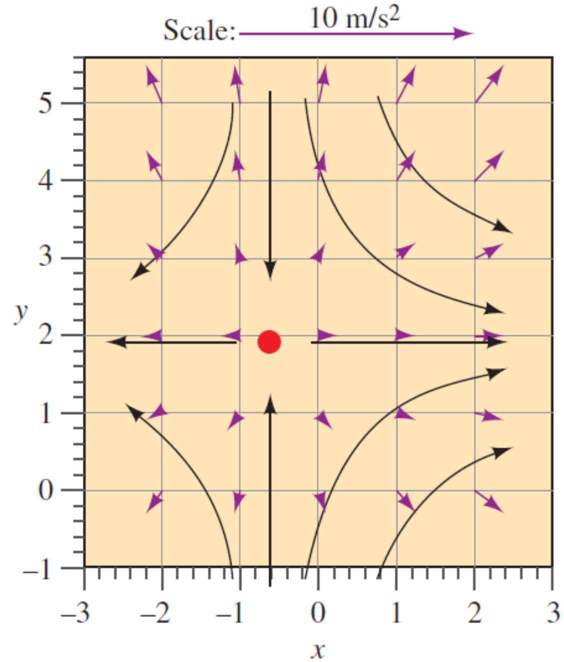
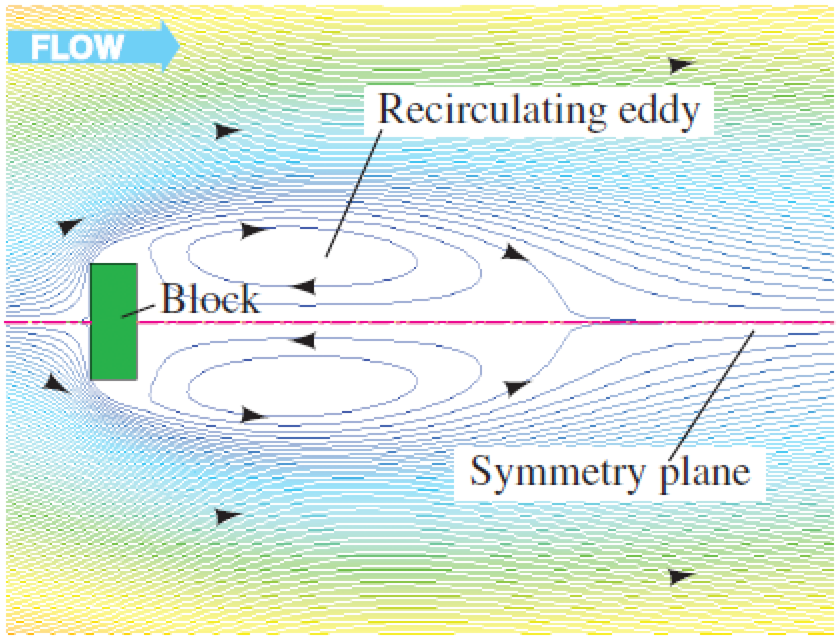
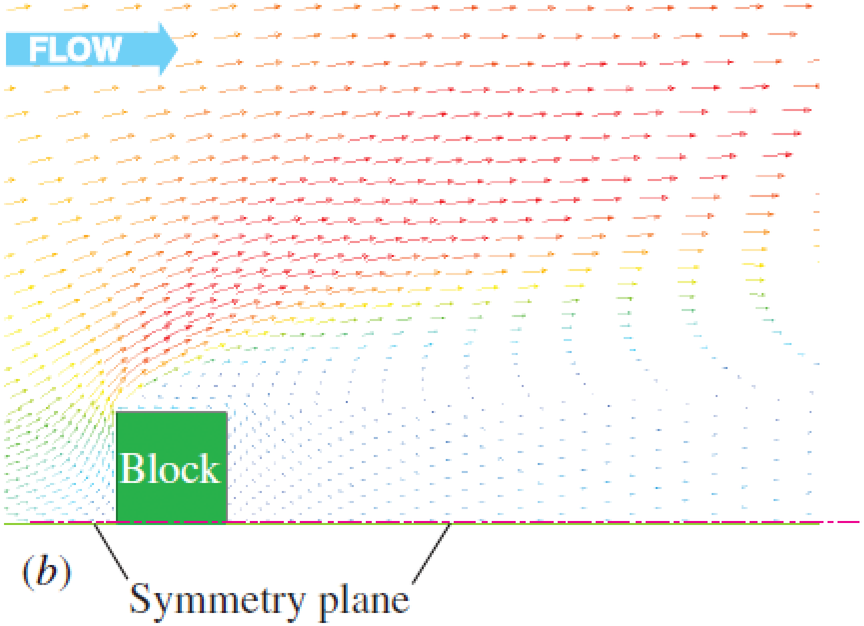


Fig. 4-4: Velocity vector plot
Fig. 4-14: Acceleration vector plot. Both generated analytically.



(*a*)



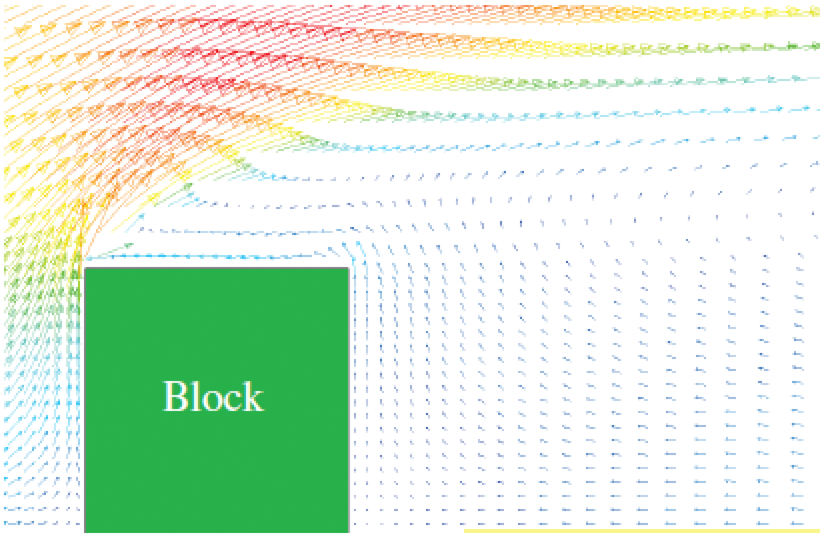
(*b*)

Results of CFD calculations of flow impinging on a block:

(*a*) streamlines

(*b*) velocity vector plot of the upper half of the flow

(*c*) velocity vector plot, close-up view revealing more details in the separated flow region



(*c*)

Contour Plots

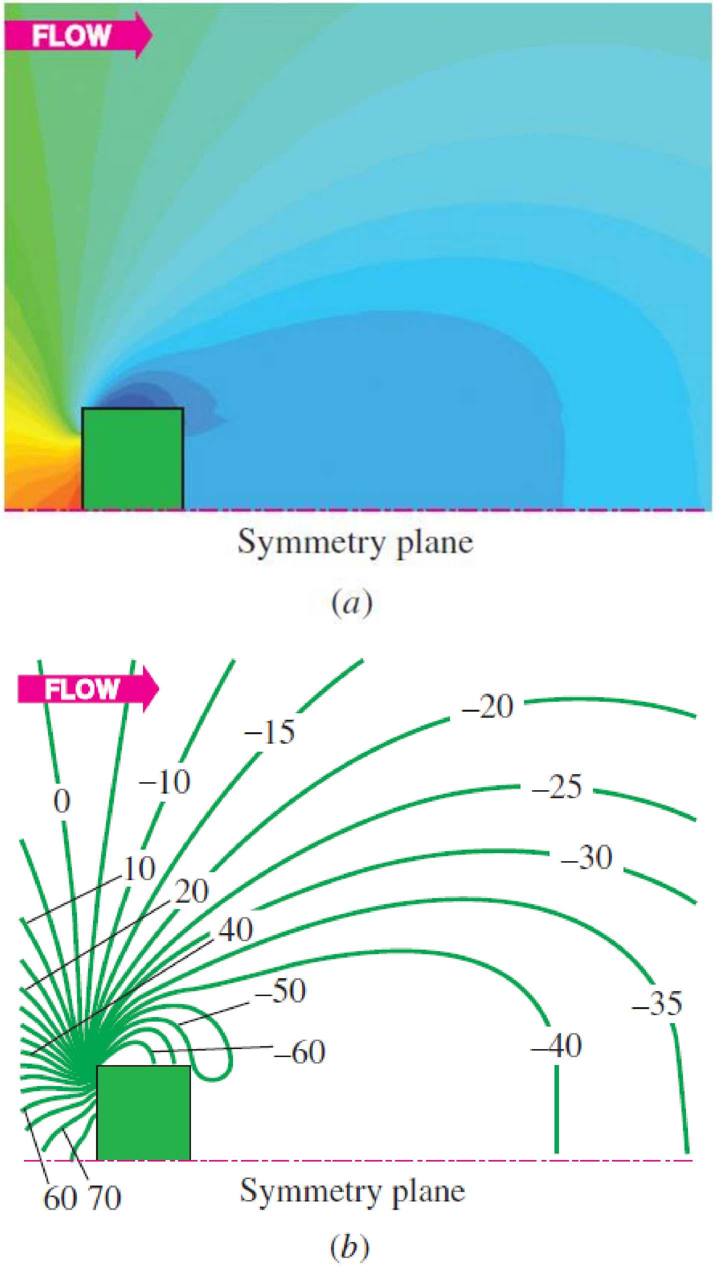
A **contour plot** shows curves of constant values of a scalar property (or magnitude of a vector property) at an instant in time.

Contour plots (also called **isocontour plots**) are generated of pressure, temperature, velocity magnitude, species concentration, properties of turbulence, etc.

A contour plot can quickly reveal regions of high (or low) values of the flow property being studied.

A contour plot may consist simply of curves indicating various levels of the property; this is called a **contour line plot**.

Alternatively, the contours can be filled in with either colors or shades of gray; this is called a **filled contour plot**.



Contour plots of the pressure field due to flow impinging on a block, as produced by CFD calculations; only the upper half is shown due to symmetry; (*a*) filled color scale contour plot and (*b*) contour line plot where pressure values are displayed in units of Pa gage pressure.

3–4 ■ OTHER KINEMATIC DESCRIPTIONS

Types of Motion or Deformation of Fluid Elements

In fluid mechanics, an element may undergo four fundamental types of motion or deformation:

(*a*) **translation**, (*b*) **rotation**,

(*c*) **linear strain** (also called **extensional strain**), and

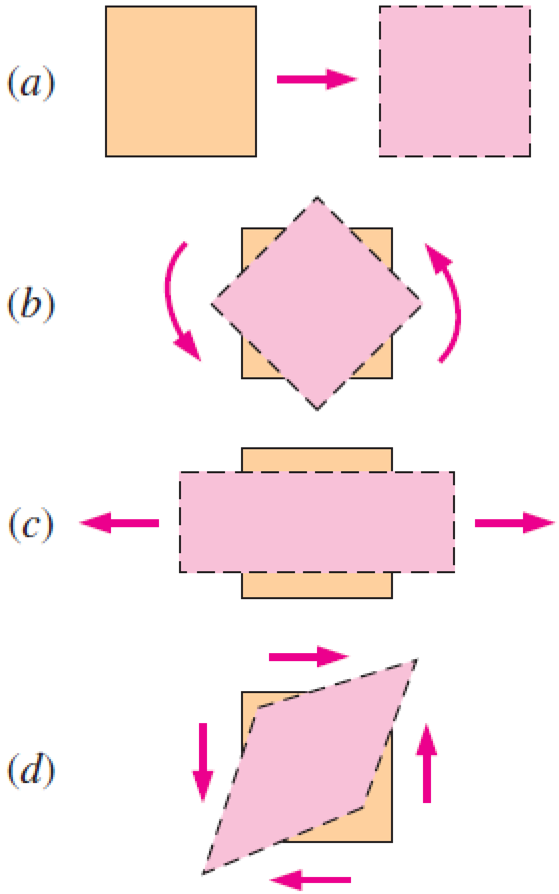
(*d*) **shear strain**.

All four types of motion or deformation usually occur simultaneously.

It is preferable in fluid dynamics to describe the motion and deformation of fluid elements in terms of *rates* such as

- *velocity* (rate of translation),
- *angular velocity* (rate of rotation),
- *linear strain rate* (rate of linear strain), and
- *shear strain rate* (rate of shear strain).

In order for these deformation rates to be useful in the calculation of fluid flows, we must express them in terms of velocity and derivatives of velocity.



Fundamental types of fluid element motion or deformation: (*a*) translation, (*b*) rotation, (*c*) linear strain, and (*d*) shear strain.

A vector is required in order to fully describe the rate of translation in three dimensions. The **rate of translation vector** is described mathematically as the **velocity vector**.

Rate of translation vector in Cartesian coordinates:

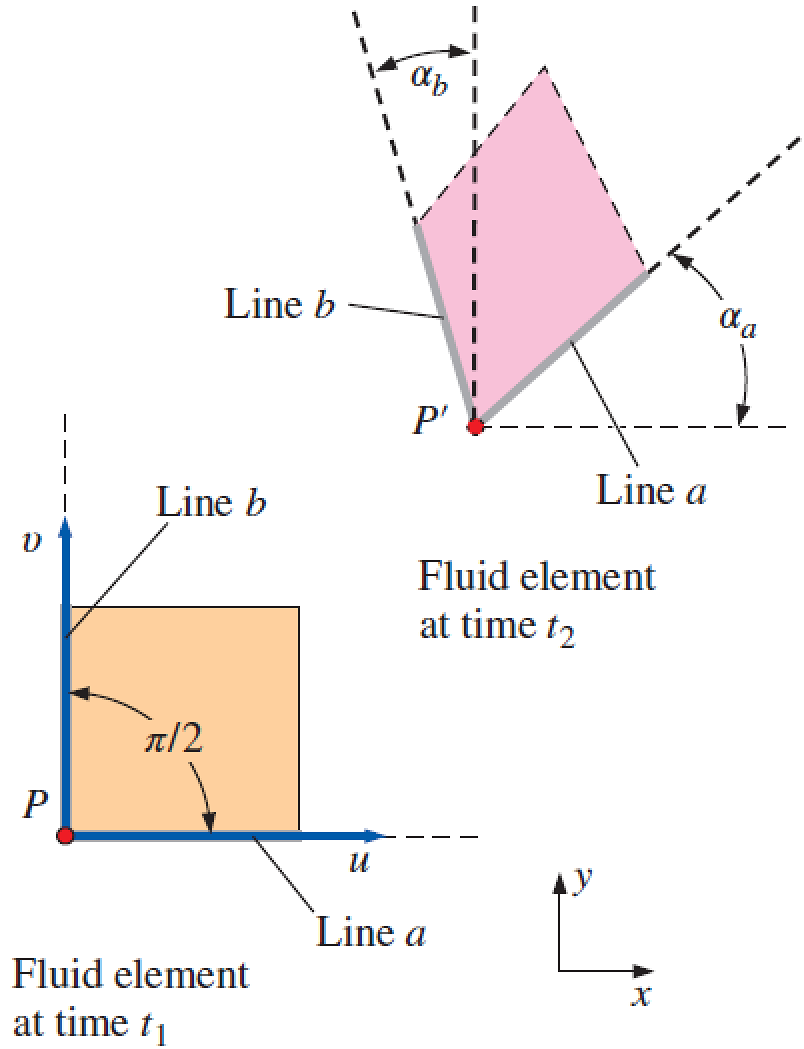
$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

Rate of rotation (angular velocity) at a point: *The average rotation rate of two initially perpendicular lines that intersect at that point*.

Rate of rotation of fluid element about point P

$$\omega = \frac{d}{dt}\left(\frac{\alpha_a + \alpha_b}{2}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For a fluid element that translates and deforms as sketched, the *rate of rotation* at point *P* is defined as the average rotation rate of two initially perpendicular lines (lines *a* and *b*).



The **rate of rotation vector** is equal to the **angular velocity vector**.

Rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Linear strain rate: *The rate of increase in length per unit length*.

Mathematically, the linear strain rate of a fluid element depends on the initial orientation or direction of the line segment upon which we measure the linear strain.

Linear strain rate in Cartesian coordinates:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Volumetric strain rate or bulk strain rate: The rate of increase of volume of a fluid element per unit volume.

This kinematic property is defined as *positive* when the volume *increases*.

Another synonym of volumetric strain rate is also called **rate of volumetric dilatation**, (the iris of your eye dilates (enlarges) when exposed to dim light).

The volumetric strain rate is the sum of the linear strain rates in three mutually orthogonal directions.

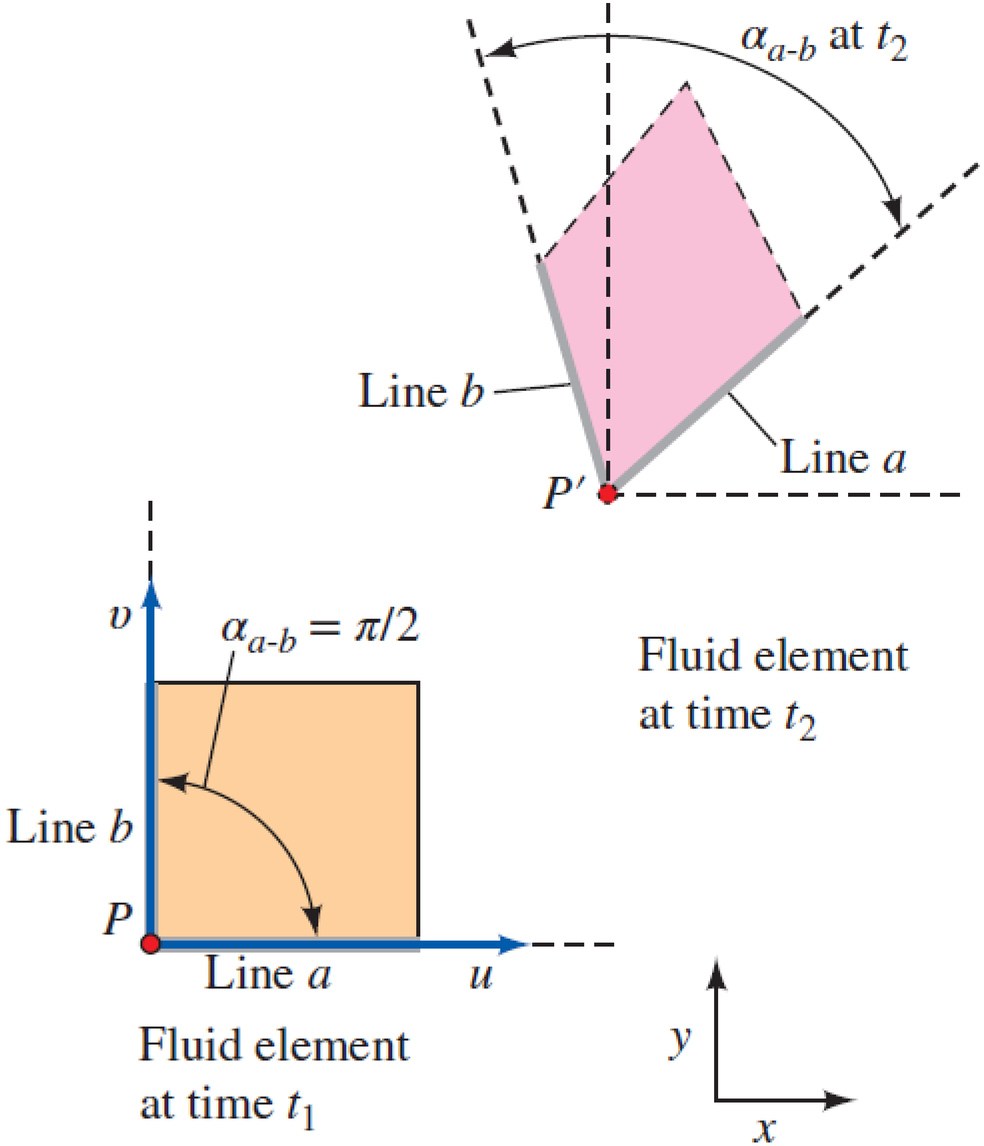
Volumetric strain rate in Cartesian coordinates:

$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The volumetric strain rate is zero in an incompressible flow.

Shear strain rate at a point: *Half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point*.

For a fluid element that translates and deforms as sketched, the *shear strain rate* at point *P* is defined as half of the rate of decrease of the angle between two initially perpendicular lines (lines *a* and *b*).



Shear strain rate, initially perpendicular lines in the x- and y-directions:

$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a-b} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Shear strain rate in Cartesian coordinates:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Strain rate tensor in Cartesian coordinates:

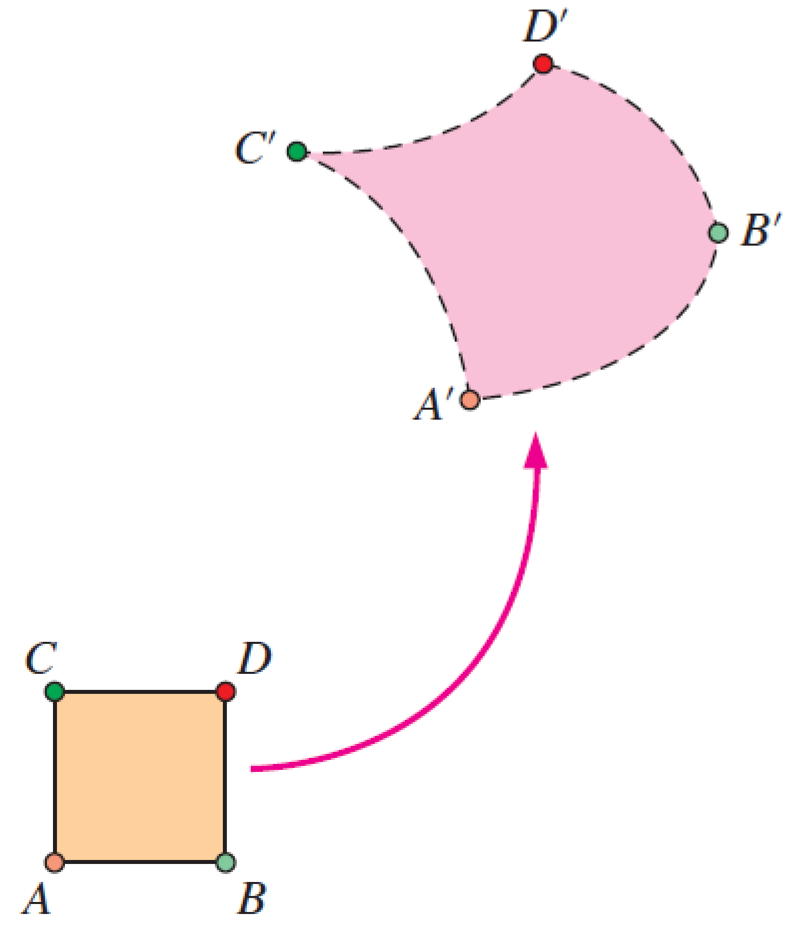
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

Figure shows a general (although two-dimensional) situation in a compressible fluid flow in which all possible motions and deformations are present simultaneously.

In particular, there is translation, rotation, linear strain, and shear strain.

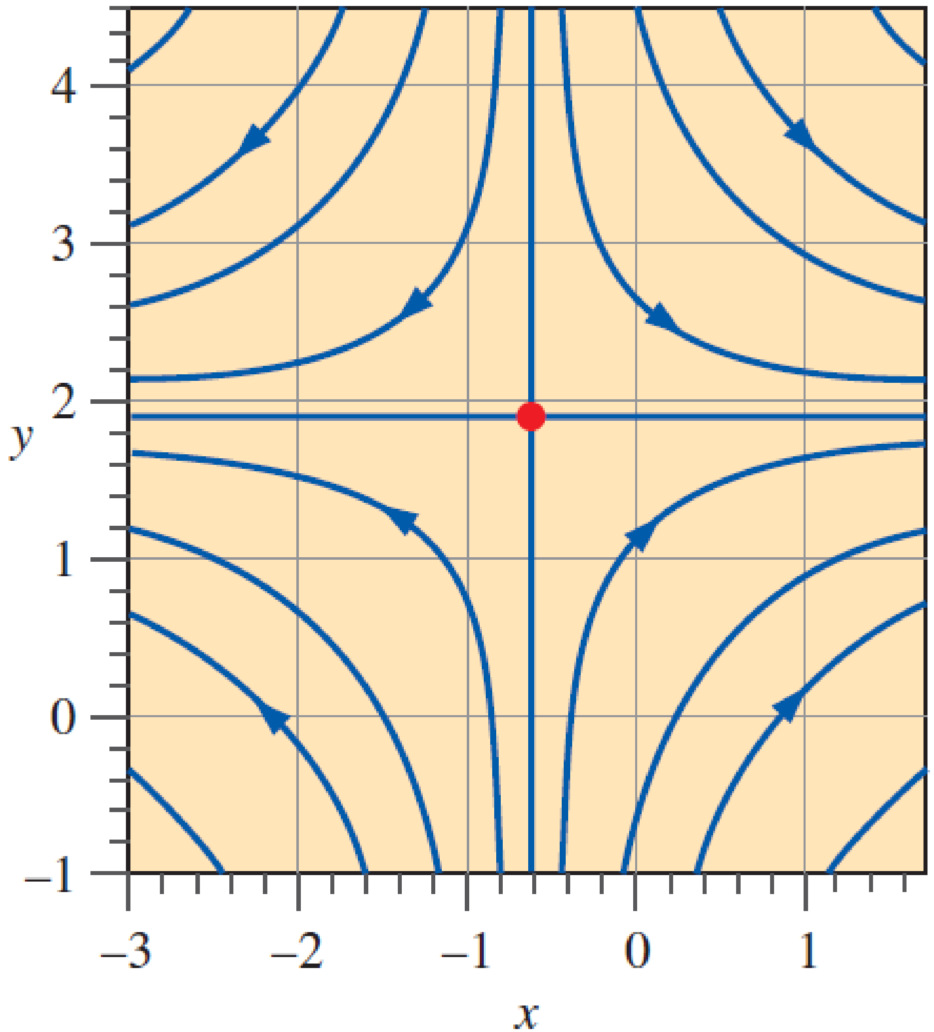
Because of the compressible nature of the fluid flow, there is also volumetric strain (dilatation).

You should now have a better appreciation of the inherent complexity of fluid dynamics, and the mathematical sophistication required to fully describe fluid motion.

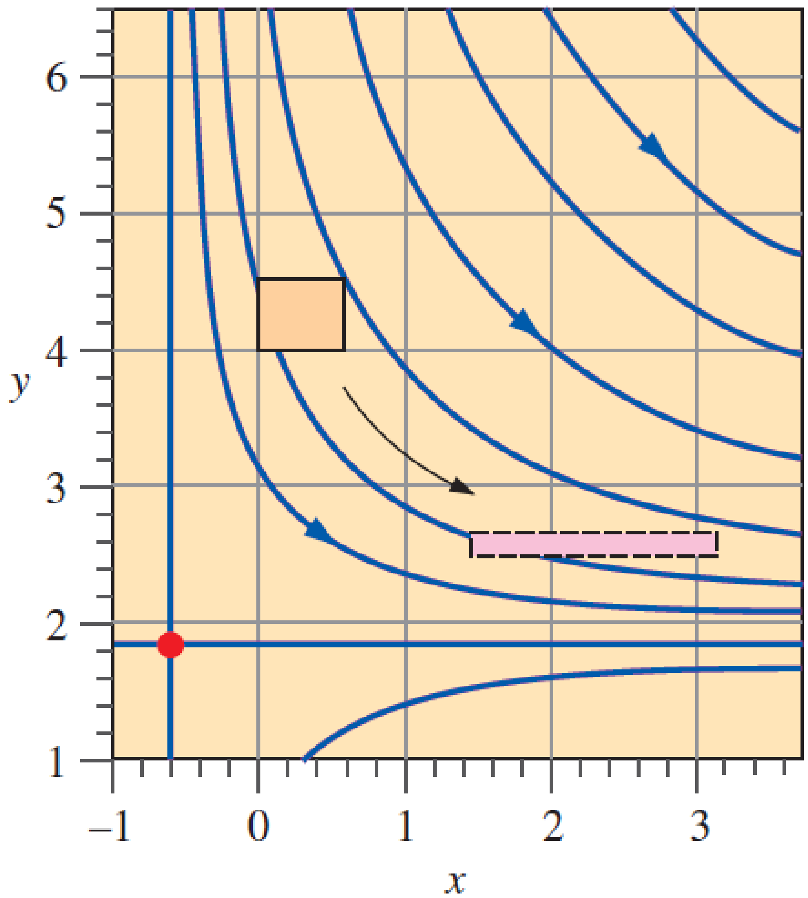


A fluid element illustrating translation, rotation, linear strain, shear strain, and volumetric strain.

$$\vec{V} = (u, v) = (0.5 + 0.8x)\,\vec{i} + (1.5 - 0.8y)\,\vec{j}$$



Streamlines for the velocity field of Example 4–6. The stagnation point is indicated by the red circle at *x* = −0.625 m and *y* = 1.875 m.



Deformation of an initially square parcel of marked fluid subjected to the velocity field of Example 4–6 for a time period of 1.5 s. The stagnation point is indicated by the red circle at $x = -0.625$ m and $y = 1.875$ m, and several streamlines are plotted.

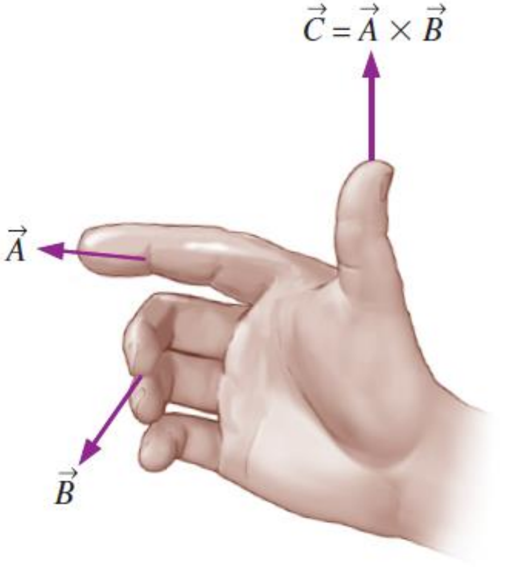
3–5 ■ VORTICITY AND ROTATIONALITY

Another kinematic property of great importance to the analysis of fluid flows is the **vorticity vector**, defined mathematically as the curl of the velocity vector

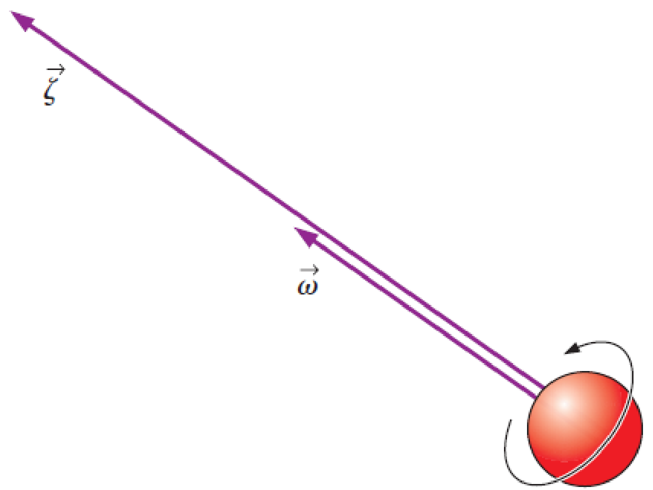
Vorticity vector: $\vec{\zeta} = \vec{\nabla} \times \vec{V} = \text{curl}(\vec{V})$

Rate of rotation vector: $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\text{curl}\left(\vec{V}\right) = \frac{\vec{\zeta}}{2}$

Vorticity is equal to twice the angular velocity of a fluid particle



The direction of a vector cross product is determined by the right-hand rule.

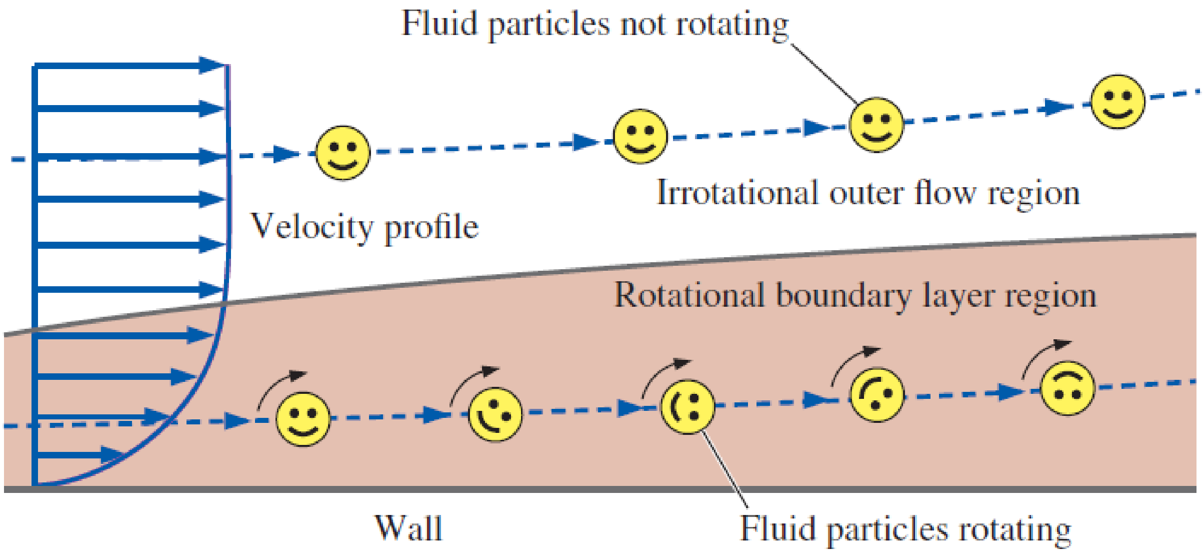


The *vorticity vector* is equal to twice the angular velocity vector of a rotating fluid particle.

If the vorticity at a point in a flow field is nonzero, the fluid particle that happens to occupy that point in space is rotating; the flow in that region is called **rotational**.

Likewise, if the vorticity in a region of the flow is zero (or negligibly small), fluid particles there are not rotating; the flow in that region is called **irrotational**.

Physically, fluid particles in a rotational region of flow rotate end over end as they move along in the flow.



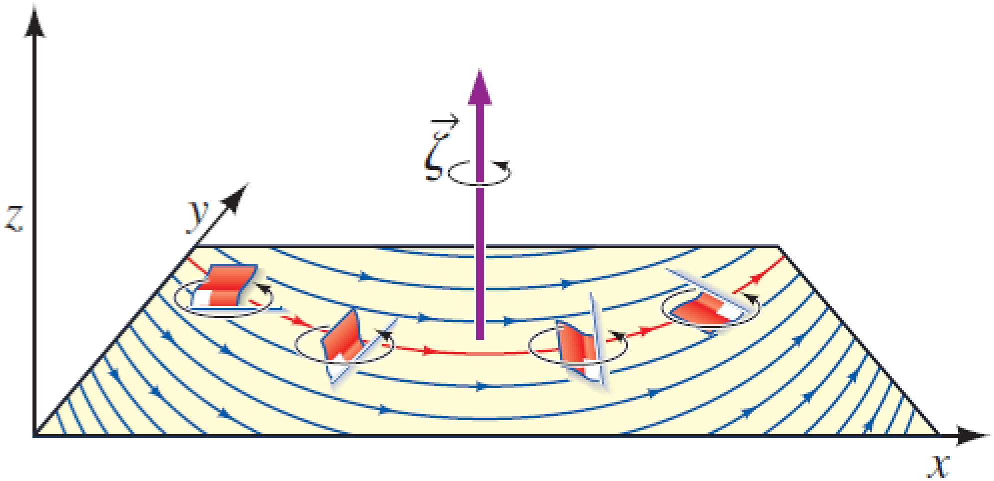
The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.

Vorticity vector in Cartesian coordinates:

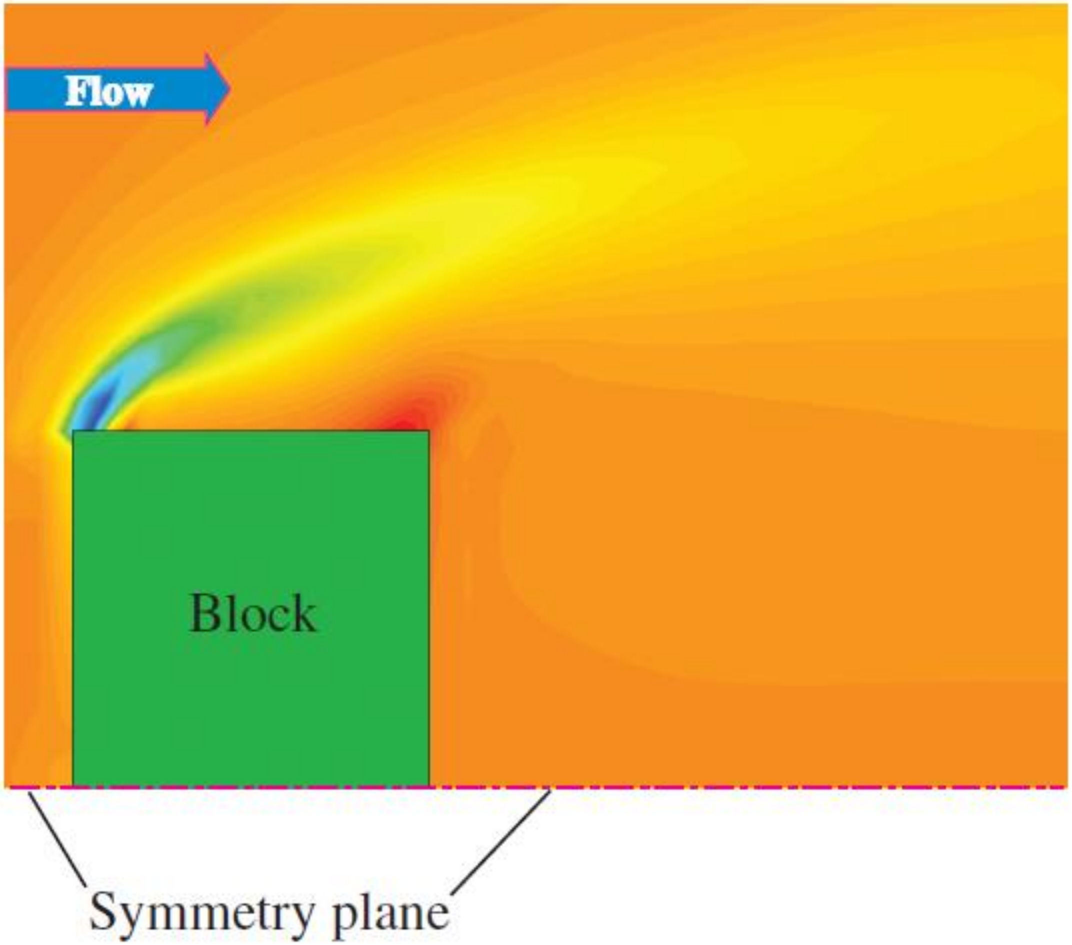
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

Two-dimensional flow in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$



For a two-dimensional flow in the *xy*-plane, the vorticity vector always points in the *z*- or *z*-direction. In this illustration, the flag-shaped fluid particle rotates in the counterclockwise direction as it moves in the *xy*-plane; its vorticity points in the positive *z*-direction as shown.



Contour plot of the vorticity field ζ_z due to flow impinging on a block, as produced by CFD calculations; only the upper half is shown due to symmetry. Blue regions represent large negative vorticity, and red regions represent large positive vorticity.

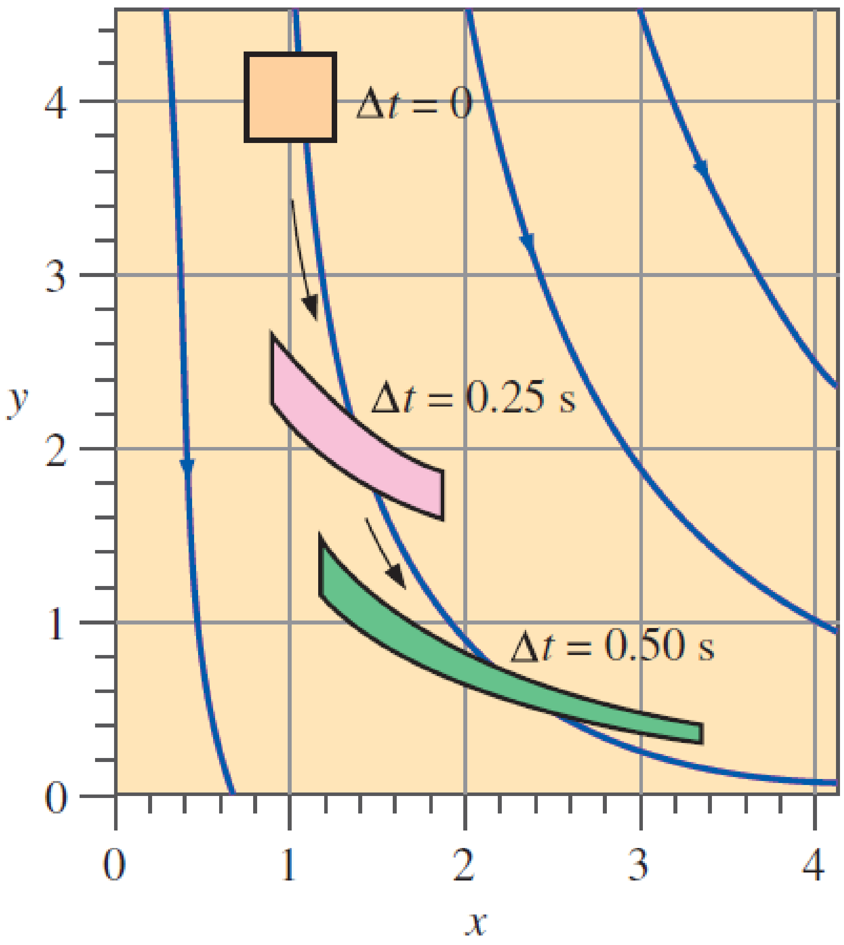
Determination of Rotationality in a Two-Dimensional Flow

steady, incompressible, two-dimensional velocity field:

$$\vec{V} = (u, v) = x^2 \vec{i} + (-2xy - 1)\vec{j}$$

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = (-2y - 0)\vec{k} = -2y\,\vec{k}$$

Vorticity



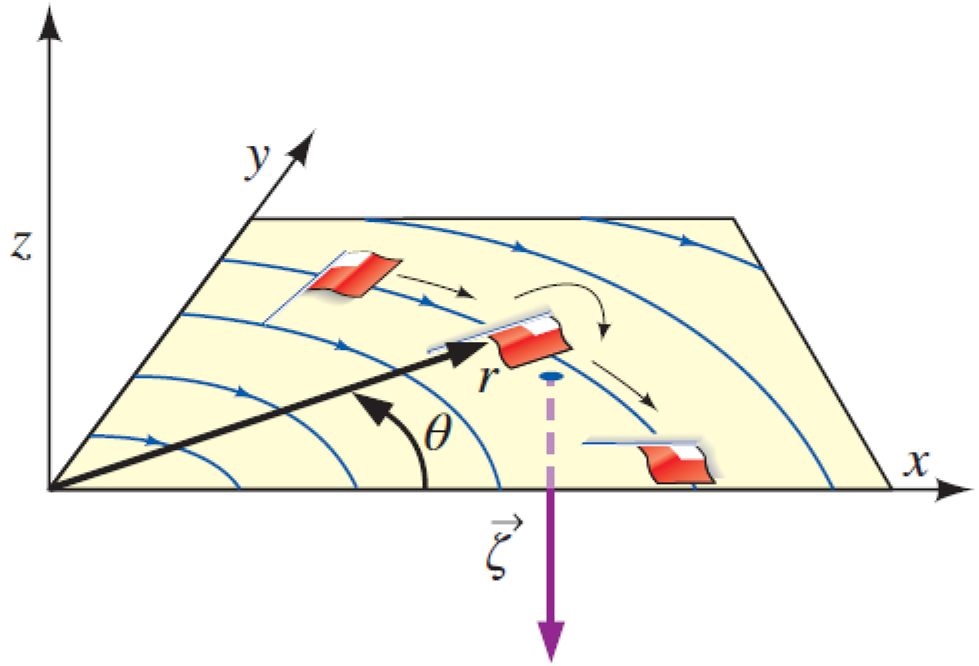
Deformation of an initially square fluid parcel subjected to the velocity field of Example 4–8 for a time period of 0.25 s and 0.50 s. Several streamlines are also plotted in the first quadrant. It is clear that this flow is *rotational*.

Vorticity vector in cylindrical coordinates:

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$$

Two-dimensional flow in cylindrical coordinates:

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{k}$$



For a two-dimensional flow in the $r\theta$-plane, the vorticity vector always points in the z (or z) direction. In this illustration, the flag-shaped fluid particle rotates in the clockwise direction as it moves in the ru-plane; its vorticity points in the z-direction as shown.

Comparison of Two Circular Flows

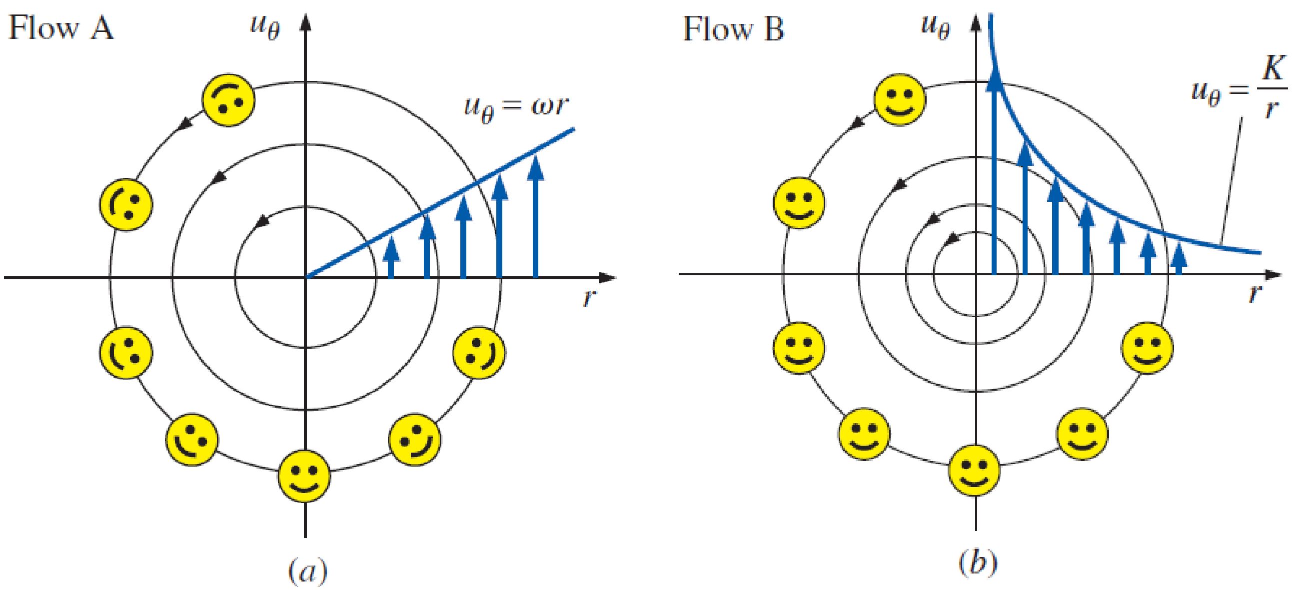
$$\text{Flow A – solid-body rotation:} \quad u_r = 0 \quad \text{and} \quad u_\theta = \omega r$$

$$\text{Flow B – line vortex:} \quad u_r = 0 \quad \text{and} \quad u_\theta = \frac{K}{r}$$

$$\text{Flow A – solid-body rotation:} \quad \vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(\omega r^2\right)}{\partial r} - 0\right)\vec{k} = 2\omega\vec{k}$$

$$\text{Flow B – line vortex:} \quad \vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(K\right)}{\partial r} - 0\right)\vec{k} = 0$$

Streamlines and velocity profiles for (*a*) flow A, solid-body rotation and (*b*) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.



The (oversized) fluid elements in flow B would also distort as they move, but in order to illustrate only particle *rotation*, such distortion is not shown here.

(*a*) (*b*)

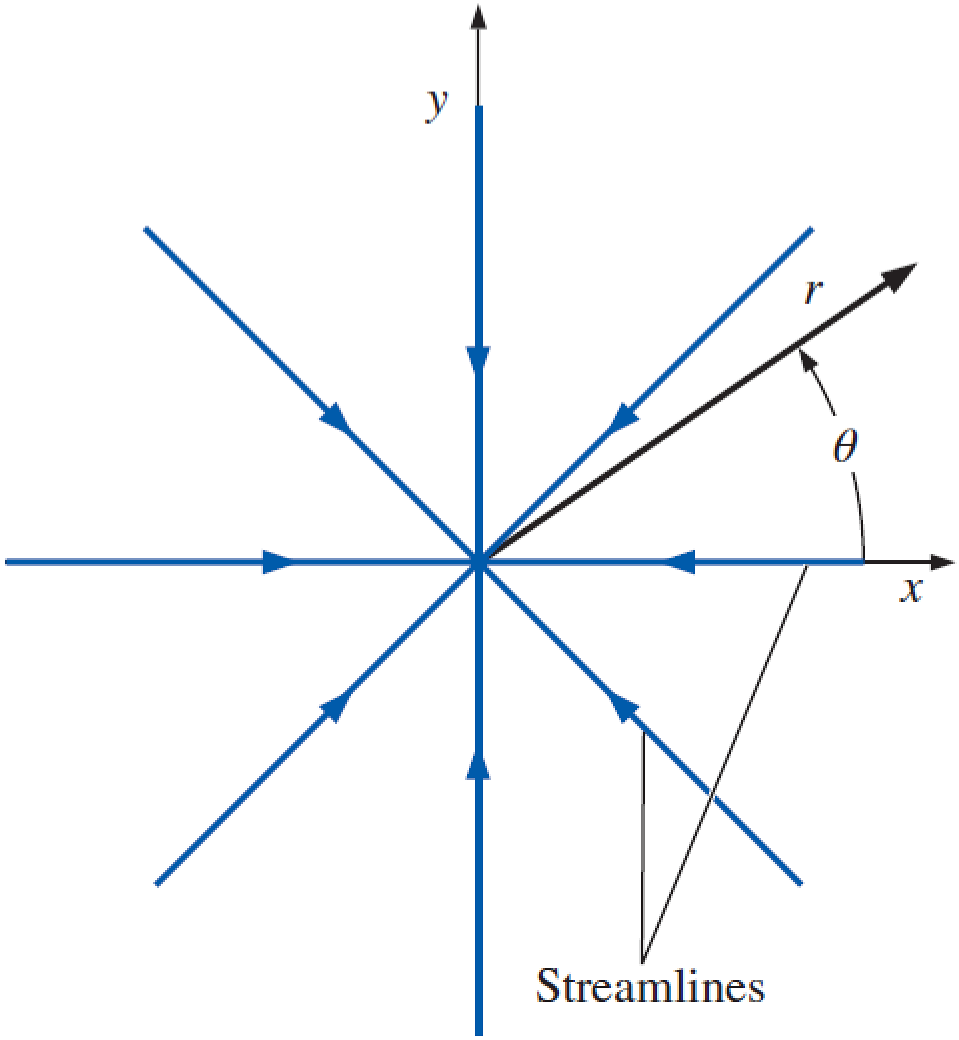
(a) © McGraw-Hill Education/Mark Dierker, photographer (b) © DAJ/Getty Images RF

A simple analogy: (*a*) *rotational* circular flow is analogous to a roundabout, while (*b*) *irrotational* circular flow is analogous to a Ferris wheel.

As children revolve around a roundabout, they also rotate at the same angular velocity as that of the ride itself. This is analogous to a rotational flow.

In contrast, children on a Ferris wheel always remain oriented in an upright position as they trace out their circular path. This is analogous to an irrotational flow.

$$Line\,sink: \qquad u_r = \frac{\dot{V}}{2\pi L}\frac{1}{r} \quad \text{and} \quad u_\theta = 0$$



Streamlines in the *rθ*-plane for the case of a line sink.